

## LINEAR QUADRATIC CONTROL PROBLEM OF STOCHASTIC SWITCHING SYSTEMS WITH DELAY

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### ABSTRACT

Linear Quadratic (LQ) problems constitute important class of optimal control problems. The solution of this problem has had a profound impact on many economics, engineering, chemical applications and many nonlinear control problems can be approximated by the LQ problems. The contribution of this paper is to investigate the stochastic optimal control problem of linear switching systems with quadratic cost functional. A necessary and sufficient condition of optimality for stochastic linear switching systems with delay is obtained.

**Keywords:** Delayed Stochastic Systems, Condition of optimality, Switching linear system, Riccati equations

## GECİKMELİ STOKASTİK GEÇİŞ SİSTEMLERİ İÇİN DOĞRUSAL KUADRATİK KONTROL PROBLEMİ

### ÖZET

Doğrusal Kuadratik (DK) problem optimal kontrol problemlerinin mühüm bir sınıfını oluşturuyor. Bu problemin çözümünün birçok ekonomi, mühendislik, kimya uygulamalarına önemli etkisi var. Aynı zamanda çok sayıda doğrusal olmayan kontrol problemleri DK problemleriyle yakınsanabilir. Bu çalışmada kuadratik amaç fonksiyonu olan doğrusal geçiş sistemleri için stokastik optimal problemi incelenmiştir. Gecikmeli stokastik doğrusal geçiş sistemlerinin optimal çözümü için gerekli ve yeterli koşullar bulunmuştur.

**Anahtar Kelimeler:** Gecikmeli Stokastik Sistemler, Optimallik koşulu, Doğrusal geçiş sistemi, Riccati denklemleri

## 1. INTRODUCTION

Randomness and time delay are associated with many real phenomena, and often they are the sources of complex behavior. Systems with stochastic uncertainties have provided a lot of interest for problems of nuclear fission, communication systems, self-oscillating systems and etc. [1,2]. Many real stochastic processes cannot be considered as Markov process, because their future behavior obviously depends not only on their present, but also on their previous states. The differential equations with time delay can be used to model the processes with a memory [3, 4]. Optimization problems for delayed stochastic control systems have attracted a lot of interest in both theoretical and applied research fields [4–10].

Although the optimal control problem for linear systems was solved, as well as the filtering one, in 1960s by Kalman [11], there exist a lot of invariant and noninvariant linear systems with still open optimal control problems. There has been a rich theory on LQ control, deterministic and stochastic alike (see [12-16]).

One of the elegant features of the LQ theory is the opportunity to give an explicit description for optimal control in a linear state feedback form in terms of Riccati equation. Riccati equations, associated with deterministic LQ problem or stochastic LQ problem with deterministic coefficients are deterministic backward ordinary differential equations. Riccati equation for deterministic systems, was essentially solved by Wonham [17] using Bellman's principle of quasilinearization (see [18]).

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Bismut performed a detailed analysis for stochastic LQ control problem with random coefficients. Riccati equation in this case is a nonlinear backward stochastic differential equations. The existence and uniqueness of solution for such class of equations was investigated in [19].

Switching systems consist of several subsystems and a switching law indicating the active subsystem at each time instantly. For general theory of stochastic switching systems, we refer to [20]. Theoretical results and applications of optimal control problems for switching systems are actual at present [21-25].

This article is concerned with optimal control problem of stochastic linear switching systems with delay.

The rest of paper is organized as follows. Section 1 formulates the main problem, presents some concepts and assumptions. The necessary and sufficient condition of optimality for stochastic linear switching systems is obtained in Section 2. In Section 3, the optimal controls are derived in a linear state feedback form via the solutions of Riccati equations. The paper is concluded in Section 4 .

