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A Note on Transitive Action of the Extended Modular Group on Rational Numbers

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Abstract: The extended modular group $\overline{\Gamma}$ is isomorphic to the amalgamated free product of two dihedral groups D_2 and D_3 with amalgamation $\mathbb{Z}_2$. This group acts on rational numbers transitively. In this study, we obtain elements in the extended modular group that are mappings between given two rationals. Also, we express these elements as a word in generators. We use interesting relations between continued fractions and the Farey graph.

Keywords: Extended modular group, continued fractions, Farey graph.

1. Introduction

The modular group $\Gamma = PSL(2,\mathbb{Z})$ is the projective special linear group of 2×2 matrices over the ring of integers with determinant one. This group is the quotient group $SL(2,\mathbb{Z})/\pm I$, hence, each matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ represents the same element with its negative $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$. Modular group acts on the upper half plane $\mathbb{H}$ via linear fractional transformations $z \to \frac{az+b}{cz+d}$. These transformations are orientation preserving isometries of $\mathbb{H}$. Modular group is generated by two elements,

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

By taking $S = TU = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, the presentation of Γ is

$$\Gamma =< T, S : T^2 = S^3 = I > \cong \mathbb{Z}_2 * \mathbb{Z}_3.$$

Let us denote the set $G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = -1 \right\}$. The corresponding transformations of elements in G are anti-automorphisms. Thus, the extended modular group can be

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defined as $\overline{\Gamma} = PSL(2,\mathbb{Z}) \cup G$. The extended modular group is isomorphic to free product of two dihedral groups of order 4 and 6, amalgamated with a cyclic group of order 2, i.e.,

$$\overline{\Gamma} =< T,S,R : T^2 = S^3 = R^2 = (TR)^2 = (SR)^2 = I >\cong D_2 *_{\mathbb{Z}_2} D_3,$$

where $R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a reflection map.

In this study, we focus on the action of the extended modular group $\overline{\Gamma}$ on rational numbers. Every rational number has a reduced fraction $\frac{p}{q} = \frac{-p}{-q}$, where $p,q \in \mathbb{Z}$ and $(p,q) = 1$. We represent ∞ as $\frac{1}{0} = \frac{-1}{0}$. Consider the element $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overline{\Gamma}$ and the corresponding Möbius transformation $V(z) = \frac{az+b}{cz+d}$. The image of $\frac{p}{q}$ is

$$V(\frac{p}{q}) = \frac{ap+bq}{cp+dq}.$$

Here $\frac{ap+bq}{cp+dq}$ is also a reduced fraction. Additionally, the Diophantine equation $px - qy = \pm 1$ is solvable since $(p,q) = 1$. Hence, it is possible to find an element $W = \begin{pmatrix} p & x \\ q & y \end{pmatrix} \in \overline{\Gamma}$ such that $W(\infty) = \frac{p}{q}$. As a result, the action of the extended modular group on rationals is transitive [14]. Our aim is to find an element $V \in \overline{\Gamma}$ for given two rationals $\frac{p}{q}, \frac{p'}{q'}$ such that $V(\frac{p}{q}) = \frac{p'}{q'}$. Also, we represent V as a word in generators.

2. Motivation and Background Materials

In this section, we give some information about continued fractions, the Farey sequence, the Farey graph and relations to the extended modular group. For more information see [1, 2, 6].

There are impressive relations between the modular group and continued fractions. Let $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = U^{r_0}.T.U^{r_1}.\cdots.U^{r_n}.T^i \in \Gamma$ where $r_j \in \mathbb{Z}$ and $i = 0,1$. The corresponding Möbius transformation of this element is

$$V(z) = U^{r_0}.T.U^{r_1}.\cdots.U^{r_n}.T(z) = r_0 - \cfrac{1}{r_1 - \cfrac{1}{r_2 - \cfrac{1}{\ddots r_{n-1} - \cfrac{1}{r_n - \frac{1}{z}}}}}. \tag{1}$$

In addition, the image of infinity is a continued fraction expansion of $\frac{a}{c}$. This expansion is the *Rosen continued fraction* defined in [11] for $\lambda = 1$, and it is called *integer continued fraction expansion*.

