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# THE REAL MATRIX REPRESENTATIONS OF SEMI-OCTONIONS

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**ABSTRACT**

Rosenfeld's book [6] is a wonderful introduction to the normed division algebras: the real numbers, the complex numbers, the quaternions, and the octonions. A brief introduction of the semi-octonions is provided in this book. In [3], we studied some fundamental properties of the semi-octonions, $O_s$, and show that the set of unit semi-octonions is a subgroup of $O_s$. In this paper, we give a complete investigation to real matrix representations of semi-octonions, and consider a relation between the powers of these matrices. The De Moivre's formula implies that there are uncountably many matrices of the unit semi-octonions $A$ satisfying $A^n = I_8$ for every integer $n \geq 3$.

**Keywords:** Alternativity, De-Moivre's formula, Euler's formula, Semi-octonion

## YARI-OKTONYONLARIN REEL MATRİS GÖSTERİMLERİ

**ÖZET**

Rosenfeld'in kitabında normlu bölüm cebirleri, reel sayılar, kompleks sayılar, kuaterniyonlar ve oktonyonlara harika bir giriş yapılmıştır [6]. Yarı-oktonyonlara bir ufak giriş bu kitapta bulunabilir. Biz daha önce yarı-oktonyonların ($O_s$) bazı temel özelliklerini inceledik ve gösterdik ki birim yarı-oktonyonların kümesi, $O_s$'nin bir alt-kümesidir [3]. Bu makalede, yarı-oktonyonların reel matris gösterimini inceleyip aralarındaki bazı ilişkileri verdik. De-Moivre formülü, birim yarı-oktonyonlara karşılık gelen sayılamaz sayıda $A$ matrisinin her $n \geq 3$ tam sayısı için $A^n = I_8$ şeklinde var olduğunu söylemektedir.

**Anahtar Kelimeler:** Alternatiflik, De Moivre's formülü, Euler's formülü, Yarı-oktonyon

## 1. INTRODUCTION

Quaternions and octonions are useful tool for the representations and generalizations of quantities in the high-dimensional physical theory. These algebraic structures are used in areas such as quantum physics, classical electrodynamics, the representations of robotic systems, kinematics, acoustics, wave and group theory, supersymmetric quantum mechanics etc. While quaternions have a four dimensional non-commutative but associative algebraic structure, the octonions possess eight components and both non-commutative and non-associative algebraic properties.
Let $x$ be an octonion, expressed as,

$$x = (a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) = \sum_{n=0}^{7} a_n e_n$$

where terms $a_n$ are real number coefficients of the octonion and the $\boldsymbol{e}_n$ 's are its basis elements [4]. In [3], we studied some algebraic properties of semi-octonions, $O_s$, and show that the set of unit semi-octonions is a subgroup of $O_s$. By using De-Moivre's formula, any powers of these semi-octonions can be obtained. In this paper, after a review of some algebraic properties of $O_s$, we investigate real matrix representations of semi-octonions. Moreover, by using De-Moivre's formula for this matrix, from which the *n-th* power of such a matrix can be determined. We give some examples for the purpose of more clarification.

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## 2. SEMI-OCTONIONS ALGEBRA

A semi-octonion $x$ has an expression of the form

$$x = a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 + a_7 e_7 = a_0 e_0 + \sum_{i=1}^{7} a_i e_i,$$

where $a_0, ..., a_7$ are real numbers and $e_i, (0 \le i \le 7)$ are octonionic units satisfying the equalities that are given in the table below;

| . | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $-1$ | $e_3$ | $-e_2$ | $e_5$ | $-e_4$ | $-e_7$ | $e_6$ |
| $e_2$ | $-e_3$ | $-1$ | $e_1$ | $e_6$ | $e_7$ | $-e_4$ | $-e_5$ |
| $e_3$ | $e_2$ | $-e_1$ | $-1$ | $e_7$ | $-e_6$ | $e_5$ | $-e_4$ |
| $e_4$ | $-e_5$ | $-e_6$ | $-e_7$ | 0 | 0 | 0 | 0 |
| $e_5$ | $e_4$ | $-e_7$ | $e_6$ | 0 | 0 | 0 | 0 |
| $e_6$ | $e_7$ | $e_4$ | $-e_5$ | 0 | 0 | 0 | 0 |
| $e_7$ | $-e_6$ | $e_5$ | $e_4$ | 0 | 0 | 0 | 0 |

