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Research Article

The Moore and Bilikam model and Burr XII sub-model under progressively type-II censoring scheme

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Abstract

The Moore and Bilikam family includes lifetime distributions, hence there is a need for a meticulous investigation of the proposed family. We evaluate different estimation procedures for both parameters and reliability function of the Moore and Bilikam family comprehensively, including the maximum likelihood, Bayesian and E-Bayesian estimation methods. The estimation methods of the Moore and Bilikam family are compared via the simulation data, whereas simulation results of the Burr XII sub-model are reported. Based on the simulation approach, we concluded the estimates of the Moore and Bilikam family are convergent to the corresponding parameters, and the root mean square error values derived by the E-Bayesian method are less than other estimators. The analysis of the time between failures of secondary reactor pumps data set has been represented for illustrative purposes, which confirmed simulation results.

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1. Introduction

The statistical distributions play an important role in characterizing attributes of natural phenomena in different fields, such as engineering, environmental, actuarial, medical sciences, biological studies, economics, hydrology, finance, and insurance. Among different families of lifetime distribution, the Moore and Bilikam (MB) family has a significant role in the modeling of lifetime data sets, which covers several distributions, such as exponential, Weibull, Pareto and Rayleigh, Burr XII, Lomax, linear exponential, Gompertz, etc. The MB family of lifetime distribution is introduced by [15], and the Bayesian estimators of the scale parameter and reliability function are obtained based on the type-II censored data.

Mukhopadhyay et al. [16] prepared the two-stage procedures based on the maximum likelihood (ML) and uniformly minimum variance unbiased estimators for the bounded risk point estimation of the parameter and hazard rate in the MB family. Chaturvedi

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et al. [9] focused on the reliability and stress-strength functions of the MB family under the record data sets. Chaturvedi et al. [8] considered the robust Bayesian analysis of parameter, reliability and hazard functions of the MB family under the ε-contamination class of priors under the type-II censoring scheme with squared error loss (SEL) and general entropy loss functions.

Several reasons lead to unaccomplished sample information, such as lack of time, insufficient sources, and the need for high-level reliability of products. Therefore, the censoring scheme is a significant practical design in lifetime data analysis, which means the test is terminated before all items have failed. Among the different schemes, progressive censoring is the most applicable, due to its flexibility in removing surviving units during the experiment. We consider the progressively type-II censoring scheme that can be illustrated as follows. The experiment commences with n units. After each m failure occurred, respectively, $r_1, r_2, \ldots, r_{m-1}$ and r_m, surviving units are randomly withdrawn from the test. At the last failure, the remaining $r_m = n - m - r_1 - \ldots - r_{m-1}$ units are excluded from the experiment. To read more about progressive type-II censoring, we refer to [12] and [1].

The Bayesian estimators for the finite mixture of Rayleigh and mixed exponential based on the censored data are obtained by [22] and [14], respectively. Asl et al. [4] concentrated on the estimation of the Lomax distribution under the progressively type-I hybrid censoring scheme.

Aslam et al. [5] presented the Bayesian estimation of the shifted exponential distribution based on progressively type-II censoring with random removals. The parameters of extended odd Weibull exponential distribution are estimated by [3] under the progressively type-II censoring scheme with random removal with maximum product spacing and ML estimation methods.

The expected Bayesian (E-Bayesian) estimation is a new extension to the Bayesian estimation first proposed by [13], which is an expectation of the Bayesian estimator regarding the distributions of hyperparameters of the prior distribution parameter. Reyad and So [20] obtained the Bayesian and E-Bayesian estimates of the shape parameter of Kumaraswamy distribution under type-II censoring based on the SEL, Linex, Degroot and quadratic loss functions. Yin and Liu [23] estimated the reliability function of the geometric distribution with both hierarchical Bayesian and E-Bayesian estimation methods under the scaled SEL function. Algarni et al. [2] focused on the E-Bayesian estimation of the scale parameter, reliability and hazard rate functions of Chen distribution under the type-I censoring scheme. They considered the balanced squared error loss function and Gamma distribution as a conjugate prior for the E-Bayesian estimators. Rabie and Li [19] investigated the ML, Bayesian and E-Bayesian estimation of the parameter and reliability for Burr-X distribution under hybrid generalized type-II censored.

Although the MB family is very common as the probability model, it is less dealing in literature, so we focus on studying it in the area of lifetime data analysis. Here we focus on the properties of the MB family with a comprehensive investigation of the parameters and reliability function estimations. In particular, we obtain the ML (Newton-Raphson (NR), expectation-maximization (EM) and stochastic EM (SEM) algorithms), Bayesian and E-Bayesian estimators. The Bayesian and E-Bayesian estimators are considered under the SEL function with inverse-gamma and gamma priors for scale and shape parameters, respectively.

The hazard rate function of several distributions has only increasing, decreasing or constant shapes. Thus, they may not be used to model lifetime data with a unimodal hazard function. The most popular MB distributions, including the exponential, Weibull, Rayleigh, Pareto and Gompertz, have monotonic hazard rate functions, whereas the Burr XII covers both monotone and non-monotonic hazard rate functions. The Burr XII of the MB family has several flexible properties, which can be applied to data sets with both monotone and unimodal hazard rate functions, varying degrees of skewness and kurtosis and a wide variety of shapes. Therefore, as a paradigm from the MB family, we concentrated on the Burr XII distribution, and compared the estimation approaches of parameters and reliability function of the proposed distribution via the root mean squared error (RMSE) factor.

The rest of the paper is organized as follows. In section 2, the ML (NR, EM and SEM algorithms), Bayesian and E-Bayesian estimators of the parameter of the proposed family under progressively type-II censoring are represented when both unknown scale and shape parameters. In section 3, the ML, Bayesian and E-Bayesian estimators of the reliability function are obtained. The efficiency of different estimation techniques of the parameters and reliability function of the proposed family are evaluated through the Monte-Carlo simulation study for Burr XII distribution in section 4. Finally, the time between failures of secondary reactor pumps data set is conducted in Section 5 to illustrate the findings of the study.

2. Estimation of the parameters of the MB family

The MB family of lifetime distributions included many probabilistic models, hence the applicability of MB family is evident. The probability density function (PDF) and cumulative distribution function (CDF) of MB family are represented, respectively, as

$$\begin{aligned} f(x,\beta,\theta) &= \frac{\beta}{\theta} g'(x) g^{\beta-1}(x) \exp\big(-\frac{g^{\beta}(x)}{\theta}\big), \qquad x,\beta,\theta \geq 0, \\ F(x,\beta,\theta) &= 1-\exp\big(-\frac{g^{\beta}(x)}{\theta}\big), \end{aligned}$$

where $g(x)$ is a real-valued, strictly increasing function of x with $g(0^+) = 0$, $g(\infty) = \infty$, and $g'(x)$ denotes the derivative of $g(x)$ with respect to x.

Progressive censoring is useful in many fields of science that allows the removal of surviving experimental units before the termination of the test. Here, we concentrate on the progressively type-II censoring scheme.