In this expansion, for $i \leq n$, $C_i = \frac{p_i}{q_i} = [r_0; r_1, ..., r_i]$ is called ith convergent of the expansion. It can be seen by calculation $p_i.q_{i-1} - q_i.p_{i-1} = \pm 1$. On the other hand, it is possible to make connections between integer continued fractions and the Farey sequence.

The Farey sequence of order n is a complete and ordered set of reduced rational numbers in the interval $[0,1]$ which have denominators less than or equal to n.

$$F_1 = \left\{ \frac{0}{1}, \frac{1}{1} \right\},$$

$$F_2 = \left\{ \frac{0}{1}, \frac{1}{2}, \frac{1}{1} \right\},$$

$$F_3 = \left\{ \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \right\},$$

$$F_4 = \left\{ \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1} \right\}.$$

It can be seen that if $\frac{a}{c}$ and $\frac{b}{d}$ appears one after another in some F_n, then $ad - bc = \pm 1$. We called such rationals Farey neighbours. All Farey neighbours of a rational x is denoted by $\mathcal{N}(x)$. The Farey sum of $\frac{a}{c}$ and $\frac{b}{d}$ defined as,

$$\frac{a}{c} \oplus \frac{b}{d} = \frac{a+b}{c+d}.$$

All Farey neigbours of a rational number can be obtained by Farey sum. More clearly, if a rational $\frac{p}{q}$ first appears in F_n by Farey sum of $\frac{a}{c}$ and $\frac{b}{d}$ in F_{n-1}, i.e., $\frac{a}{c} \oplus \frac{b}{d} = \frac{a+b}{c+d} = \frac{p}{q}$, then $\frac{a}{c}$ and $\frac{b}{d}$ are Farey neighbours of $\frac{p}{q}$. Here $\frac{a}{c}$ and $\frac{b}{d}$ are called Farey parents of $\frac{p}{q}$ and conversely, $\frac{p}{q}$ is called Farey child of $\frac{a}{c}$ and $\frac{b}{d}$. If $\frac{a_i}{c_i}$ is a Farey neighbour of $\frac{p}{q}$, then $\frac{a_i}{c_i} \oplus \frac{p}{q}$ is also a Farey neighbour of $\frac{p}{q}$.

Observe that every F_n includes F_{n-1} and new members are obtained by Farey sum of its neighbours. For instance $\frac{1}{2} \in F_2$ is the Farey sum of $\frac{0}{1}$ and $\frac{1}{1}$ in F_1. This rule is known as the mediant rule. It should be noted that if the denominator of Farey sum of two neighbours in F_{n-1} exceeds n, then this rational number will not appear in F_n since the definition of the Farey sequence. Definition of the Farey sequence can be extended to $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ by assuming $\infty = \frac{1}{0}$. Hence, for a given rational $\frac{a}{c}$, it is known that $\frac{a}{c}$ has finite integer continued fraction expansion. In addition, $\frac{b}{d}$ is the penultimate convergent of the integer continued fraction expansion of $\frac{a}{c}$. This yields $ad - bc = \pm 1$, in other words, $\frac{a}{c}$ and $\frac{b}{d}$ are Farey neighbours. As a result, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overline{\Gamma}$.