A semi-octonion $x$ can also be written as

$$x = q + q' e,$$

where $e^2 = 0$ and $q = a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3$ , $q' = a_4 e_0 + a_5 e_1 + a_6 e_2 + a_7 e_3$ are the real quaternion division algebras. The set of all semi-octonions is denoted by $\mathbb{O}_s$. A semi-octonion $x$ can be decomposed in terms of its scalar $(S_x)$ and vector $(\vec{V}_x)$ parts as

$$S_x = a_0 e_0, \quad \vec{V}_x = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 + a_7 e_7.$$

Multiplication of two semi-octonions can be described by a matrix-vector product as

$$x.\omega = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 & 0 & 0 & 0 & 0 \\ a_1 & a_0 & -a_3 & a_2 & 0 & 0 & 0 & 0 \\ a_2 & a_3 & a_0 & -a_1 & 0 & 0 & 0 & 0 \\ a_3 & -a_2 & a_1 & a_0 & 0 & 0 & 0 & 0 \\ a_4 & a_5 & a_6 & a_7 & a_0 & -a_1 & -a_2 & -a_3 \\ a_5 & -a_4 & a_7 & -a_6 & a_1 & a_0 & a_3 & -a_2 \\ a_6 & -a_7 & -a_4 & a_5 & a_2 & -a_3 & a_0 & a_1 \\ a_7 & a_6 & -a_5 & -a_4 & a_3 & a_2 & -a_1 & a_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \end{bmatrix},$$

where $x, \omega \in \mathbb{O}_s$.

Semi-octonions multiplication is not associative, since

$$e_1(e_2 e_4) = e_1 e_6 = -e_7,$$
$$(e_1 e_2) e_4 = e_3 e_4 = e_7.$$

But it has the property of *alternativity*, that is, any two elements in it generate an associative subalgebra isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{C}^0$, $\mathbb{H}$, $\mathbb{H}_S$, $\mathbb{H}^0$.

$e_0$ and $e_i$ $(1 \le i \le 3)$ generate a subalgebra isomorphic to C (complex numbers).
$e_0$ and $e_i$ $(4 \le i \le 7)$ generate a subalgebra isomorphic to $C^0$ (dual numbers).
Subalgebra with bases $e_0, e_1, e_2, e_3$ is isomorphic to H, and the subalgebra with bases $e_0, e_1, e_4, e_5$ is isomorphic to $H_S$ (semi-quaternions, see [5]).
Subalgebra with bases $e_0, e_4, e_5, e_6$ is isomorphic to $H^0$ (quasi-quaternions, see [2]).

**Corollary 1.** $O_S$ is a non-commutative non-associative algebra which is isomorphic to the subalgebra of the algebra $C^0 \otimes O$ [6].

## 3. SOME PROPERTIES OF SEMI-OCTONIONS

1) The *conjugate* of semi-octonion $x = a_0 e_0 + \sum_{i=1}^{7} a_i e_i = S_x + \vec{V}_x$ is

$$\bar{x} = a_0 e_0 - \sum_{i=1}^{7} a_i e_i = S_x - \vec{V}_x.$$

Conjugate of product of two semi-octonions and its own are described as

$$\overline{xy} = \bar{y}\,\bar{x}, \quad \bar{\bar{x}} = x$$

It is clear the scalar and vector parts of $x$ is denoted by $S_x = \dfrac{x+\bar{x}}{2}$ and $\vec{V}_x = \dfrac{x-\bar{x}}{2}$.