Consider the sample size n and $t_1 < \ldots < t_m$ be a progressively type-II censored sample with sample size m, where the ordered lifetimes have a MB family of distribution with a specified number of removal, as $(r_1, \ldots, r_m)$, then the likelihood function is given by

$$\begin{aligned} L(\beta,\theta,T) &= C\prod_{i=1}^{m} f(t_i,\beta,\theta)[1-F(t_i,\beta,\theta)]^{r_i} \qquad (2.1) \\ &= C\Big(\frac{\beta}{\theta}\Big)^m \prod_{i=1}^{m}\Big[g'(t_i)g^{\beta-1}(t_i)\Big]\exp\Big(\frac{-\sum_{i=1}^{m} g^{\beta}(t_i)}{\theta}\Big)\prod_{i=1}^{m}\Big[\exp(-\frac{g^{\beta}(t_i)}{\theta})\Big]^{r_i} \\ &= C\Big(\frac{\beta}{\theta}\Big)^m \prod_{i=1}^{m}\Big[g'(t_i)g^{\beta-1}(t_i)\Big]\exp\Big(\frac{-\sum_{i=1}^{m}(r_i+1) g^{\beta}(t_i)}{\theta}\Big), \end{aligned}$$

where $C = n(n-1-r_1)\ldots(n-r_1-\ldots-r_{m-1}-m+1)$ and both θ and β parameters are unknown.

We consider an especial case of the MB family with $g(x) = \ln(1+x^b),\ b > 0, \beta = 1$ called Burr XII, with the following PDF and CDF

$$\begin{aligned} f(x,b,\theta) &= \frac{bx^{b-1}}{\theta(1+x^b)^{\frac{1}{\theta}+1}}, \qquad b,\theta > 0, x \geq 0 \\ F(x,b,\theta) &= 1-\frac{1}{(1+x^b)^{\frac{1}{\theta}}}. \end{aligned}$$

In the following several statistical properties of the MB family and especially the Burr XII distribution are provided.

2.1. Some statistical properties of the MB family

In this section, some statistical properties of the MB family and Burr XII distribution are provided, including the r-th moments, moment generating function (MGF) and incomplete moments.

Proposition 2.1. *Consider the MB family of lifetime distributions and the especial case Burr XII distribution, then*

(i) *The r-moments are provided as*

$$E(X^r) = \int_0^\infty e^{-u}\Big[g^{-1}\Big(\theta u\Big)^{\frac{1}{\beta}}\Big)\Big]^r du,$$

where $g^{-1}(.)$ is the inverse of $g(.)$. For Burr XII, consider $u = \frac{1}{1+x^b}$, so we have

$$E(X^r) = \int_0^\infty \frac{bx^{r+b-1}}{\theta(1+x^b)^{\frac{1}{\theta}+1}}dx = \frac{1}{\theta}\int_0^1 u^{\frac{1}{\theta}-1}\Big(\frac{1}{u}-1\Big)^{\frac{r}{b}}du = \frac{1}{\theta}B\Big(\frac{1}{\theta}-\frac{r}{b},\frac{r}{b}+1\Big),$$

where $B(.,.)$ is the Beta function as $B(\alpha_1,\alpha_2) = \int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1}du$.

(ii) *The MGF of the MB family is given by*

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{-u}e^{t(g^{-1}(\theta u)^{\frac{1}{\beta}})}du,$$

and for Burr XII distribution, the MGF is derived as

$$\begin{aligned} E(e^{tX}) &= \int_0^\infty \frac{bx^{b-1}e^{tx}}{\theta(1+x^b)^{\frac{1}{\theta}+1}}dx = \frac{1}{\theta}\sum_{h=0}^\infty \frac{t^h}{h!}\int_0^1 u^{\frac{1}{\theta}-1}\Big(\frac{1}{u}-1\Big)^{\frac{h}{b}}du \\ &= \sum_{h=0}^\infty \frac{t^h\Gamma(\frac{1}{\theta}-\frac{h}{b})\Gamma(\frac{h}{b}+1)}{h!\Gamma(\frac{1}{\theta})}. \end{aligned}$$

(iii) *The incomplete moment is represented as*

$$m_r(y) = \int_0^{\frac{g^\beta(y)}{\theta}} e^{-u}\Big[g^{-1}\Big(\theta u\Big)^{\frac{1}{\beta}}\Big)\Big]^r du,$$

where $m_r(y) = \int_0^y x^r f(x,\beta,\theta)dx$ and for Burr XII distribution, we have

$$\begin{aligned} m_r(y) &= \int_0^y \frac{bx^{r+b-1}}{\theta(1+x^b)^{\frac{1}{\theta}+1}}dx = \frac{1}{\theta}\int_{\frac{1}{1+y^b}}^1 u^{\frac{1}{\theta}-1}\Big(\frac{1}{u}-1\Big)^{\frac{r}{b}}du \\ &= \frac{1}{\theta}\Big(1 - B\Big(\frac{1}{1+y^b},\frac{1}{\theta}-\frac{r}{b},\frac{r}{b}+1\Big)\Big) \end{aligned}$$

where $B(.,.,.)$ is the incomplete Beta function as $B(z,\alpha_1,\alpha_2) = \int_0^z u^{\alpha_1-1}(1-u)^{\alpha_2-1}du$.

2.2. The maximum likelihood estimation approaches

Consider the MB family with the type II censoring scheme and the likelihood function (2.1), by ignoring the additive constant C, the log-likelihood function is obtained as

$$\mathcal{L}(\beta,\theta,T) \propto m\ln\beta - m\ln\theta + \sum_{i=1}^m \ln g'(t_i) + (\beta-1)\sum_{i=1}^m \ln g(t_i) - \sum_{i=1}^m (r_i+1)\frac{g^\beta(t_i)}{\theta}. \quad (2.2)$$

The ML estimator of β and θ can be obtained by setting the score functions of (2.2) equal zero, as follows

$$\frac{\partial \mathcal{L}}{\partial \theta} = -\frac{m}{\theta} + \sum_{i=1}^{m}(r_i+1)\frac{g^{\beta}(t_i)}{\theta^2} = 0, \tag{2.3}$$

$$\frac{\partial \mathcal{L}}{\partial \beta} = \frac{m}{\beta} + \sum_{i=1}^{m}\ln g(t_i) - \sum_{i=1}^{m}(r_i+1)\frac{\ln(g(t_i))g^{\beta}(t_i)}{\theta} = 0. \tag{2.4}$$

From (2.3), we have

$$\hat{\theta}(\beta) = \frac{1}{m}\sum_{i=1}^{m}(r_i+1)g^{\beta}(t_i), \tag{2.5}$$

by substituting (2.5) into (2.4), the equation will be reduced to the following relation

$$\frac{m}{\beta} = -\sum_{i=1}^{m}\ln g(t_i) + \sum_{i=1}^{m}(r_i+1)\frac{\ln(g(t_i))g^{\beta}(t_i)}{\hat{\theta}(\beta)}.$$

The last equation does not have an analytical solution, hence numerical methods such as NR are applied to find the ML estimate of β. Subsequently by inserting this value in (2.5), the ML estimate of θ is provided.

The ML estimates obtained via the NR method are sensitive to the initial parameter values and converge slowly to the real values of the parameters in certain cases. Moreover, the NR estimators under the censoring scheme are significantly biased. All these deficiencies persuade us to discuss the different estimation approaches.

In the following, the EM and SEM algorithms are provided. Moreover, the Bayesian and E-Bayesian estimation procedures are investigated for more accurate discussion.

2.2.1. EM algorithm. The EM algorithm was recommended by [10] to estimate any missing or incomplete data. Some superiorities of the EM algorithm to NR are the facility in running and computational stability with the appropriate convergence rate. Also, the asymptotic behavior of the EM algorithm estimates can be obtained.
The progressive type-II censoring can be considered as an incomplete data set, and therefore, the EM algorithm is recommended instead of the NR method to find the ML estimators.