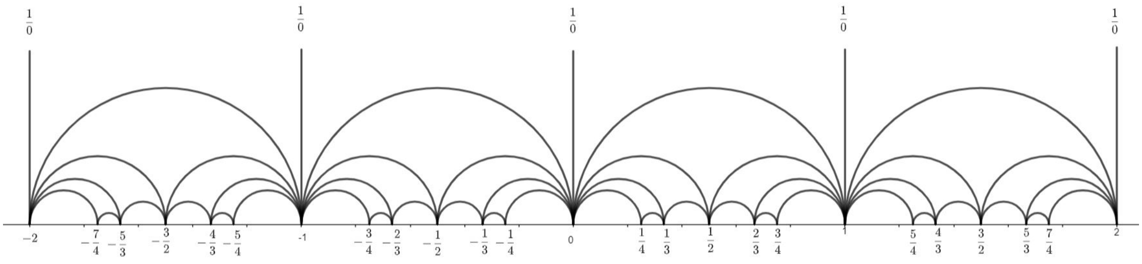

Figure 1: The Farey graph

The Farey graph is a graph with vertex set $\hat{\mathbb{Q}}$. Two reduced fractions $\frac{p}{q}$ and $\frac{r}{s}$ are adjacent if and only if $ps - rq = \pm 1$, i.e., they are Farey neighbours. The edge between two vertices is drawn by a hyperbolic line in $\mathbb{H}$. The edges between $\frac{1}{0} = \infty$ and every integer a are vertical lines. To construct the graph, first join the vertices $\frac{1}{0}, \frac{0}{1}$ and $\frac{1}{1}$ and obtain a big triangle. By induction, if the endpoints of a long edge are $\frac{a}{c}$ and $\frac{b}{d}$, then the label of the third vertex of the triangle is $\frac{a}{c} \oplus \frac{b}{d} = \frac{a+b}{c+d}$, see in Figure 1.

In recent years, many studies have contributed the continued fractions related to the action of some subgroups Möbius transformations. In [2], integer continued fraction expansions and geodesic expansions are studied from the perspective of graph theory. Short and Walker used Rosen continued fractions as paths in a class of graphs in hyperbolic geometry [13]. Same authors also studied connections between even integer continued fractions and the Farey graph [12]. Relations between cusp points and Fibonacci numbers are studied in [7] using Farey graph and continued fractions. Algebraic and combinatorial properties of continued fractions and modular group related with Farey graph are given in [10]. Besides that some relations between elliptic elements and circuits in graph for normalisers of subgroups of $PSL(2,\mathbb{R})$ are examined in [4, 5].

3. Main Results

Firstly, we obtain matrix representation of the elements in $\overline{\Gamma}$ which the corresponding transformation is a mapping between given two rationals. For a given reduced rational $x = \frac{p}{q}$ and a neighbour $y = \frac{r}{s} \in \mathcal{N}(x)$, we know $ps - rq = \pm 1$. Thus, we have $\begin{pmatrix} p & r \\ q & s \end{pmatrix} \in \overline{\Gamma}$. In addition, if $y = \frac{r}{s}$ is on the left side of $x = \frac{p}{q}$ in the Farey graph that is $y < x$, then $ps - rq = 1$ and $\begin{pmatrix} p & r \\ q & s \end{pmatrix} \in \Gamma$. In other words, the corresponding transformation $\frac{pz+r}{qz+s}$ is an automorphism. In this case, it is possible to construct an anti-automorphism element by taking $-r$ for r and $-s$ for s, i.e., $\begin{pmatrix} p & -r \\ q & -s \end{pmatrix}$. Similar observations can be done for the case $y > x$. For convenience throughout this paper, we need to define a location function $\mu_x : \mathcal{N}(x) \to \{-1, +1\}$ for neighbours of a rational:

$$\mu_x(y) = \begin{cases} 1 & , y < x \\ -1 & , y > x \end{cases}.$$

Now we are ready to obtain a mapping between two rationals.

Lemma 3.1 *Let $\frac{p}{q}, \frac{p'}{q'} \in \mathbb{Q}$ and $\frac{r}{s} \in \mathcal{N}\left(\frac{p}{q}\right)$, $\frac{r'}{s'} \in \mathcal{N}\left(\frac{p'}{q'}\right)$. Then the corresponding transformation of the element*

$$V = \begin{pmatrix} p's - r'q & pr' - p'r \\ q's - qs' & ps' - q'r \end{pmatrix}$$

maps the rational $\frac{p}{q}$ to $\frac{p'}{q'}$. Moreover,

- *If $\mu_{\frac{p}{q}}\left(\frac{r}{s}\right).\mu_{\frac{p'}{q'}}\left(\frac{r'}{s'}\right) = 1$, then the corresponding transformation of V is an automorphism.*
- *If $\mu_{\frac{p}{q}}\left(\frac{r}{s}\right).\mu_{\frac{p'}{q'}}\left(\frac{r'}{s'}\right) = -1$, then the corresponding transformation of V is an anti-automorphism.*