2) The *norm* of $x$ is

$$N_x = x\bar{x} = \bar{x}x = \sum_{i=0}^{3} a_i^2.$$

It satisfies the following property

$$N_{xy} = N_x N_y = N_y N_x$$

If $N_x = 1$, then $x$ is called a unit semi-octonion. We will use $O_S^1$ to denote the set of unit semi-octonions.

3) The *inverse* of $x$ with $N_x \neq 0$, is

$$x^{-1} = \frac{1}{N_x}\bar{x}.$$

4) Every nonzero semi-octonion $x = a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 + a_7 e_7$ can be written in the polar form

$$x = \sqrt{N_x}(\cos\theta + \vec{w}\sin\theta)$$

where $\cos\theta = \dfrac{a_0}{\sqrt{N_q}}$ and $\sin\theta = \dfrac{\left(\sum_{i=1}^{3} a_i^2\right)^{1/2}}{\sqrt{N_q}}$. The unit vector $\vec{w}$ is given by

$$\vec{w} = (w_1, w_2, ..., w_7) = \frac{1}{(\sum_{i=1}^{3} a_i^2)^{1/2}}(a_1, a_2, ..., a_7).$$

**Example 1.** The polar form of the semi-octonions $x_1 = \dfrac{\sqrt{2}}{2} + (\dfrac{1}{2}, \dfrac{1}{\sqrt{8}}, -\dfrac{1}{\sqrt{8}}, -1, 2, 1, 1)$ is

$$x_1 = \cos\frac{\pi}{4} + \vec{w}_1 \sin\frac{\pi}{4}$$

and $x_2=\sqrt{3}+(\frac{1}{2},-\frac{1}{\sqrt{2}},\frac{1}{2},-1,2,0,1)$ is

$$x_2=2(\cos\frac{\pi}{6}+\vec{w}_2\sin\frac{\pi}{6}),$$

where

$$\vec{w}_1=(\frac{1}{\sqrt{2}},\frac{1}{2},-\frac{1}{2},-\sqrt{2},2\sqrt{2},\sqrt{2},\sqrt{2}) \text{ and } w_2=(\frac{1}{2},-\frac{1}{\sqrt{2}},\frac{1}{2},-1,2,0,1).$$

It is clear that $N_{\vec{w}_1}=N_{\vec{w}_2}=1.$

**Theorem 1.** (De-Moivre's formula) Let $x=\sqrt{N_x}(\cos\theta+\vec{w}\sin\theta)$ be a semi-octonion. Then for any integer $n$;

$$x^n=(\sqrt{N_x})^n.(\cos n\theta+\vec{w}\sin n\theta)$$

**Proof:** The proof will be by induction on $n$ [3]. ■

**Example 2.** Let $x=\sqrt{3}+(\frac{1}{2},-\frac{1}{\sqrt{2}},\frac{1}{2},-1,2,0,1)$ be a semi-octonion. Every power of this octonion is found with the aid of Theorem 1. For example, *20-th* and *83-th* powers are

$$\begin{aligned} x^{20}&=2^{20}(\cos 20\frac{\pi}{6}+\vec{w}\sin 20\frac{\pi}{6}) \\ &=2^{20}(-\frac{1}{2}-\vec{w}\frac{\sqrt{3}}{2})=-2^{19}[1+\sqrt{3}(\frac{1}{2},-\frac{1}{\sqrt{2}},\frac{1}{2},-1,2,0,1)], \end{aligned}$$

and

$$\begin{aligned} x^{83}&=2^{83}(\cos 83\frac{\pi}{6}+\vec{w}\sin 83\frac{\pi}{6}) \\ &=2^{82}(\sqrt{3}-\vec{w}). \end{aligned}$$

**Theorem 2.** De Moivre's formula implies that there are uncountably many unit semi-octonion $x$ satisfying $x^n=1$ for $n\geq 3$.