Let $Z = (Z_1, Z_2, \ldots, Z_m)$ with $Z_j = (Z_{j1}, Z_{j2}, \ldots, Z_{jR_j})$, $j = 1, \ldots, m$, be the censored data and suppose the censored data as missing and observed data as $T = (t_1, \ldots, t_m)$. Therefore, the complete data set X is constructed from the combination of $(T, Z) = X$. The joint likelihood function of the complete sample is represented as

$$\begin{aligned}
L_c(\beta,\theta,X) &= \prod_{i=1}^{m} f(t_i,\beta,\theta)\prod_{k=1}^{r_i} f(z_{ik},\beta,\theta)\\
&= \Big(\frac{\beta}{\theta}\Big)^n \prod_{i=1}^{m}\Big[g'(t_i)g^{\beta-1}(t_i)\exp\big(-\frac{g^{\beta}(t_i)}{\theta}\big)\prod_{k=1}^{r_i} g'(z_{ik})g^{\beta-1}(z_{ik})\exp\big(-\frac{g^{\beta}(z_{ik})}{\theta}\big)\Big]\\
&= \Big(\frac{\beta}{\theta}\Big)^n \prod_{i=1}^{m}\Big[g'(t_i)g^{\beta-1}(t_i)\prod_{k=1}^{r_i} g'(z_{ik})g^{\beta-1}(z_{ik})\Big]\exp\big(-\sum_{i=1}^{m}\frac{g^{\beta}(t_i)}{\theta}\big)\\
&\quad \exp\big(-\sum_{i=1}^{m}\sum_{k=1}^{r_i}\frac{g^{\beta}(z_{ik})}{\theta}\big).
\end{aligned}$$

So, the complete log-likelihood function based on X is given by

$$\mathcal{L}_c(\beta,\theta,X) \propto n\ln\beta - n\ln\theta + \sum_{i=1}^{m}\ln g'(t_i) + \sum_{i=1}^{m}\sum_{k=1}^{r_i}\ln g'(z_{ik}) + (\beta-1)\sum_{i=1}^{m}\ln g(t_i) \tag{2.6}$$
$$+ (\beta-1)\sum_{i=1}^{m}\sum_{k=1}^{r_i}\ln g(z_{ik}) - \sum_{i=1}^{m}\frac{g^{\beta}(t_i)}{\theta} - \sum_{i=1}^{m}\sum_{k=1}^{r_i}\frac{g^{\beta}(z_{ik})}{\theta}.$$

The ML estimators of the parameters θ and β for complete sample X can be achieved by equating the score functions of the log-likelihood function (2.6) to zero, as below

$$\frac{\partial\mathcal{L}_c}{\partial\theta} = -\frac{n}{\theta} + \frac{\sum_{i=1}^{m} g^{\beta}(t_i) + \sum_{i=1}^{m}\sum_{k=1}^{r_i} g^{\beta}(z_{ik})}{\theta^2} = 0, \tag{2.7}$$
$$\frac{\partial\mathcal{L}_c}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{m}\ln g(t_i) + \sum_{i=1}^{m}\sum_{k=1}^{r_i}\ln g(z_{ik}) \tag{2.8}$$
$$-\frac{1}{\theta}\Big[\sum_{i=1}^{m}\ln(g(t_i))g^{\beta}(t_i) + \sum_{i=1}^{m}\sum_{k=1}^{r_i}\ln(g(z_{ik}))g^{\beta}(z_{ik})\Big] = 0.$$

The EM algorithm has the two step, E-step and M-step. In the E-step, any function of Z_{ik} must be replaced by $E(h(Z_{ik})|Z_{ik} > t_i)$. Therefore, (2.7) and (2.8) can be represented as

$$\frac{\partial\mathcal{L}_c}{\partial\theta} = -\frac{n}{\theta} + \frac{\sum_{i=1}^{m} g^{\beta}(t_i) + \sum_{i=1}^{m}\sum_{k=1}^{r_i} E(g^{\beta}(z_{ik})|Z_{ik} > t_i)}{\theta^2}, \tag{2.9}$$
$$\frac{\partial\mathcal{L}_c}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{m}\ln g(t_i) + \sum_{i=1}^{m}\sum_{k=1}^{r_i} E(\ln g(z_{ik})|Z_{ik} > t_i) \tag{2.10}$$
$$-\frac{1}{\theta}\Big[\sum_{i=1}^{m}\ln(g(t_i))g^{\beta}(t_i) + \sum_{i=1}^{m}\sum_{k=1}^{r_i} E(\ln g(z_{ik})g^{\beta}(z_{ik})|Z_{ik} > t_i)\Big].$$

Given $T_i = t_i$, the conditional distribution of Z_{ik} follows a truncated MB distribution with left truncation at t_i as

$$f_{z|t}(z_{ik}|T) = \frac{f(z_{ik})}{1-F(t_i)} = \frac{\beta}{\theta}g'(z_{ik})g^{\beta-1}(z_{ik})\exp(-\frac{g^{\beta}(z_{ik})}{\theta})\exp(\frac{g^{\beta}(t_i)}{\theta}),\ z_{ik} > t_i. \tag{2.11}$$

Hence, by (2.11), the conditional expectations in equations (2.9) and (2.10) can be computed as follows

$$\begin{aligned}
E_1(t,\theta,\beta) &= E(g^{\beta}(z_{ik})|Z_{ik} > t) \qquad (2.12)\\
&= \frac{\beta}{\theta}\exp(\frac{g^{\beta}(t)}{\theta})\int_t^{\infty} g^{2\beta-1}(z_{ik})g'(z_{ik})\exp(-\frac{g^{\beta}(z_{ik})}{\theta})dz_{ik},\\
E_2(t,\theta,\beta) &= E(\ln(g(z_{ik}))|Z_{ik} > t)\\
&= \frac{\beta}{\theta}\exp(\frac{g^{\beta}(t)}{\theta})\int_t^{\infty}\ln(g(z_{ik}))g^{\beta-1}(z_{ik})g'(z_{ik})\exp(-\frac{g^{\beta}(z_{ik})}{\theta})dz_{ik},\\
E_3(t,\theta,\beta) &= E(\ln(g(z_{ik}))g^{\beta}(z_{ik})|Z_{ik} > t)\\
&= \frac{\beta}{\theta}\exp(\frac{g^{\beta}(t)}{\theta})\int_t^{\infty}\ln(g(z_{ik}))g^{2\beta-1}(z_{ik})g'(z_{ik})\exp(-\frac{g^{\beta}(z_{ik})}{\theta})dz_{ik}.
\end{aligned}$$

In the M-step, at (k+1)-th iteration, the value of $\hat{\theta}^{(k+1)}$ of the EM algorithm, is obtained by solving the following equation

$$\frac{\partial\mathcal{L}_c}{\partial\theta} = -\frac{n}{\hat{\theta}^{(k+1)}} + \frac{\sum_{i=1}^{m} g^{\beta^{(k)}}(t_i) + \sum_{i=1}^{m} r_i E_1(t_i,\hat{\theta}^{(k)},\beta^{(k)})}{\hat{\theta}^{2(k+1)}} = 0, \tag{2.13}$$

once $\hat{\theta}^{(k+1)}$ is obtained, then $\hat{\beta}^{(k+1)}$ is obtained by solving the equation

$$\frac{n}{\hat{\beta}^{(k+1)}} + \sum_{i=1}^{m} \ln g(t_i) + \sum_{i=1}^{m} r_i E_2(t_i, \hat{\theta}^{(k+1)}, \hat{\beta}^{(k)}) - \frac{1}{\hat{\theta}^{(k+1)}} \Big[\sum_{i=1}^{m} \ln(g(t_i)) g^{\hat{\beta}^{(k+1)}}(t_i) + \sum_{i=1}^{m} r_i E_3(t_i, \hat{\theta}^{(k+1)}, \hat{\beta}^{(k)}) \Big] = 0. \tag{2.14}$$

$(\hat{\theta}^{(k+1)}, \hat{\beta}^{(k+1)})$ is implemented as a new value of (θ, β) in the subsequent iteration and the steps are repeated until reach convergence. The iterative procedure in Equations (2.13) and (2.14) are stopped on reaching the convergence as $|\hat{\theta}^{(k+1)} - \hat{\theta}^{(k)}| + |\hat{\beta}^{(k+1)} - \hat{\beta}^{(k)}| < \varepsilon$, where $\varepsilon > 0$ is an arbitrary small value. Cancho et al. [7] investigated the uniqueness and existence of the ML estimates.