## 2. NOTATION AND PROBLEM FORMULATION

In this section we fix notation and definition used throughout this paper. Let  $N$  be some positive constant,  $R^n$  denotes the  $n$  - dimensional real vector space,  $|\cdot|$  denotes the Euclidean norm and  $\langle \cdot, \cdot \rangle$  denotes scalar product in  $R^n$ .  $E$  represents expectation ; the set of integer numbers  $l, \dots, s$  is denoted by  $\overline{1, s}$ ; ' (the prime) denotes derivative; \* is the matrix transposition operation; by a.c. we denote almost certainly.

Let  $w_t^1, w_t^2, \dots, w_t^s$  are independent Wiener processes that generate the filtrations  $F_t^l = \overline{\sigma}(w_q^l, t_{l-1} \leq t \leq t_l)$ ,  $l = \overline{1, s}$ ;  $(\Omega, F^l, P), l = \overline{1, s}$  be a probability spaces with corresponding filtrations  $\{F_t^l, t \in [t_{l-1}, t_l]\}$ . By  $L_{F^l}^2(a, b; R^n)$  we denote the space of all predictable processes  $x(t, \omega) \equiv x_t(\omega)$  such that:  $E \int_a^b |x_t(\omega)|^2 dt < +\infty$ .  $R^{m \times n}$  is the space of all linear transformations from  $R^m$  to  $R^n$ .

Let  $O_l \subset R^{n_l}, Q_l \subset R^{m_l}$  be open sets;  $T = [0, T]$  be a finite interval and  $0 = t_0 < t_1 < \dots < t_s = T$ . Unless specified otherwise we use the following notation:  $\mathbf{t} = (t_0, t_1, \dots, t_s)$ ,  $\mathbf{u} = (u^1, u^2, \dots, u^s)$ ,  $\mathbf{x} = (x^1, x^2, \dots, x^s)$ .

Consider following linear controlled system with variable structure:

$$dx_t^l = (A_t^l x_t^l + B_t^l x_{t-h}^l + C_t^l u_t^l) dt + D_t^l x_t^l dw_t^l, \quad t \in (t_{l-1}, t_l], \quad l = \overline{1, s} \quad (1)$$

$$x_t^l = S_t^l, \quad t \in [t_l - h, t_l), \quad l = \overline{1, s}, \quad (2)$$

$$x_{t_l}^{l+1} = \Phi_{t_l}^l x_{t_l}^l + K_{t_l}^l, \quad l = \overline{1, s-1}; \quad x_{t_0}^1 = x_0 \quad (3)$$

$$u_t^l \in U_{\delta}^l \equiv \{u^l(\cdot, \cdot) \in L_F^2(t_{l-1}, t_l; R^m) | u^l(t, \cdot) \in U^l \subset R^m\} \quad (4)$$

where  $U^l, l = \overline{1, s}$  are non-empty bounded sets. The elements of  $U_{\delta}^l$  are called the admissible controls.

The problem is concluded to find an optimal solution  $(x^1, x^2, \dots, x^s, u^1, u^2, \dots, u^s)$  and a switching sequence  $t_1, t_2, \dots, t_s$  which minimize the cost functional:

$$J(u) = E \left\{ \langle Gx_T^s, x_T^s \rangle + \sum_{l=1}^s \int_{t_{l-1}}^{t_l} (\langle M_t^l x_t^l, x_t^l \rangle + \langle N_t^l u_t^l, u_t^l \rangle) dt \right\} \quad (5)$$

The elements of matrices  $A^l, B^l, C^l, M^l, N^l, \Phi^l, K^l, l = \overline{1, s}$  are a bounded continuous functions.  $G, N^l, l = \overline{1, s}$  are a positively semi-defined matrices, and  $N^l, l = \overline{1, s}$  are a positively defined matrices. Let  $\Lambda_l, l = \overline{1, s}$  be the set of piecewise continuous functions  $S^l: [t_{l-1}-h, t_{l-1}] \rightarrow N_l \subset O_l$  and  $h \geq 0$ .