**Proof:** For every unit vector $\vec{w}$, the unit semi-octonion

$$x=\cos\frac{2\pi}{n}+\vec{w}\sin\frac{2\pi}{n},$$

is of order $n$. For $n=1$ or $n=2$, the semi-octonion $x$ is independent of $\vec{w}$. ■

**Example 3.** $x=\frac{\sqrt{2}}{2}+(\frac{1}{2},\frac{1}{\sqrt{8}},-\frac{1}{\sqrt{8}},-1,2,1,1)$ is of order 8 and $x=\frac{1}{2}+\frac{1}{2}(1,-1,1,-2,2,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$ is of order 6.

**Theorem 3.** [4] Let $x=\cos\theta+\vec{w}\sin\theta$ be a unit semi-octonion. The equation $a^n=x$ has $n$ roots, and they are

$$a_k=\cos(\frac{\theta+2k\pi}{n})+\vec{w}\sin(\frac{\theta+2k\pi}{n}), \qquad k=0,1,2,...,n-1.$$

**Example 4.** Let $x = \frac{\sqrt{3}}{2} + \frac{1}{4}(1, -\sqrt{2}, 1, 0, \frac{1}{\sqrt{2}}, -1, 0) = \cos\frac{\pi}{6} + \vec{w}\sin\frac{\pi}{6}$ be a unit semi-octonion. The cube roots of the octonion $x$ are

$$x_k^{1/3} = \cos(\frac{\pi/6 + 2k\pi}{3}) + \vec{w}\sin(\frac{\pi/6 + 2k\pi}{3}), \qquad k = 0, 1, 2.$$

For $k = 0$, the first root is $x_0^{1/3} = \cos\frac{\pi}{18} + \vec{w}\sin\frac{\pi}{18} = 0.98 + 0.17\vec{w}$, and the second one for $k = 1$ is $x_1^{1/3} = \cos\frac{13\pi}{18} + \vec{w}\sin\frac{13\pi}{18} = -0.64 + 0.76\vec{w}$ and third one is $x_2^{1/3} = \cos\frac{25\pi}{18} + \vec{w}\sin\frac{25\pi}{18} = -0.34 - 0.93\vec{w}$. Also, it is easy to see that $x_0^{\frac{1}{3}} + x_1^{\frac{1}{3}} + x_2^{\frac{1}{3}} = 0$.

The relation between the powers of semi-octonions can be found in the following Theorem.

**Theorem 5.** Let $x$ be a unit semi-octonion with the polar form $x = \cos\theta + \vec{u}\sin\theta$. If $p = \frac{2\pi}{\theta} \in \mathrm{Z}^+ - \{1\}$, then $x^n = x^m$ if and only if $n \equiv m \pmod{p}$.

**Proof:** Let $n \equiv m \pmod{p}$. Then we have $n = ap + m$, where $a \in \mathrm{Z}$.

$$\begin{aligned} x^n &= \cos n\theta + \vec{u}\sin n\theta \\ &= \cos(ap+m)\theta + \vec{u}\sin(ap+m)\theta \\ &= \cos(a\frac{2\pi}{\theta} + m)\theta + \vec{u}\sin(a\frac{2\pi}{\theta} + m)\theta \\ &= \cos(m\theta + a2\pi) + \vec{u}\sin(m\theta + a2\pi) \\ &= \cos m\theta + \vec{u}\sin m\theta \\ &= x^m. \end{aligned}$$