Here, the closed-form of the expectations (2.12) are obtained for Burr XII distribution. For $g(x) = \ln(1 + x^b)$, consider $u = \ln(1 + z_{ik}^b)$, the conditional expectations in (2.12) are computed as follows

$$\begin{aligned}
E_1(t, \theta, \beta) &= E\big(g(z_{ik}) | Z_{ik} > t\big) = \frac{(1+t^b)^{\frac{1}{\theta}}}{\theta} \int_t^{\infty} \frac{b z_{ik}^{b-1} \ln(1+z_{ik}^b)(1+z_{ik}^b)^{-\frac{1}{\theta}}}{1+z_{ik}^b} dz_{ik} \\
&= \frac{(1+t^b)^{\frac{1}{\theta}}}{\theta} \int_{\ln(1+t^b)}^{\infty} u e^{\frac{-u}{\theta}} du = \ln(1+t^b) + \theta, \\
E_2(t, \theta, \beta) &= E\Big(\ln(g(z_{ik})) | Z_{ik} > t\Big) = \frac{(1+t^b)^{\frac{1}{\theta}}}{\theta} \int_t^{\infty} \frac{b z_{ik}^{b-1} \ln\big(\ln(1+z_{ik}^b)\big)(1+z_{ik}^b)^{-\frac{1}{\theta}}}{1+z_{ik}^b} dz_{ik} \\
&= \frac{(1+t^b)^{\frac{1}{\theta}}}{\theta} \int_{\ln(1+t^b)}^{\infty} \ln(u) e^{\frac{-u}{\theta}} du \\
&= \ln\big(\frac{\ln(1+t^b)}{\theta}\big) - (1+t^b)^{\frac{1}{\theta}} \mathrm{Ei}\big(-\frac{\ln(1+t^b)}{\theta}\big) + \ln(\theta), \\
E_3(t, \theta, \beta) &= E\Big(\ln(g(z_{ik})) g(z_{ik}) | Z_{ik} > t\Big) \\
&= \frac{(1+t^b)^{\frac{1}{\theta}}}{\theta} \int_t^{\infty} \frac{b z_{ik}^{b-1} \ln\big(\ln(1+z_{ik}^b)\big)(1+z_{ik}^b)^{-\frac{1}{\theta}}}{1+z_{ik}^b} dz_{ik} \\
&= \frac{(1+t^b)^{\frac{1}{\theta}}}{\theta} \int_{\ln(1+t^b)}^{\infty} u \ln(u) e^{\frac{-u}{\theta}} du \\
&= \Big(\ln(\theta) + \ln\Big(\frac{\ln(1+t^b)}{\theta}\Big)\Big)\Big(\ln(1+t^b) + \theta\Big) - \theta(1+t^b)^{\frac{1}{\theta}} \mathrm{Ei}\big(-\frac{\ln(1+t^b)}{\theta}\big) + \theta,
\end{aligned}$$

where Ei(.) is the exponential integral function defined as $\mathrm{Ei}(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t} dt$.

2.2.2. SEM algorithm. The SEM algorithm is a stochastic version of the EM algorithm, which replaces each missing observation by a value randomly generated from the distribution conditional on results from the previous step and the M-step is a complete-data ML estimation with convenient computation. Nielsen [18] proved that the SEM algorithm always converges to some local optimum.
In the SEM algorithm, r_i number of samples of z_{ik} must be generated from the following conditional CDF for $i = 1, 2, \ldots, m,\ k = 1, 2, \ldots, r_i$

$$F_{z|t}(z_{ik}|t_i) = \frac{F(z_{ik}) - F(t_j)}{1 - F(t_j)} = 1 - \exp\big(-\frac{g^{\beta}(z)}{\theta}\big) \exp\big(\frac{g^{\beta}(t)}{\theta}\big), \qquad z_{ik} > t_i.$$

Now, using (2.7) and (2.8), the estimators of θ at $k+1$-step of the algorithm can be obtained as follows

$$\hat{\theta}^{(k+1)}_{SEM} = \frac{\sum_{i=1}^{m} g^{\hat{\beta}^{(k)}_{SEM}}(t_i) + \sum_{i=1}^{m}\sum_{k=1}^{r_i} g^{\hat{\beta}^{(k)}_{SEM}}(z_{ik})}{n},$$

and the estimators of β at the $k+1$-step of the algorithm is obtained by solving the following equation

$$\begin{aligned} \frac{\partial \mathcal{L}_c}{\partial \beta} &= \frac{n}{\beta} + \sum_{i=1}^{m} \ln g(t_i) + \sum_{i=1}^{m}\sum_{k=1}^{r_i} \ln g(z_{ik}) \\ &- \frac{1}{\hat{\theta}^{(k+1)}_{SEM}} \Big[\sum_{i=1}^{m} \ln(g(t_i)) g^{\beta}(t_i) + \sum_{i=1}^{m}\sum_{k=1}^{r_i} \ln(g(z_{ik})) g^{\beta}(z_{ik}) \Big] = 0. \end{aligned}$$

2.3. Bayesian estimation

In this section, the Bayesian estimators of parameters θ and β of the MB family of distribution under the progressively type-II censoring scheme are determined when both θ and β are unknown, under the SEL function.

Consider independent inverse-Gamma prior for the parameter θ and Gamma prior for the parameter β with the PDFs as

$$\pi_1(\theta|\alpha_1, \lambda_1) = \frac{\lambda_1^{\alpha_1} \theta^{-(\alpha_1+1)} e^{-\frac{\lambda_1}{\theta}}}{\Gamma(\alpha_1)}, \tag{2.15}$$

$$\pi_2(\beta|\alpha_2, \lambda_2) = \frac{\lambda_2^{\alpha_2} \beta^{\alpha_2-1} e^{-\lambda_2 \beta}}{\Gamma(\alpha_2)}, \tag{2.16}$$

where $\alpha_i, \lambda_i,\ i = 1, 2$ are known hyperparameters and chosen to reflect prior knowledge about θ and β. The prior distributions of θ and β are given by

$$\pi(\theta, \beta) = \pi_1(\theta|\alpha_1, \lambda_1)\pi_2(\beta|\alpha_2, \lambda_2).$$