Proof Let $\frac{p}{q}$ be a reduced rational and $\frac{r}{s}$ be a Farey neighbour of $\frac{p}{q}$. Then, we have from the definition of Farey neighbour $ps - rq = \pm 1$. Hence, $V_1 = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in \overline{\Gamma}$. In addition, cusp point of this element is $\frac{p}{q}$. Similarly, we have the element $V_2 = \begin{pmatrix} p' & r' \\ q' & s' \end{pmatrix} \in \overline{\Gamma}$ with cusp point $\frac{p'}{q'}$. Finally, $V = V_2.V_1^{-1}$ is the element $V\left(\frac{p}{q}\right) = \frac{p'}{q'}$.

The equality $\mu_{\frac{p}{q}}\left(\frac{r}{s}\right).\mu_{\frac{p'}{q'}}\left(\frac{r'}{s'}\right) = 1$ tells us both V_1 and V_2 are automorphism or anti-automorphism simultaneously. This yields V is an automorphism. The case $\mu_{\frac{p}{q}}\left(\frac{r}{s}\right).\mu_{\frac{p'}{q'}}\left(\frac{r'}{s'}\right) = -1$ can be interpreted similarly. □

Obtaining an element that maps $\frac{p}{q}$ to $\frac{p'}{q'}$ via Lemma 3.1 requires Farey neighbours one for each. Following corollary is an answer to what if $\frac{p}{q}$ and $\frac{p'}{q'}$ are adjacent.

Corollary 3.2 *Let the reduced rationals $\frac{p}{q}$ and $\frac{p'}{q'}$ be adjacent in the Farey graph. Then, the corresponding transformation of the element*

$$V = \begin{pmatrix} p'q' - pq & p^2 - p'^2 \\ q'^2 - q^2 & pq - p'q' \end{pmatrix}$$

maps $\frac{p}{q}$ to $\frac{p'}{q'}$. Furthermore, V is a reflection.

Proof Since $\frac{p}{q}$ and $\frac{p'}{q'}$ are adjacent in the Farey graph, we can take $V_1 = \begin{pmatrix} p & p' \\ q & q' \end{pmatrix}$ and $V_2 = \begin{pmatrix} p' & p \\ q' & q \end{pmatrix}$. Proof follows similar to the proof of Lemma 3.1. It can be seen easily that $\mu_{\frac{p}{q}}\left(\frac{p'}{q'}\right).\mu_{\frac{p'}{q'}}\left(\frac{p}{q}\right) = -1$. This equality proves that V is an anti-automorphism. It is obvious that V is a reflection since $Tr(V) = 0$. □

Considering the arguments mentioned before Lemma 3.1, we can map $\frac{p}{q}$ to $\frac{p'}{q'}$ via an elliptic element. It is enough to take $V_2 = \begin{pmatrix} p' & -p \\ q' & -q \end{pmatrix}$ in the proof of Corolary 3.2. Therefore, we omit the proof of the following corollary.

Corollary 3.3 *Let the reduced rationals $\frac{p}{q}$ and $\frac{p'}{q'}$ be adjacent in the Farey graph. Then, the corresponding transformation of the element*

$$V = \begin{pmatrix} p'q' + pq & -p^2 - p'^2 \\ q'^2 + q^2 & -pq - p'q' \end{pmatrix}$$

maps $\frac{p}{q}$ to $\frac{p'}{q'}$. Furthermore, V is an elliptic element of order 2 in Γ.