Now suppose $x^n = \cos n\theta + \vec{u}\sin n\theta$ and $x^m = \cos m\theta + \vec{u}\sin m\theta$.
If $x^n = x^m$ then we get $\cos n\theta = \cos m\theta$ and $\sin n\theta = \sin m\theta$, which means

$$n\theta = m\theta + 2\pi a,\ a \in \mathrm{Z}.$$

Thus $n = m + \frac{2\pi}{\theta}a$ or $n \equiv m \pmod{p}$. ■

**Example 5.** Let $x = \frac{\sqrt{2}}{2} + (\frac{1}{2}, \frac{1}{\sqrt{8}}, -\frac{1}{\sqrt{8}}, -1, 2, 1, 1)$ be a unit semi-octonion. From Theorem 5, $p = \frac{2\pi}{\pi/4} = 8$, so we have

$$\begin{aligned} & x = x^9 = x^{17} = \ldots \\ & x^2 = x^{10} = x^{18} = \ldots \\ & x^3 = x^{11} = x^{19} = \ldots \\ & x^4 = x^{12} = x^{20} = \ldots = -1 \\ & \vdots \\ & x^8 = x^{16} = x^{24} = \ldots = 1. \end{aligned}$$

## 4. RESULTS

### 4.1. 8×8 Real Matrix Representations of Semi-Octonions

We introduce the $\mathbb{R}$-linear transformation representing left and right multiplication in $\mathbb{O}_s$ and look for also the De-Moivre's formula for corresponding matrix representations.
Let $x = q + q' e_4$ be a semi-octonion and $L_x : \mathbb{O}_s \to \mathbb{O}_s$ and $R_x : \mathbb{O}_s \to \mathbb{O}_s$ are defined as follows:

$$L_x(\omega) = x\omega, \qquad R_x(\omega) = \omega x, \qquad \omega \in \mathbb{O}_s$$

The transformations $L$ and $R$ are, respectively, defined as

$$\varphi(x) = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 & 0 & 0 & 0 & 0 \\ a_1 & a_0 & -a_3 & a_2 & 0 & 0 & 0 & 0 \\ a_2 & a_3 & a_0 & -a_1 & 0 & 0 & 0 & 0 \\ a_3 & -a_2 & a_1 & a_0 & 0 & 0 & 0 & 0 \\ a_4 & a_5 & a_6 & a_7 & a_0 & -a_1 & -a_2 & -a_3 \\ a_5 & -a_4 & a_7 & -a_6 & a_1 & a_0 & a_3 & -a_2 \\ a_6 & -a_7 & -a_4 & a_5 & a_2 & -a_3 & a_0 & a_1 \\ a_7 & a_6 & -a_5 & -a_4 & a_3 & a_2 & -a_1 & a_0 \end{bmatrix}, \tag{1}$$

and

$$\psi(x) = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 & 0 & 0 & 0 & 0 \\ a_1 & a_0 & a_3 & -a_2 & 0 & 0 & 0 & 0 \\ a_2 & -a_3 & a_0 & a_1 & 0 & 0 & 0 & 0 \\ a_3 & a_2 & -a_1 & a_0 & 0 & 0 & 0 & 0 \\ a_4 & -a_5 & -a_6 & -a_7 & a_0 & -a_1 & -a_2 & -a_3 \\ a_5 & a_4 & -a_7 & a_6 & a_1 & a_0 & -a_3 & a_2 \\ a_6 & a_7 & a_4 & -a_5 & a_2 & a_3 & a_0 & -a_1 \\ a_7 & -a_6 & a_5 & a_4 & a_3 & -a_2 & a_1 & a_0 \end{bmatrix}, \tag{2}$$

or equality

$$\varphi(x) = \begin{bmatrix} \overset{+}{H}(q) & 0 \\ \overset{+}{H}(q')\mathrm{K} & \overset{-}{H}(q) \end{bmatrix}, \quad \psi(x) = \begin{bmatrix} \overset{-}{H}(q) & 0 \\ \overset{+}{H}(q') & \overset{-}{H}(\overline{q}) \end{bmatrix}, \tag{3}$$