It follows from (2.2), (2.15) and (2.16) that the joint posterior density function of θ and β given T is shown by

$$\begin{aligned} \pi(\theta, \beta|T) &\propto \beta^{\alpha_2+m-1} e^{-\lambda_2 \beta} \theta^{-(\alpha_1+m+1)} e^{-\frac{\lambda_1 + \sum_{i=1}^{m}(r_i+1) g^{\beta}(t_i)}{\theta}} \prod_{i=1}^{m} g^{\beta}(t_i) \qquad (2.17) \\ &\propto \mathrm{G}_{\beta}(\alpha_2+m, \lambda_2) \mathrm{IG}_{\theta|\beta}\Big(\alpha_1+m, \lambda_1 + \sum_{i=1}^{m}(r_i+1) g^{\beta}(t_i)\Big) Q(\beta), \end{aligned}$$

where G(., .) and IG(., .) represent the PDF of Gamma and inverse-Gamma distribution, respectively, and $Q(\beta, \theta)$ is given by

$$Q(\beta) = \prod_{i=1}^{m} g^{\beta}(t_i).$$

The Bayesian estimator of a function $g(\eta)$ under the SEL function is given by

$$\hat{g}_{SEL}(\eta) = E(g(\eta|X)).$$

The above expression does not have a closed-form, so the importance sampling (IS) technique is implemented to achieve the Bayesian estimators. Based on the Burr XII distribution, the IS algorithm is represented in the following, where parameters θ and b have Gamma(α_1, λ_1) and inverse-Gamma(α_2, λ_2) priors, respectively.

IS algorithm for Bayesian estimates

1. Generate the parameter b from the $\mathrm{G}_b(\alpha_2+m,\lambda_2)$.
2. Based on the value of b, generate θ from $\mathrm{IG}_{\theta|b}\Big(\alpha_1+m,\lambda_1+\sum_{i=1}^m(r_i+1)\ln(1+t_i^b)\Big)$ distribution.
3. Repeat s times steps 1 and 2, to reach $\{(b_1,\theta_1),\dots,(b_s,\theta_s)\}$.
4. Compute the Bayesian estimates of b and θ under SEL function as follows

$$\hat{\theta}_B=\frac{\sum_{i=1}^s\theta_iQ(b_i)}{\sum_{i=1}^sQ(b_i)},\qquad \hat{b}_B=\frac{\sum_{i=1}^sb_iQ(b_i)}{\sum_{i=1}^sQ(b_i)}.$$

where $Q(b_i)=\prod_{i=1}^m\frac{t_i^{b-1}}{1+t_i^b}$.

2.4. The E-Bayesian estimators

This section deals with the E-Bayesian estimation of the parameters of the MB family based on the progressively type-II censoring scheme. Here, we consider the inverse-Gamma and Gamma priors for parameters θ and β, respectively.

According to [13], the prior parameters θ and β should be selected to guarantee that the prior $\pi_1(\theta|\alpha_1,\lambda_1)$ and $\pi_2(\beta|\alpha_2,\lambda_2)$ is a decreasing function of θ and β, respectively. To ensure this condition is met, the first derivative of $\pi_1(\theta|\alpha_1,\lambda_1)$ and $\pi_2(\beta|\alpha_2,\lambda_2)$ with respect to θ and β are represented in the following

$$\begin{aligned}\frac{\partial\pi_1(\theta|\alpha_1,\lambda_1)}{\partial\theta}&=\frac{\lambda_1^{\alpha_1}\theta^{-\alpha_1-2}e^{-\frac{\lambda_1}{\theta}}}{\Gamma(\alpha_1)}\Big[-\alpha_1-1+\frac{\lambda_1}{\theta}\Big],\\ \frac{\partial\pi_2(\beta|\alpha_2,\lambda_2)}{\partial\beta}&=\frac{\lambda_2^{\alpha_2}\beta^{\alpha_2-2}e^{-\lambda_2\beta}}{\Gamma(\alpha_2)}\Big[(\alpha_2-1)-\lambda_2\beta\Big].\end{aligned}$$

Thus, for $0<\lambda_1,\alpha_2<1$ and $\alpha_1,\lambda_2>0$, the priors $\pi_1(\theta|\alpha_1,\lambda_1)$ and $\pi_2(\beta|\alpha_2,\lambda_2)$ are decreasing function of the parameters. It is worth mentioning that here we suppose $\theta>1$.

Assuming that the hyperparameters $\alpha_i,\lambda_i,\ i=1,2$ are independent random variables and their density functions are $\pi'(\alpha_1),\pi'(\lambda_1),\pi'(\alpha_2)$ and $\pi'(\lambda_2)$, respectively. The joint bivariate density function of the hyperparameters can be represented

$$\pi_1'(\alpha_1,\lambda_1)=\pi'(\alpha_1)\pi'(\lambda_1),\qquad \pi_2'(\alpha_2,\lambda_2)=\pi'(\alpha_2)\pi'(\lambda_2),$$

then, the E-Bayesian estimate of the parameters θ and β, according to [13], can be obtained as follows

$$\begin{aligned}\hat{\theta}_{EB}&=E(\hat{\theta}_B|T)=\int_0^1\int_0^{s_1}\hat{\theta}_B\pi_1'(\alpha_1,\lambda_1)d\alpha_1\,d\lambda_1,\\ \hat{\beta}_{EB}&=E(\hat{\beta}_B|T)=\int_0^1\int_0^{s_2}\hat{\beta}_B\pi_2'(\alpha_2,\lambda_2)d\lambda_2\,d\alpha_2,\end{aligned}$$

where $\hat{\theta}_B$ and $\hat{\beta}_B$ are the Bayesian estimate of the parameters θ and β under SEL.

Based on the following prior distributions of the hyperparameters, the E-Bayesian estimates of the parameters θ and β can be obtained. The selected prior distributions are given by

$$\pi_{11}'(\alpha_1,\lambda_1)=\frac{\lambda_1^{u_1-1}(1-\lambda_1)^{v_1-1}}{s_1B(u_1,v_1)},\qquad \pi_{21}'(\alpha_2,\lambda_2)=\frac{\alpha_2^{u_2-1}(1-\alpha_2)^{v_2-1}}{s_2B(u_2,v_2)},\tag{2.18}$$

These prior distributions are used to guarantee that priors are decreasing functions of the parameters θ and β. Same as the Bayesian estimators, the E-Bayesian estimators do not have a closed-form.

The E-Bayesian estimators of the parameters of Burr XII distribution are computed stepwise, through the IS algorithm.

IS algorithm for the E-Bayesian estimates

1. Generate α_1 and λ_1 hyperparameters from Uniform$(0, s_1)$ and Beta(u_1, v_1) distributions, respectively.
2. Generate α_2 and λ_2 hyperparameters from Beta(u_2, v_2) and Uniform$(0, s_2)$ distributions, respectively.
3. Regarding the hyperparameters (α_2, λ_2), obtained in step 2, the parameter b is generated from the $\mathrm{G}_\beta\Big(\alpha_2 + m, \lambda_2\Big)$ distribution.
4. Regarding the hyperparameters (α_1, λ_1), obtained in step 1, the parameter θ, given the values of the parameters b, is generated from $\mathrm{G}_{\theta|b}\Big(\alpha_1 + m, \lambda_1 + \sum_{i=1}^m (r_i + 1)\ln(1 + t_i^b)\Big)$ distribution.