Now our aim is to obtain a generalization of Lemma 3.1. For doing this, we need more information about Farey neighbours. As we mentioned in the motivation section, the Farey sequence of level n is a complete and ordered set of reduced rationals which have denominators less than or equal to n. Every F_n includes F_{n-1}. New members obtained via mediant rule. More clearly, if $\frac{a}{c}$ and $\frac{b}{d}$ is contained in F_{n-1}, then the mediant of these two terms $\frac{a}{c} \oplus \frac{b}{d} = \frac{a+b}{c+d}$ is contained in F_n on one condition that $c + d \leq n$. If a reduced rational $\frac{p}{q}$ first appears in F_n via Farey sum of $\frac{a}{c}$ and $\frac{b}{d}$ in F_{n-1}, then $\frac{a}{c}$ and $\frac{b}{d}$ is called Farey parents of $\frac{p}{q}$. After that all Farey neighbours of $\frac{p}{q}$ will be of the form,

$$\frac{a}{c} < \frac{a}{c} \oplus \frac{p}{q} = \frac{p+a}{q+c} < \frac{p+a}{q+c} \oplus \frac{p}{q} = \frac{2p+a}{2q+c} < \ldots < \frac{p}{q} < \ldots \frac{p}{q} \oplus \frac{b}{d} = \frac{p+b}{q+d} < \frac{b}{d}.$$

A basic result of this, is the following lemma.

Lemma 3.4 *Let $\frac{p}{q}$ be a reduced rational number and $\frac{r}{s}, \frac{r'}{s'} \in \mathcal{N}\left(\frac{p}{q}\right)$. Then, there exists an integer k such that*

$$\begin{pmatrix} r' \\ s' \end{pmatrix} = \begin{pmatrix} p & r \\ q & s \end{pmatrix} . \begin{pmatrix} k \\ 1 \end{pmatrix}.$$

Here we consider the element $\begin{pmatrix} p & r \\ q & s \end{pmatrix} \in \overline{\Gamma}$ as an automorphism (or anti-automorphism). Using Lemma 3.4, we can construct another automorphism (anti-automorphism) element with cusp point $\frac{p}{q}$.

Lemma 3.5 *Let $V_1, V_2 \in \overline{\Gamma}$ with same cusp point. Then there exists an integer k such that*

$$V_1.U^k = V_2.$$

Here $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is the parabolic generator of $\overline{\Gamma}$.

Proof Suppose the common cusp point is $\frac{p}{q}$. For neighbours $\frac{r}{s}, \frac{r'}{s'} \in \mathcal{N}\left(\frac{p}{q}\right)$, we can think V_1 and V_2 as

$$V_1 = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \text{ and } V_2 = \begin{pmatrix} p & r' \\ q & s' \end{pmatrix}.$$

From Lemma 3.4, we have an integer k such that

$$\frac{r'}{s'} = \frac{kp + r}{kq + s}.$$

Hence, we complete the proof by considering the generator $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$,

$$\begin{pmatrix} p & r' \\ q & s' \end{pmatrix} = \begin{pmatrix} p & r \\ q & s \end{pmatrix} . \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k .$$

□

The above lemma tells us V_1 and $V_1.U^k$ have common cusp point for every integer k, and that is the key for the following theorem which is a generalization of Lemma 3.1.

Theorem 3.6 *Let $\frac{p}{q}, \frac{p'}{q'}$ be reduced fractions in $\mathbb{Q}$ and $\frac{r}{s} \in \mathcal{N}\left(\frac{p}{q}\right)$, $\frac{r'}{s'} \in \mathcal{N}\left(\frac{p'}{q'}\right)$. Then for every $k \in \mathbb{Z}$, the corresponding transformation of the element*

$$V = \begin{pmatrix} p's - r'q - kp'q & pr' - p'r + kpp' \\ q's - qs' - kqq' & ps' - q'r + kpq' \end{pmatrix}$$

maps the rational $\frac{p}{q}$ to $\frac{p'}{q'}$.