where $\overset{+}{H}, \overset{-}{H}$ are Hamilton operators and $K = diag(1, -1, -1, -1)$.
(See [4] for Hamilton operators in $\mathbb{R}^4$)
Using the definitions of $\varphi(x)$ and $\psi(x)$, the multiplication of the two semi-octonions $x, \omega$ is given by

$$x\omega = \varphi(x)\omega, \quad \omega x = \psi(x)\omega$$

**Theorem 6**. Let $x, \omega \in \mathbb{O}_s$ and $\lambda \in \mathbb{R}$ be given. Then

1. $x = \omega \Leftrightarrow \varphi(x) = \varphi(\omega) \Leftrightarrow \psi(x) = \psi(\omega)$
2. $\varphi(x+\omega) = \varphi(x) + \varphi(\omega), \ \psi(x+\omega) = \psi(x) + \psi(\omega)$
3. $\varphi(\lambda x) = \lambda\varphi(x), \ \psi(\lambda x) = \lambda\psi(x)$
4. $\varphi(\overline{x}) = [\varphi(x)]^T, \quad \psi(\overline{x}) = [\psi(x)]^T$
5. $\mathrm{tr}\,\varphi(x) = 8a_0, \ \mathrm{tr}\,\psi(x) = 8a_0$

**Proof:** Follows from a direct verification. ■

**Theorem 7**. Let $x \in \mathbb{O}_s$ be given. Then

$$x E_8 = E_8\, \varphi(x), \qquad and \qquad E_8^* x = \varphi(x) E_8^*$$

where $E_8 = [e_0 \quad e_1 \; ... \; e_7]$, $E_8^* = [e_0 \quad -e_1 \; ... \; -e_7]^T$.

**Proof:** Follows from a direct verification. ■

**Theorem 8**. Let $x \in \mathbb{O}_s$ be given. Then

$$\det\, \varphi(x) = \det\, \psi(x) = (N_x)^4.$$

**Proof:** Write $x = q + q'e$. Then we easily find by Eq. (3) and Theorem1 of Ref. [1], that

$$\det \varphi(x) = \begin{vmatrix} \overset{+}{H}(q) & 0 \\ \overset{+}{H}(q')\mathrm{K} & \overset{-}{H}(q) \end{vmatrix} = \det[\overset{+}{H}(q)\overset{-}{H}(q)] = \det \overset{+}{H}(q).\det \overset{-}{H}(q) = (N_x)^2.(N_x)^2 = (N_x)^4$$

$$\det \psi(x) = \begin{vmatrix} \overset{-}{H}(q) & 0 \\ \overset{+}{H}(q') & \overset{-}{H}(\bar{q}) \end{vmatrix} = \det[\overset{-}{H}(q)\overset{-}{H}(\bar{q})] = \det \overset{-}{H}(q).\det \overset{-}{H}(\bar{q}) = (N_x)^2.(N_x)^2 = (N_x)^4.$$

**Theorem 9.** Let $x$ be a unit semi-octonion. Matrix generated by operator $L_x$ is a quasi-orthogonal matrix, *i.e.*

$$\varphi(x)\,\varepsilon\,[\varphi(x)]^T = [\varphi(x)]^T\,\varepsilon\varphi(x) = \varepsilon,$$

$\det \varphi(x) = 1,$ where $\varepsilon = \begin{bmatrix} I_4 & 0 \\ 0 & 0 \end{bmatrix}.$

**Example 6.** Let $x = \frac{\sqrt{3}}{2} + \frac{1}{4}(1, \sqrt{2}, 1, 0, \frac{1}{\sqrt{2}}, -1, 1) = \cos\frac{\pi}{6} + \vec{w}\sin\frac{\pi}{6}$ be a unit semi-octonion. The matrix corresponding to this semi-octonion is