This procedure can be repeated z times to reach a sample of $\{(b_1, \theta_1), \ldots, (b_z, \theta_z)\}$. Accordingly, the E-Bayesian estimates of b and θ under SEL function are indicated as below

$$\hat{b}_{EB} = \frac{\sum_{i=1}^z b_i Q(b_i)}{\sum_{i=1}^z Q(b_i)}, \qquad \hat{\theta}_{EB} = \frac{\sum_{i=1}^z \theta_i Q(b_i)}{\sum_{i=1}^z Q(b_i)}.$$

3. The reliability analysis

The reliability function is widely utilized in reliability analysis and lifetime data, and the estimation of reliability function is important for several reasons. For instance, the reliability function can be applied for stochastic orderings and estimation of the unreliability, conditional reliability, hazard rate and cumulative risk functions. The reliability function of the MB family is represented as

$$R(t) = P(X \geq t) = e^{\frac{-g^\beta(t)}{\theta}}, \qquad t,\ \beta,\ \theta \geq 0.$$

In this section, we focused on the estimation of the reliability function via the ML, Bayesian and E-Bayesian estimation methods, where both parameters are unknown.

Corollary 3.1. *Due to the invariant property of the ML estimators of the model's parameters, the reliability function estimator of the MB family based on ML methods is derived as*

$$\hat{R}_{ML}(t) \quad = \quad \exp\Big(\frac{-g^{\hat{\beta}_{ML}}(t)}{\hat{\theta}_{ML}}\Big),$$

where $(\hat{\beta}_{ML}, \hat{\theta}_{ML})$ are the corresponding ML estimators.

Corollary 3.2. *Consider the SEL and independent inverse-Gamma and Gamma priors for the parameters θ and β, respectively. Regarding the IS method for the Bayesian estimation of parameters, the Bayesian estimator of the reliability function is represented as*

$$\hat{R}_B(t) = \frac{\sum\limits_{i=1}^z Q(\beta_i)\exp\Big(\frac{-g^{\beta_i}(t)}{\theta}\Big)}{\sum\limits_{i=1}^z Q(\beta_i)}.$$

Corollary 3.3. *Consider SEL and independent inverse-Gamma and Gamma priors for the parameters θ and β, respectively. Set $0 < \lambda_1, \alpha_2 < 1$ and $\alpha_1, \lambda_2 > 0$, assuming that the hyperparameters $\alpha_i, \lambda_i,\ i = 1, 2$ are independent random variables. The E-Bayesian estimator of the reliability function of the MB family is obtained as follows*

$$\hat{R}_{EB}(t) = E(\hat{R}_B(t)|T) = \int_0^1 \int_0^{s_1} \int_0^1 \int_0^{s_2} \hat{R}_B(t)\pi'(\alpha_1, \lambda_1)\pi'(\alpha_2, \lambda_2) d\lambda_2\, d\alpha_2 d\alpha_1\, d\lambda_1,$$

where $\hat{R}_B(t)$ is the Bayesian estimate of the reliability function under SEL. According to the IS method for the E-Bayesian estimation of the parameters, the E-Bayesian estimate of the reliability function of the MB family is provided as

$$\hat{R}_{EB}(t) = \frac{\sum_{i=1}^{z} Q(\beta_i) \exp\left(\frac{-g^{\beta}(t)}{\theta}\right)}{\sum_{i=1}^{z} Q(\beta_i)}.$$

4. Simulation study

In this section, we represent the Monte-Carlo simulation results, which have been conducted to evaluate the performance of the estimates using the ML (via NR, EM, SEM), Bayesian and E-Bayesian estimation methods of the parameters of the MB family under progressive type-II censoring for the different combinations of parameters and sample sizes. The estimation procedures are repeated $h = 1000$ times and the simulation program is written by statistical software R.

The progressively type-II censored data from MB family is generated based on the algorithm proposed by [6] with censoring schemes $r_1 = (1^2, 0^{m-7}, 2^4,\ n-m-10)$, $r_2 = (2^2, 0^2, 1, 2^2, n-m-9, 0^{m-8})$ and $(n, m) = (30, 10), (30, 15), (70, 30), (70, 50)$. For arbitrary function $g(x)$, we consider the Burr XII distribution as $g(x) = \ln(1 + x^b)$, with $\beta = 1, b = 0.5, 1.5,\ \theta = 1.5, 2.5$. It is worth mentioning, the simulation comparisons of other sub-models of MB family, such as Weibull and Pareto, are conducted which leads to the same results as Burr XII. So, extra tables are removed due to reducing the manuscript page numbers.

The Burr XII distribution has decreasing (when $0 < b < 1$) and inverted bathtub-shaped (when $b > 1$) hazard rate function, which is depicted in Figure 1.

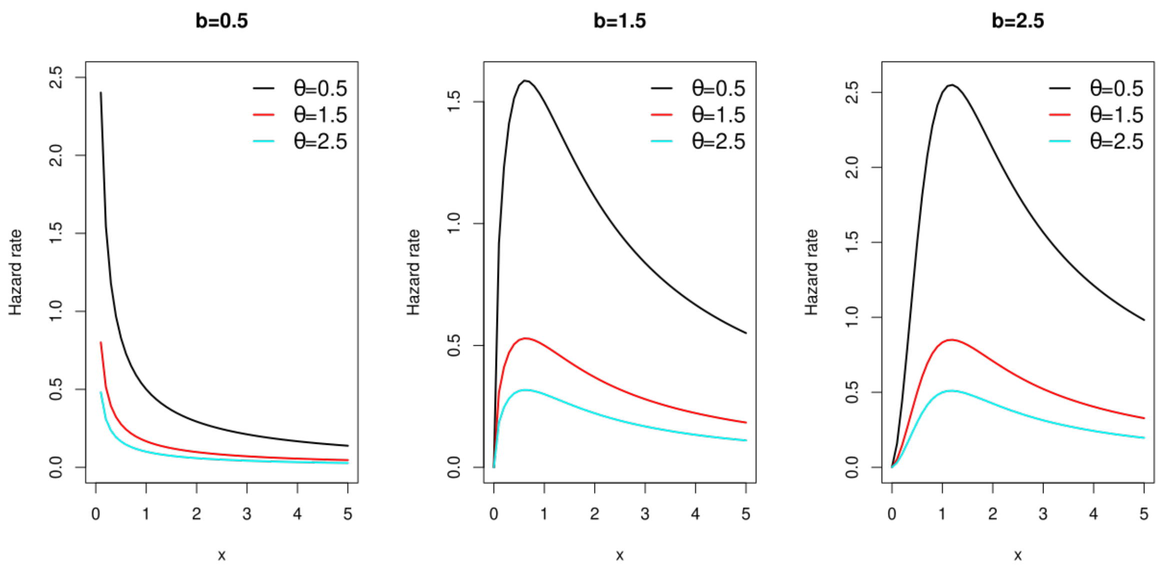

Figure 1. The hazard rate plot for Burr XII distribution

In order to obtain the hyperparameters of informative prior, we first generate 1000 samples from the complete MB family and derive the ML estimates of parameters based on each sample. Subsequently, we compare the mean and variance of samples and considered priors [11,21]. The RMSE is calculated to compare the estimators. The estimation results are demonstrated in Table 1 and Table 2.

For the reliability function, we consider the ML, Bayesian and E-Bayesian estimation approaches and the estimation results of the reliability function are represented in Table 3.

Table 1 and Table 2 represent the mean and RMSE (in brackets) of the ML, Bayesian and E-Bayesian estimates of the parameters of the Burr XII distribution under the progressively type-II censoring for different combinations of the parameters. Consequently, it can be observed that the estimates are convergent to the real values of the parameters. Moreover, increasing the sample size implies smaller RMSE values.