Proof We use a similar technique of the proof of Lemma 3.1. From Lemma 3.5, the elements

$$V_1 = \begin{pmatrix} p & r \\ q & s \end{pmatrix} . \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{k_1} \text{ and } V_2 = \begin{pmatrix} p' & r' \\ q' & s' \end{pmatrix} . \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{k_2}$$

have cusp points $\frac{p}{q}$ and $\frac{p'}{q'}$, respectively, for $k_1, k_2 \in \mathbb{Z}$. Then, $V_2.V_1^{-1}$ is the desired element V for $k_2 - k_1 = k \in \mathbb{Z}$. □

4. Words in Generators

In this section, we consider relations between Farey paths and integer continued fractions. Using these relations, we obtain extended modular group elements as words in terms of generators that correspond a transformation between given two rationals.

Theorem 4.1 *Let* $\frac{p}{q} = [r_0, r_1, ..., r_n]$ *and* $\frac{p'}{q'} = [s_0, s_1, ..., s_m]$ *be reduced rationals. Then, the automorphism in the extended modular group that maps* $\frac{p}{q}$ *to* $\frac{p'}{q'}$ *has the word form*

$$W(U,T,R) = U^{s_0}.T.U^{s_1}.T.\cdots.U^{s_m}.T.U^{k}.T.U^{-r_n}.T.U^{-r_{n-1}}.T.\cdots.T.U^{-r_0} \tag{2}$$

for every integer k. *In addition, the anti-automorphism has the word form*

$$W'(U,T,R) = U^{s_0}.T.U^{s_1}.T.\cdots.U^{s_m}.R.U^{k}.T.U^{-r_n}.T.U^{-r_{n-1}}.T.\cdots.T.U^{-r_0}. \tag{3}$$

Proof First we map $\frac{p}{q}$ to 0. Considering the equality (1),

$$U^{-r_n}.T.U^{-r_{n-1}}.T.\cdots.T.U^{-r_0}(\frac{p}{q}) = 0.$$

The two ordered elliptic generator T maps 0 to infinity. Then, the parabolic generator U fixes infinity. Finally, the cusp point of the element

$$U^{s_0}.T.U^{s_1}.T.\cdots.U^{s_m}.T$$

is $\frac{p'}{q'}$ which proves the result. The second part of the proof can be done by considering the element with cusp point $\frac{p'}{q'}$ as $U^{s_0}.T.U^{s_1}.T.\cdots.U^{s_m}.R$. □

Since the modular group is isomorphic to the free product of the cyclic groups of orders 2 and 3, every element can be expressed as a word in T and S. Considering $U = TS$, we get two blocks,

$$TS = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad and \quad TS^2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Hence, every element $V \in \overline{\Gamma}$ has a word form,

$$V = S^i.(TS)^{m_0}.(TS^2)^{n_0}.(TS)^{m_1}.(TS^2)^{n_1}.\cdots.(TS)^{m_k}.(TS^2)^{n_k}.T^j.R^t,$$

where $i = 0, 1, 2$, $j, t = 0, 1$. The powers of the blocks are positive integers but m_0 and n_k may be zero. This form is called *block reduced form.* Every element in the extended modular group has block reduced form. For instance, $RTS^2RTSTS^2TS^2RTS$ can be expressed as $(TS)^2.(TS^2)^3R$. Trace classes of the modular group and the extended modular group were studied in [3, 8]. Corresponding transformations of these blocks are related to simple continued fraction expansions. Here we obtain block forms of the words given in Theorem 4.1.

Theorem 4.2 *The block reduced form of the elements given in (2) and (3) are*

$$\begin{aligned} W_{BRF} &= (TS)^{s_0-1}.(TS^2).(TS)^{s_1-2}.(TS^2).\cdots.(TS)^{s_m-2}.(TS^2).(TS)^{k-1}. \\ &\quad (TS^2)^{r_n-1}.(TS).(TS^2)^{r_{n-1}-2}.(TS).\cdots.(TS).(TS^2)^{r_1-2}.(TS).(TS^2)^{r_0-1}.T \end{aligned}$$

$$\begin{aligned} W'_{BRF} &= (TS)^{s_0-1}.(TS^2).(TS)^{s_1-2}.(TS^2).\cdots.(TS)^{s_{m-1}-2}.(TS^2).(TS)^{s_m-1}. \\ &\quad (TS^2)^k.(TS)^{r_n-1}.(TS^2).(TS)^{r_{n-1}-2}.(TS^2).\cdots.(TS)^{r_1-2}.(TS^2).(TS)^{r_0-1}.T.R, \end{aligned}$$

respectively.