$$\varphi(x) = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{4} & -\frac{1}{2\sqrt{2}} & -\frac{1}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{\sqrt{3}}{2} & -\frac{1}{4} & \frac{1}{2\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{1}{2\sqrt{2}} & \frac{1}{4} & \frac{\sqrt{3}}{2} & -\frac{1}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & -\frac{1}{2\sqrt{2}} & \frac{1}{4} & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4\sqrt{2}} & -\frac{1}{4} & \frac{1}{4} & \frac{\sqrt{3}}{2} & -\frac{1}{4} & -\frac{1}{2\sqrt{2}} & -\frac{1}{4} \\ \frac{1}{4\sqrt{2}} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{\sqrt{3}}{2} & \frac{1}{4} & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{4} & -\frac{1}{4} & 0 & \frac{1}{4\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{4} & \frac{\sqrt{3}}{2} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4\sqrt{2}} & 0 & \frac{1}{4} & \frac{1}{2\sqrt{2}} & -\frac{1}{4} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$\varphi(x)$ is a quasi-orthogonal matrix.

Let $x$ be a unit semi-octonion with the polar form $x = \cos\theta + \vec{w}\sin\theta.$ The polar form of matrix (1) is

$$\begin{bmatrix} \cos\theta & -w_1\sin\theta & -w_2\sin\theta & -w_3\sin\theta & 0 & 0 & 0 & 0 \\ w_1\sin\theta & \cos\theta & -w_3\sin\theta & w_2\sin\theta & 0 & 0 & 0 & 0 \\ w_2\sin\theta & w_3\sin\theta & \cos\theta & -w_1\sin\theta & 0 & 0 & 0 & 0 \\ w_3\sin\theta & -w_2\sin\theta & w_1\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ w_4\sin\theta & w_5\sin\theta & w_6\sin\theta & w_7\sin\theta & \cos\theta & -w_1\sin\theta & -w_2\sin\theta & -w_3\sin\theta \\ w_5\sin\theta & -w_4\sin\theta & w_7\sin\theta & -w_6\sin\theta & w_1\sin\theta & \cos\theta & w_3\sin\theta & -w_2\sin\theta \\ w_6\sin\theta & -w_7\sin\theta & -w_4\sin\theta & w_5\sin\theta & w_2\sin\theta & -w_3\sin\theta & \cos\theta & w_1\sin\theta \\ w_7\sin\theta & w_6\sin\theta & -w_5\sin\theta & -w_4\sin\theta & w_3\sin\theta & w_2\sin\theta & -w_1\sin\theta & \cos\theta \end{bmatrix}.$$

**Theorem 10.** (De Moivre's formula for matrices of semi-octonions) For an integer $n$ and matrix

$$A=\begin{bmatrix} \cos\theta & -w_1\sin\theta & -w_2\sin\theta & -w_3\sin\theta & 0 & 0 & 0 & 0 \\ w_1\sin\theta & \cos\theta & -w_3\sin\theta & w_2\sin\theta & 0 & 0 & 0 & 0 \\ w_2\sin\theta & w_3\sin\theta & \cos\theta & -w_1\sin\theta & 0 & 0 & 0 & 0 \\ w_3\sin\theta & -w_2\sin\theta & w_1\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ w_4\sin\theta & w_5\sin\theta & w_6\sin\theta & w_7\sin\theta & \cos\theta & -w_1\sin\theta & -w_2\sin\theta & -w_3\sin\theta \\ w_5\sin\theta & -w_4\sin\theta & w_7\sin\theta & -w_6\sin\theta & w_1\sin\theta & \cos\theta & w_3\sin\theta & -w_2\sin\theta \\ w_6\sin\theta & -w_7\sin\theta & -w_4\sin\theta & w_5\sin\theta & w_2\sin\theta & -w_3\sin\theta & \cos\theta & w_1\sin\theta \\ w_7\sin\theta & w_6\sin\theta & -w_5\sin\theta & -w_4\sin\theta & w_3\sin\theta & w_2\sin\theta & -w_1\sin\theta & \cos\theta \end{bmatrix}, \qquad (2)$$