Let  $U = U^1 \times U^2 \times \dots \times U^s$ , and consider the sets:  $A_i = T^{i+1} \times \prod_{j=1}^i O_j \times \prod_{j=1}^i \Lambda_j \times \prod_{j=1}^i U^j$  with the elements  $\pi^i = (t_0, \dots, t_i, x_1^1, \dots, x_i^1, S^1, \dots, S^i, u^1, \dots, \mu^i)$ .

Now we introduce the following definitions.

**Definition 1.** The set of functions  $\{x_t^l = x^l(t, \pi^l), t \in [t_{l-1}-h, t_l], l = \overline{1, s}\}$  is said to be a solution of the linear stochastic differential equation (1) corresponding to an element  $\pi^s \in A_s$ , if the function  $x_t^l \in O_l$  on the interval  $[t_l-h, t_l]$  satisfies conditions (2),(3), it is absolutely continuous a.c. and satisfies the equation (1) almost everywhere while on the interval  $[t_{l-1}, t_l]$ .

**Definition 2.** The element  $\pi^s \in A_s$  is said to be admissible if the pairs  $(x_t^l, u_t^l), t \in [t_{l-1}-h, t_l], l = \overline{1, s}$  are the solutions of switching system (1)-(4).

$A_s^0$  be the set of admissible elements.

**Definition 3.** The element  $\tilde{\pi}^s \in A_s^0$ , is said to be an optimal solution of problem (1)-(5) if there exist admissible controls  $\tilde{u}_t^l, l = \overline{1, s}$  and corresponding solutions  $\tilde{x}_t^l, l = \overline{1, s}$  of system (1)-(4) such that the pairs  $(\tilde{x}_t^l, \tilde{u}_t^l), l = \overline{1, s}$  minimize the functional (5).

### 3. LINEAR QUADRATIC OPTIMAL CONTROL PROBLEM

Necessary and sufficient conditions for an optimal solution, play an important role in the analysis of control problems. In this section optimality condition for stochastic control problem of linear switching systems is obtained.

**Theorem.** Let there exist random processes  $(\psi_t^l, \beta_t^l) \in L_{F^l}^2(t_{l-1}, t_l; R^{n_l}) \times L_{F^l}^2(t_{l-1}, t_l; R^{n_l \times n_l})$  that are the solutions of the following adjoint equations:

$$\begin{cases} d\psi_t^l = -[A_t^{l*} \psi_t^l + D_t^{l*} \beta_t^l + B_{t+h}^{l*} \psi_{t+h}^l - M_t x_t^l] dt + \beta_t^l dw_t, & t \in [t_{l-1}, t_l - h), l = \overline{1, s} \\ d\psi_t^l = -[A_t^{l*} \psi_t^l + D_t^{l*} \beta_t^l - M_t x_t^l] dt + \beta_t^l dw_t, & t \in [t_l - h, t_l], l = \overline{1, s} \\ \psi_{t_l}^l = \psi_{t_l}^{l+1} \Phi_{t_l}^l, l = \overline{1, s-1} \\ \psi_T^s = -G x_T^s; \end{cases} \quad (6)$$

The element  $\pi^s = (t_0, \dots, t_s, x_1^1, \dots, x_i^s, S^1, \dots, S^s, u^1, \dots, \mu^s)$  is an optimal solution of problem (1)-(5) only and only if:

a) the candidate optimal controls  $u_t^l$  are defined by

$$N_t^{l*} u_t^l = C_t^{l*} \psi_t^l, t \in [t_{l-1}, t_l], l = \overline{1, s}; \quad (7)$$

b) the following transversality conditions hold:

$$a_l \psi_{t_l}^{l+1} (\Phi_{t_l}^l x_{t_l}^l + K_{t_l}^l) = 0, l = \overline{1, s-1}. \quad (8)$$

**Proof.** Let  $u_t^l$  and  $\bar{u}_t^l, l = \overline{1, s}$  be some admissible controls; call the vectors  $\Delta \bar{u}_t^l = \bar{u}_t^l - u_t^l$  be an admissible increments of the controls  $u_t^l$ . By (1)-(3), the trajectories  $x_t^l, \bar{x}_t^l, l = \overline{1, s}$  correspond to the controls  $u_t^l, \bar{u}_t^l$ .