Proof First we take $U = T.S$ in (2).

$$\begin{aligned} W_{BRF} &= (TS)^{s_0}.T.(TS)^{s_1}.T.\cdots.(TS)^{s_m}.T.(TS)^k.T.(TS)^{-r_n}.T.(TS)^{-r_{n-1}}.T.\cdots.T.(TS)^{-r_0} \\ &= (TS)^{s_0-1}.TS.T.TS.(TS)^{s_1-2}TS.T.\cdots.T.TS.(TS)^{s_m-2}.TS.T.TS.(TS)^{k-1}. \\ &\quad T.(S^2T)^{r_n}.T.(S^2T)^{r_{n-1}}.T.\cdots.T.(S^2T)^{r_0}. \end{aligned}$$

Since the elliptic generator T is of order 2 and S is of order 3, we have the block reduced form of the word as stated. The second part of the proof can be obtained similarly with relations

$$RS = S^2R,$$

$$TR = RT.$$

□

Before we sum up all our results, we make connections with Farey paths. A path in a graph consists of consecutive adjacent vertices. So, a Farey path $< v_1, v_2, ..., v_n >$ is a path such that $v_i = \frac{p_i}{q_i}$ for $i = 1, 2, ..., n$ are reduced rationals and since the consecutive v_i's are adjacent, we have $p_i.q_{i-1} - q_i.p_{i-1} = \pm 1$. As Farey graph is connected, there always be a path between two rationals.

For a given reduced rational $\frac{p}{q} = [r_0; r_1, ..., r_n]$, the ith convergent of the integer continued fraction expansion of $\frac{p}{q}$ defined as $C_i = \frac{p_i}{q_i} = [r_0; r_1, ..., r_i]$ for $0 \leq i \leq n$, where $C_0 = \frac{p_0}{q_0} = \frac{r_0}{1}$ and $C_n = \frac{p_n}{q_n} = \frac{p}{q}$. Furthermore, we know that $p_i.q_{i-1} - q_i.p_{i-1} = \pm 1$. Hence, every consecutive pair C_i and C_{i-1} are Farey neighbours. Also, this situation can be thought as $< \infty, C_0, C_1, ..., C_{n-1}, C_n >$ is a path from ∞ to $\frac{p}{q}$. Finally, every integer continued fraction expansion of a rational $\frac{p}{q}$ is related to a path from ∞ to $\frac{p}{q}$. Moreover, the shortest integer continued fraction of $\frac{p}{q}$ is related to a geodesic path from ∞ to $\frac{p}{q}$.

Now we give an example to explain all our results.

Example 4.3 *Let the given reduced rationals be* $\frac{7}{11}$ *and* $\frac{12}{7}$. *We find elements* $V \in \overline{\Gamma}$ *that the corresponding transformation* $V(z)$ *maps* $\frac{7}{11}$ *to* $\frac{12}{7}$, *i.e.,* $V(\frac{7}{11}) = \frac{12}{7}$. *We observe the following two paths from infinity to rationals* $\frac{7}{11}$ *and* $\frac{12}{7}$,

$$v =< \infty, 1, \frac{2}{3}, \frac{7}{11} >,$$

$$v' =< \infty, 2, \frac{9}{5}, \frac{7}{4}, \frac{12}{7} > .$$

The penultimate vertex $\frac{2}{3}$ *in path* v, *is the neighbour of* $\frac{7}{11}$ *such that* $\mu_{\frac{7}{11}}(\frac{2}{3}) = -1$. *Similarly, the neighbour of* $\frac{12}{7}$ *is* $\frac{7}{4}$, $\mu_{\frac{12}{7}}(\frac{7}{4}) = -1$. *By Lemma 3.1, we have the hyperbolic element,*