Regarding comparing different estimation approaches, it can be seen that the RMSE values derived by the E-Bayesian estimation approach are less than ML and Bayesian estimators. Also, the higher values of n lead to better estimates in the sense close to the true parameter values and have smaller RMSE values.

The reliability function estimates of the MB family and RMSE values under the progressively type-II censoring for $t = 1, 2, 3, 4, 5$ are represented in Table 3, which demonstrated that as the sample size n and the effective sample size m increase, RMSEs decrease. The E-Bayesian estimators are better than Bayesian and ML estimators of the reliability function, which have the smallest values of the RMSE.

Table 1. Estimation of parameters of Burr XII with RMSE in brackets for r_1

		$\hat{b}$					$\hat{\theta}$				
n	**m**	**NR**	**EM**	**SEM**	**Bayes**	**E-Bayes**	**NR**	**EM**	**SEM**	**Bayes**	**E-Bayes**
					$(b, \theta) = (0.5, 1.5)$						
30	15	0.79232	0.54622	0.53118	0.49084	0.50627	1.85278	1.55541	1.478903	1.49170	1.49441
		(0.03541)	(0.03157)	(0.03232)	(0.02817)	(0.02113)	(0.04761)	(0.03881)	(0.04167)	(0.03697)	(0.03271)
	20	0.764995	0.53337	0.48136	0.49118	0.49411	1.80664	1.54837	1.53395	1.49232	1.49529
		(0.03304)	(0.03072)	(0.02954)	(0.02592)	(0.01968)	(0.04240)	(0.03755)	(0.03901)	(0.03525)	(0.02909)
70	30	0.55676	0.50508	0.49530	0.49080	0.496429	1.53398	1.48838	1.523490	1.488709	1.49584
		(0.03148)	(0.02387)	(0.02548)	(0.02115)	(0.01359)	(0.03702)	(0.03640)	(0.03814)	(0.02777)	(0.01839)
	50	0.52327	0.49062	0.50115	0.49581	0.49826	1.514239	1.49535	1.49091	1.49878	1.49981
		(0.02838)	(0.01965)	(0.01856)	(0.01434)	(0.00851)	(0.03174)	(0.02724)	(0.03136)	(0.02179)	(0.01192)
					$(b, \theta) = (1.5, 2.5)$						
30	15	1.73632	1.48093	1.48633	1.486530	1.49403	2.36568	2.52217	2.5144	2.49016	2.49264
		(0.05985)	(0.05387)	(0.05482)	(0.04553)	(0.04044)	(0.09109)	(0.08466)	(0.08619)	(0.07956)	(0.07271)
	20	1.69118	1.48665	1.48898	1.48944	1.49414	2.39887	2.49347	2.49279	2.49399	2.49573
		(0.05503)	(0.05287)	(0.05384)	(0.04224)	(0.03855)	(0.08813)	(0.08296)	(0.08541)	(0.07221)	(0.06925)
70	30	1.55345	1.49329	1.49236	1.49536	1.50053	2.52025	2.51407	2.49448	2.49232	2.49612
		(0.04209)	(0.03729)	(0.03926)	(0.03592)	(0.02444)	(0.06227)	(0.05725)	(0.05849)	(0.05186)	(0.04232)
	50	1.52171	1.49411	1.49349	1.49602	1.50019	2.50708	2.49302	2.49571	2.49737	2.49878
		(0.03602)	(0.03066)	(0.03318)	(0.02828)	(0.01949)	(0.05733)	(0.05118)	(0.05201)	(0.04512)	(0.03736)

Table 2. Estimation of parameters of Burr XII with RMSE in brackets for r_2

		$\hat{b}$					$\hat{\theta}$				
n	**m**	**NR**	**EM**	**SEM**	**Bayes**	**E-Bayes**	**NR**	**EM**	**SEM**	**Bayes**	**E-Bayes**
						$(b,\theta)=(0.5,1.5)$					
30	15	0.62798 (0.03759)	0.48209 (0.03227)	0.51172 (0.03595)	0.48466 (0.02951)	0.48663 (0.02575)	1.25208 (0.05414)	1.46456 (0.04729)	1.47945 (0.04982)	1.48501 (0.04277)	1.49103 (0.03783)
	20	0.57459 (0.03604)	0.48629 (0.03068)	0.48361 (0.03248)	0.48524 (0.02593)	0.48946 (0.02315)	1.32377 (0.05169)	1.47109 (0.04429)	1.48788 (0.04516)	1.48848 (0.03757)	1.49256 (0.03417)
70	30	0.51781 (0.02709)	0.49397 (0.02206)	0.49221 (0.02587)	0.49496 (0.01878)	0.49515 (0.01519)	1.52029 (0.03844)	1.48911 (0.03191)	1.48665 (0.03418)	1.50809 (0.02954)	1.49697 (0.02045)
	50	0.50843 (0.02298)	0.4425 (0.01837)	0.01599 (0.01919)	0.49066 (0.01577)	0.49797 (0.01003)	1.48686 (0.03505)	1.49087 (0.02876)	1.50616 (0.03175)	1.49668 (0.02645)	1.49758 (0.01304)
						$(b,\theta)=(1.5,2.5)$					
30	15	1.75280 (0.06288)	1.47818 (0.05313)	1.46656 (0.05939)	1.48137 (0.04907)	1.48736 (0.04287)	2.79979 (0.10281)	2.47841 (0.09201)	2.46422 (0.09626)	2.48247 (0.08439)	2.48802 (0.07606)
	20	1.7254 (0.05831)	1.48401 (0.05188)	1.47407 (0.05479)	1.48289 (0.05001)	1.48942 (0.04044)	2.71129 (0.09102)	2.47927 (0.08289)	2.46905 (0.08913)	2.50181 (0.08165)	2.49393 (0.07354)
70	30	1.5384 (0.04538)	1.48601 (0.03874)	1.48969 (0.04127)	1.49022 (0.03424)	1.49357 (0.02656)	2.5775 (0.07231)	2.48767 (0.06055)	2.48760 (0.06497)	2.49075 (0.04981)	2.49554 (0.04454)
	50	1.51405 (0.03903)	1.49183 (0.03089)	1.49202 (0.03246)	1.49563 (0.02764)	1.49708 (0.02045)	2.52393 (0.06611)	2.48997 (0.05404)	2.50208 (0.05698)	2.49637 (0.04288)	2.49853 (0.03953)

5. Real data analysis

In this section, the application of the MB family is represented under progressively type-II censoring, where we consider several sub-models of the proposed family. Here, we focus on the Exponential ($g(x)=x, \beta=1$), Weibull ($g(x)=x$), Rayleigh ($g(x)=x, \beta=2$), Lomax ($g(x)=\ln(1+\frac{x}{\nu}), \beta=1$), modified Weibull ($g(x)=x^{\nu}e^{\nu x}, \beta=1$), Gompertz ($g(x)=\frac{\alpha}{\beta}(e^{bx}-1), \beta=1$), Chen ($g(x)=e^{x^b}-1, \beta=1$) distributions and also Burr XII distribution.

The data set represents the time between failures (thousands of hours) of secondary reactor pumps with $n=23$ [17]. The performance of the estimation procedures is evaluated by the real data set under the progressively type-II censoring scheme, which verifies the simulation results.

Some statistical indices of the failures are demonstrated in Table 4, which indicates that the empirical distribution is right-skewed and leptokurtic.

Based on the sub-models of the MB family, the ML estimators of the parameters and Kolmogorov-Smirnov (K-S) distance results are represented in Table 5. Also, the goodness of fit statistics (Akaike information criterion (AIC)) of the failure data set under several distributions of the MB family are provided in Table 5.