Consider two sequence laws  $\mathbf{t} = (0, t_1, t_2, \dots, t_{s-1}, T)$  and  $\bar{\mathbf{t}} = (0, \bar{t}_1, \bar{t}_2, \dots, \bar{t}_{s-1}, T)$ . Increment of cost functional (5) along admissible control  $\bar{\mathbf{u}} = (\bar{u}_1^1, \bar{u}_1^2, \dots, \bar{u}_1^s)$  looks like:

$$\langle J'(\mathbf{u}), \bar{\mathbf{u}} - \mathbf{u} \rangle = E \left\{ \langle Gx_T^s, \bar{x}_T^s - x_T^s \rangle + \sum_{l=1}^s \int_{\bar{t}_{l-1}}^{\bar{t}_l} \left( \langle M_l^l x_t^l, \bar{x}_t^l - x_t^l \rangle + \langle N_l^l u_t^l, \bar{u}_t^l - u_t^l \rangle \right) dt \right\} \quad (9)$$

Taking into consideration (1)-(3) we obtain:

$$\begin{cases} d(\bar{x}_t^l - x_t^l) = [A_t^l(\bar{x}_t^l - x_t^l) + B_t^l(\bar{x}_{t-h}^l - x_{t-h}^l) + C_t^l(\bar{u}_t^l - u_t^l)]dt + D_t^l(\bar{x}_t^l - x_t^l)dw_t^l, t \in [\bar{t}_{l-1}, \bar{t}_l] \\ x_{\bar{t}_l}^{l+1} = \bar{x}_{\bar{t}_l}^{l+1} - x_{\bar{t}_l}^{l+1} = \Phi_{\bar{t}_l}^l \bar{x}_{\bar{t}_l}^l - \Phi_{\bar{t}_l}^l x_{\bar{t}_l}^l. \end{cases} \quad (10)$$

According to Ito's formula for each  $t \in [\bar{t}_{l-1}, \bar{t}_l]$ ,  $l = \bar{1}, \bar{s}$  we have:

$$d\langle \psi_t^l, (\bar{x}_t^l - x_t^l) \Delta t_l \rangle = \langle d\psi_t^l, (\bar{x}_t^l - x_t^l) \Delta t_l \rangle + \langle \psi_t^l, (\bar{x}_t^l - x_t^l) \Delta t_l \rangle + \langle \psi_t^l, d(\bar{x}_t^l - x_t^l) \Delta t_l \rangle + \langle \beta_t^l, D_t^l(\bar{x}_t^l - x_t^l) \Delta t_l \rangle dt$$

Integrating this equality and taking expectation of both side, in to account (10) as follows:

$$\begin{aligned} E\langle \psi_{\bar{t}_l}^l, (\bar{x}_{\bar{t}_l}^l - x_{\bar{t}_l}^l) \Delta t_l \rangle - E\langle \psi_{\bar{t}_{l-1}}^l, (\bar{x}_{\bar{t}_{l-1}}^l - x_{\bar{t}_{l-1}}^l) \Delta t_{l-1} \rangle &= E \int_{\bar{t}_{l-1}}^{\bar{t}_l} \langle d\psi_t^l + A_t^{l*} \psi_t^l + D_t^{l*} \beta_t^l, (\bar{x}_t^l - x_t^l) \Delta t_l \rangle + \\ &E \int_{\bar{t}_{l-1}}^{\bar{t}_l} \langle B_t^{l*} \psi_t^l, (\bar{x}_t^l - x_t^l) \Delta t_l \rangle + E \int_{\bar{t}_{l-1}}^{\bar{t}_l} \langle C_t^{l*} \psi_t^l, (\bar{u}_t^l - u_t^l) \Delta t_l \rangle dt. \end{aligned}$$