$$V = \begin{pmatrix} -41 & 25 \\ -23 & 14 \end{pmatrix} \in \Gamma.$$

The corresponding transformation is $V(z) = \frac{-41z+25}{-23z+14}$. *Hence, we obtain* $V(\frac{7}{11}) = \frac{12}{7}$. *For the neighbour* $\frac{2}{3}$, *taking* -2 *for* 2 *and* -3 *for* 3 *in Lemma 3.1, we have the element,*

$$V_1 = \begin{pmatrix} 113 & -73 \\ 65 & -42 \end{pmatrix}$$

which the corresponding transformation $V_1(z) = \frac{113\overline{z}-73}{65\overline{z}-42}$ *is a glide-reflection. To express* V *and* V_1 *as words in generators we need the integer continued fraction expansions of* $\frac{7}{11}$ *and* $\frac{12}{7}$. *The consecutive vertices in Farey path are the concecutive convergents of the integer continued fraction expansion. The convergents of* $\frac{7}{11}$ *are* $1, \frac{2}{3}$ *and* $\frac{7}{11}$. *For* $\frac{12}{7}$, *we have the convergents* $2, \frac{9}{5}, \frac{7}{4}$ *and* $\frac{12}{7}$. *Hence, one can calculate the integer continued fractions*

$$\frac{7}{11} = [1, 3, 4],$$

$$\frac{12}{7} = [2, 5, 1, 3].$$

From Theorem 4.1, we have the words

$$W = U^2.T.U^5.T.U.T.U^3.T.U^k.T.U^{-4}.T.U^{-3}.T.U^{-1},$$

$$W' = U^2.T.U^5.T.U.T.U^3.R.U^k.T.U^{-4}.T.U^{-3}.T.U^{-1}.$$

The elements V and V_1 have the word forms W and W' for $k = 0$, respectively. Finally, we express W and W' in block reduced forms by Theorem 4.2,

$$W_{BRF} = (TS).(TS^2).(TS)^3.(TS^2).(TS)^{-1}.(TS^2).(TS).(TS^2).(TS)^{k-1}.(TS^2)^3.(TS).(TS^2).(TS).T,$$

$$W'_{BRF} = (TS).(TS^2).(TS)^3.(TS^2).(TS)^{-1}.(TS^2).(TS)^2.(TS^2)^k.(TS)^3.(TS^2).(TS).(TS^2).T.R.$$

We substitute S^2T for the fifth term $(TS)^{-1}$ in each word,

$$W_{BRF} = (TS).(TS^2).(TS)^2.(TS^2)^2.(TS)^{k-1}.(TS^2)^3.(TS).(TS^2).(TS).T,$$

$$W'_{BRF} = (TS).(TS^2).(TS)^2.(TS^2).(TS).(TS^2)^k.(TS)^3.(TS^2).(TS).(TS^2).T.R.$$

5. Conclusion

For given two reduced rationals $\frac{p}{q}$ and $\frac{p'}{q'}$, we obtain elements $V \in \overline{\Gamma}$ such that $V(\frac{p}{q}) = \frac{p'}{q'}$. We use the relations between paths in the Farey graph and continued fractions to get V as a word in terms of generators. We also obtain the block reduced form that is a word contains only finite ordered elements. For future research one can consider the blocks,

$$f = RTS = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } h = TSR = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

defined in [9]. Powers of these matrices have only Fibonacci entries. Koruoğlu proved that every element can be written as a word in powers of f and h. This word is called *New Block Reduced Form* [9]. Obtaining new block reduced form of the words given in this study, makes relations to the Fibonacci sequence.

Declaration of Ethical Standards

The author declares that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Conflict of Interest

The author declares no conflicts of interest.

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