The P-value of the K-S test confirms the adequacy of the Burr XII distribution, among other distributions of the MB family, since the Burr XII distribution has the maximum P-value with minimum K-S distance. The Burr XII distribution has the smallest value of AIC, which confirms that the Burr XII distribution provides the best modeling among the other relevant distributions of the MB family for the failure data set.

The total time on test (TTT) plot of the failure data in Figure 2 shows the decreasing hazard rate, which is compatible with the Burr XII distribution.

Table 3. Estimation of reliability of Burr XII with RMSE in brackets for r_1

$\mathbf{R(t)}$	n	m	t	$\hat{\mathbf{R}}_{\mathbf{ML}}(\mathbf{t})$	$\hat{\mathbf{R}}_{\mathbf{B}}(\mathbf{t})$	$\hat{\mathbf{R}}_{\mathbf{EB}}(\mathbf{t})$	n	m	t	$\hat{\mathbf{R}}_{\mathbf{ML}}(\mathbf{t})$	$\hat{\mathbf{R}}_{\mathbf{B}}(\mathbf{t})$	$\hat{\mathbf{R}}_{\mathbf{EB}}(\mathbf{t})$
0.62996			1	0.64223 (0.01148)	0.63702 (0.00912)	0.62141 (0.00624)			1	0.63206 (0.00884)	0.62221 (0.00654)	0.62782 (0.00284)
0.55566			2	0.52937 (0.01085)	0.53363 (0.00747)	0.54524 (0.00514)			2	0.54104 (0.00721)	0.54872 (0.00516)	0.55169 (0.00216)
0.51169	30	15	3	0.49398 (0.00752)	0.49727 (0.00652)	0.50419 (0.00435)	70	30	3	0.50211 (0.00608)	0.51827 (0.00487)	0.50912 (0.00183)
0.48074			4	0.46101 (0.00917)	0.46529 (0.00817)	0.47146 (0.00692)			4	0.47172 (0.00749)	0.47655 (0.00563)	0.47952 (0.00209)
0.45707			5	0.42179 (0.01078)	0.42765 (0.00899)	0.43394 (0.00737)			5	0.44179 (0.00835)	0.44721 (0.00641)	0.45322 (0.00279)
0.62996			1	0.63914 (0.01103)	0.63529 (0.00872)	0.62312 (0.00596)			1	0.63017 (0.00608)	0.62748 (0.00586)	0.62812 (0.00201)
0.55566			2	0.53458 (0.01051)	0.53797 (0.00671)	0.54913 (0.00497)			2	0.54697 (0.00577)	0.55034 (0.00409)	0.55271 (0.00187)
0.51169	30	20	3	0.49742 (0.00694)	0.49954 (0.00516)	0.50672 (0.00388)	70	50	3	0.50684 (0.00478)	0.51396 (0.00343)	0.51293 (0.00104)
0.48074			4	0.46669 (0.00896)	0.46924 (0.00743)	0.47367 (0.00629)			4	0.48521 (0.00564)	0.47936 (0.00476)	0.48113 (0.00157)
0.45707			5	0.42888 (0.00905)	0.43081 (0.00822)	0.43974 (0.00692)			5	0.48819 (0.00612)	0.45199 (0.00566)	0.45503 (0.00211)

Table 4. Some statistical index of the failures data

Mean	**Median**	**Variance**	**Skewness**	**Kurtosis**
1.577	0.614	3.727	1.364	3.544

Table 5. Results of the failure data analysis

Model	**ML estimates**	**K-S distance**	**P-value**	**AIC**
Exponential	$\hat{\theta} = 1.577$	0.199	0.286	68.97
Weibull	$\hat{\beta} = 0.807,\ \hat{\theta} = 1.305$	0.118	0.883	69.02
Rayleigh	$\hat{\theta} = 6.055$	0.482	0.002	114.92
Lomax	$\hat{\theta} = 0.446,\ \hat{\nu} = 2.167$	0.101	0.958	68.99
Modified Weibull	$\hat{\theta} = 2.329,\ \hat{\nu} = 0.289$	0.204	0.259	79.72
Gompertz	$\hat{\alpha} = 1.527,\ \hat{\theta} = 1.883,\ \hat{b} = 0.141$	0.142	0.705	71.91
Chen	$\hat{\theta} = 2.472,\ \hat{b} = 0.456$	0.136	0.754	71.68
Burr XII	$\hat{b} = 1.13,\ \hat{\theta} = 0.762$	0.098	0.972	67.09

Now, the progressively type-II censored samples of size $m = 10$ are generated from the failure data set. Based on the censored data, the ML, Bayesian and E-Bayesian estimates of the parameters of the Burr XII distribution are illustrated in Table 6.

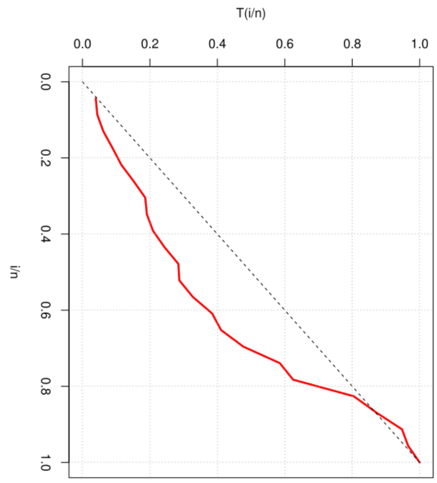

Figure 2. The TTT plot for data set of the failure data

Finally, the ML, Bayesian and E-Bayesian estimate of the reliability function of the Burr XII distribution are represented in Table 7.

Table 6. The parameters estimate of the Burr XII distribution under the progressively type-II censoring

parameters	NR	EM	SEM	Bayes	E-Bayes
$\hat{b}$	0.899	0.835	0.906	0.871	0.862
$\hat{\theta}$	1.022	0.973	1.095	1.139	1.267

Table 7. The reliability estimate of Burr XII distribution under progressively type-II censoring

t	$\hat{\mathbf{R}}_{\mathbf{ML}}$	$\hat{\mathbf{R}}_{\mathbf{B}}$	$\hat{\mathbf{R}}_{\mathbf{EB}}$
1	0.491	0.544	0.578
2	0.349	0.401	0.441
3	0.275	0.324	0.365
4	0.229	0.275	0.316
5	0.198	0.241	0.285

Conclusion

In this paper, several estimation approaches of the MB family of lifetime distributions are considered for both parameters and reliability function. The estimation approaches such as ML (NR, EM and SEM algorithms), Bayesian and E-Bayesian estimates of the parameters of the MB family are proposed, when the data are progressively type-II censored. Moreover, the reliability function estimation of the proposed family is discussed via different estimation methods. An especial case of the proposed family called Burr XII distribution is considered. By simulation attitude, the different estimations are compared, which leads to the superiority of the E-Bayesian estimation over other estimation methods. Moreover, it is observed that the reliability estimation based on the E-Bayesian is more efficient than Bayesian and ML estimators with respect to RMSE values. The real data analysis, based on failure data, also approves the findings in the simulation study. Some competitive distributions are considered in modeling the failure data. Based on the

AIC measure, the adequacy of the Burr XII distribution is confirmed among the other sub-models of the MB family. The estimation approaches can be compared based on their computer running times, which may lead to some new results. The new transformations of the MB family are not considered in previous literature, which can be interesting in future research with a concentration on the most accurate estimation method, including both RMSE and minimum running time indices.

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Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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