Due to the last expression, representation of functional increment (9) can be rewritten as:

$$\begin{aligned} \langle J'(\mathbf{u}), \bar{\mathbf{u}} - \mathbf{u} \rangle &= E\langle Gx_T^s, \bar{x}_T^s - x_T^s \rangle + E \sum_{l=1}^s \langle \psi_{\bar{t}_l}^l, (\bar{x}_{\bar{t}_l}^l - x_{\bar{t}_l}^l) \Delta t_l \rangle \\ &+ E \sum_{l=1}^s \int_{\bar{t}_{l-1}}^{\bar{t}_l} \left( \langle M_l^l x_t^l, (\bar{x}_t^l - x_t^l) \Delta t_l \rangle + \langle N_l^l u_t^l, (\bar{u}_t^l - u_t^l) \Delta t_l \rangle \right) dt \\ &- E \sum_{l=1}^s \int_{\bar{t}_{l-1}}^{\bar{t}_l} \langle d\psi_t^l + A_t^{l*} \psi_t^l + D_t^{l*} \beta_t^l, (\bar{x}_t^l - x_t^l) \Delta t_l \rangle - \\ &E \sum_{l=1}^s \int_{\bar{t}_{l-1}}^{\bar{t}_l} \langle B_t^{l*} \psi_t^l, (\bar{x}_t^l - x_{t-h}^l) \Delta t_l \rangle - E \sum_{l=1}^s \int_{\bar{t}_{l-1}}^{\bar{t}_l} \langle C_t^{l*} \psi_t^l, (\bar{u}_t^l - u_t^l) \Delta t_l \rangle dt. \end{aligned} \quad (11)$$

The stochastic processes  $\psi_t^l$ , at the points  $t_1, t_2, \dots, t_s$  can be defined as follows:

$$\psi_{\bar{t}_l}^l = \psi_{\bar{t}_l}^{l+1} \Phi_{\bar{t}_l}^l, \quad l = \bar{1}, \bar{s}-1 \quad \text{and} \quad \psi_T^s = -Gx_T^s$$

Further using equation (6) we get:

$$\langle J'(\mathbf{u}), \bar{\mathbf{u}} - \mathbf{u} \rangle = E \sum_{l=1}^s \int_{\bar{t}_{l-1}}^{\bar{t}_l} \langle N_l^l u_t^l - C_t^{l*} \psi_t^l, (\bar{u}_t^l - u_t^l) \Delta t_l \rangle dt \quad (12)$$

It is well known that the necessary and sufficient condition of optimality for convex functional given by as:  $J'(\mathbf{u})=0$ . The validity of (7)-(8), hence the necessary condition of optimality follows from the relation (12). Finally, according to the independence of increments  $\Delta \bar{x}_t^l, \Delta \bar{y}_t^l, \Delta \bar{u}_t^l$  and  $\Delta \bar{t}_t$ , sufficiency is obtained from the expression (11).

#### 4. STOCHASTIC RICCATI EQUATIONS OF DELAYED SWITCHING SYSTEMS

In the theory it is very natural to connect the LQ problem with the Riccati equation for the possible feedback design. This section is devoted to the stochastic LQ problem of switching systems and explicit representation of the optimal control is determined via a set of stochastic Riccati equations. To establish this fact we use following linear relation:

$$d\psi_t^l = -p_t^l x_t^l, \quad l = \overline{1, s}, \text{ a.c.} \quad (13)$$

At the end, we obtain the differential equations of determination the functions  $p_t^l, l = \overline{1, s}$  that are a stochastic analogue of the Riccati equations.

We will search the differential of stochastic processes  $p_t^l, l = \overline{1, s}$  in the following form.

$$dp_t^l = \alpha_t^l dt + \gamma_t^l dw_t^l, \quad l = \overline{1, s}$$

According to formula Ito it is obtained:

$$d\psi_t^l = -[dp_t^l x_t^l + p_t^l dx_t^l + \gamma_t^l D_t^l x_t^l dt] = -[dp_t^l x_t^l + p_t^l (A_t^l x_t^l + B_t^l x_{t-h}^l + C_t^l u_t^l) dt + p_t^l D_t^l x_t^l dw_t^l + \gamma_t^l D_t^l x_t^l dt] \quad (14)$$

Using (6) and (14) for each  $l = \overline{1, s}$  we have:

$$\int_{t_{i-1}}^{t_i-h} [-A_t^{l*} \psi_t^l - B_{t+h}^{l*} \psi_{t+h}^l - D_t^{l*} \beta_t^l + M_t^{l*} x_t^l] dt - \int_{t_{i-h}}^{t_i} [A_t^{l*} \psi_t^l + D_t^{l*} \beta_t^l - M_t^{l*} x_t^l] dt + \int_{t_{i-1}}^{t_i} \beta_t^l dw_t^l = - \int_{t_{i-1}}^{t_i} [\alpha_t^l x_t^l dt + \gamma_t^l x_t^l dw_t^l + p_t^l A_t^l x_t^l dt + p_t^l B_t^l x_{t-h}^l dt + p_t^l C_t^l u_t^l dt + p_t^l D_t^l x_t^l dw_t^l + \gamma_t^l D_t^l x_t^l dt] \quad (15)$$

For  $\beta_t^l, l = \overline{1, s}$  we are having next form:

$$\beta_t^l = -[\gamma_t^l x_t^l + p_t^l D_t^l x_t^l] \quad t \in [t_{i-1}, t_i] \quad l = \overline{1, s} \quad (16)$$

By means of simple transformations taking into account (16) the expression (15) can be rewritten as follows :

$$\int_{t_{i-1}}^{t_i} [\alpha_t^l + p_t^l A_t^l + A_t^{l*} p_t^l + D_t^{l*} \gamma_t^l + M_t^{l*}] x_t^l dt + \int_{t_{i-1}}^{t_i} [D_t^{l*} \beta_t^l + p_t^l C_t^l u_t^l + p_t^l B_t^l x_{t-h}^l] dt = 0, \quad l = \overline{1, s}.$$

Finally, considering (13) in expression (7), optimal control can be explicitly defined as:

$$N_t^l u_t^l = -C_t^{l*} p_t^l x_t^l \quad t \in [t_{i-1}, t_i) \quad l = \overline{1, s},$$

here random processes  $(p_t^l, \gamma_t^l)$  are the solutions of the following Riccati equations:

$$dp_t^l = -[p_t^l A_t^l + A_t^{l*} p_t^l + \gamma_t^l C_t^l + C_t^{l*} \gamma_t^l + C_t^{l*} p_t^l C_t^l + M_t^l - (p_t^l B_t^l + \gamma_t^l D_t^l + C_t^{l*} p_t^l D_t^l) (N_t^l + D_t^{l*} p_t^l D_t^l)^{-1} (B_t^{l*} p_t^l + D_t^{l*} \gamma_t^l + D_t^{l*} p_t^l C_t^l)] + \gamma_t^l dw_t^l.$$

## 5. CONCLUSIONS

This work deals with optimal control problems of the noninvariant natural systems with memory. The objective of the present paper is to give an explicit solution to the LQ problem for stochastic switching systems. The LQ controller is constructed by the solution of stochastic backward Riccati differential equation.

The condition of optimality developed in this manuscript can be viewed as a stochastic analogues of results obtained in [8,12,21]. The LQ problem considered in this manuscript is a natural improving of the problem given in [26].

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