$n$-*th* power of the matrix $A$ reads as

$$A^n=\begin{bmatrix} \cos n\theta & -w_1\sin n\theta & -w_2\sin n\theta & -w_3\sin n\theta & 0 & 0 & 0 & 0 \\ w_1\sin n\theta & \cos n\theta & -w_3\sin n\theta & w_2\sin n\theta & 0 & 0 & 0 & 0 \\ w_2\sin n\theta & w_3\sin n\theta & \cos n\theta & -w_1\sin n\theta & 0 & 0 & 0 & 0 \\ w_3\sin n\theta & -w_2\sin n\theta & w_1\sin n\theta & \cos n\theta & 0 & 0 & 0 & 0 \\ w_4\sin n\theta & w_5\sin n\theta & w_6\sin n\theta & w_7\sin n\theta & \cos n\theta & -w_1\sin n\theta & -w_2\sin n\theta & -w_3\sin n\theta \\ w_5\sin n\theta & -w_4\sin n\theta & w_7\sin n\theta & -w_6\sin n\theta & w_1\sin n\theta & \cos n\theta & w_3\sin n\theta & -w_2\sin n\theta \\ w_6\sin n\theta & -w_7\sin n\theta & -w_4\sin n\theta & w_5\sin n\theta & w_2\sin n\theta & -w_3\sin n\theta & \cos n\theta & w_1\sin n\theta \\ w_7\sin n\theta & w_6\sin n\theta & -w_5\sin n\theta & -w_4\sin n\theta & w_3\sin n\theta & w_2\sin n\theta & -w_1\sin n\theta & \cos n\theta \end{bmatrix}.$$

*Proof:* The proof is easily followed by induction on $n$. ■

**Example 7.** Let $x=\frac{\sqrt{2}}{2}+(\frac{1}{2},\frac{1}{\sqrt{8}},-\frac{1}{\sqrt{8}},-1,2,1,1)=\cos\frac{\pi}{4}+\vec{w}\sin\frac{\pi}{4}$ be a unit semi-octonion. The matrix corresponding to this semi-octonion is

$$A=\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & \frac{\sqrt{2}}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & \frac{1}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 \\ -1 & 2 & 1 & 1 & \frac{\sqrt{2}}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} \\ 2 & 1 & 1 & -1 & \frac{1}{2} & \frac{\sqrt{2}}{2} & -\frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} \\ 1 & -1 & 1 & 2 & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{\sqrt{2}}{2} & \frac{1}{2} \\ 1 & 1 & -2 & 1 & -\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & -\frac{1}{2} & \frac{\sqrt{2}}{2} \end{bmatrix},$$

every power of this matrix with the aid of Theorem 10 is found to be expressible similarly, for example, *52-th* and *135-th* powers are

$$A^{52}=\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}=-I_8.$$

$$A^{135}=\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{\sqrt{2}}{2} & -\frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{\sqrt{2}}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & -\frac{1}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 \\ 1 & -2 & -1 & -1 & \frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} \\ -2 & -1 & -1 & 1 & -\frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} \\ -1 & 1 & -1 & -2 & -\frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & \frac{\sqrt{2}}{2} & -\frac{1}{2} \\ -1 & -1 & 2 & -1 & \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & \frac{1}{2} & \frac{\sqrt{2}}{2} \end{bmatrix},$$

**Theorem 11.** De Moivre's formula implies that there are uncountably many matrices of the unit octonions $A$ satisfying $A^n = I_8$ for every integer $n \geq 3$.
**Proof:** The proof is similar proof of Theorem 2.

### 4.2. Euler's Formula for Matrices Associated with Semi-Octonions

Let

$$W=\begin{bmatrix} 0 & -w_1 & -w_2 & -w_3 & 0 & 0 & 0 & 0 \\ w_1 & 0 & -w_3 & w_2 & 0 & 0 & 0 & 0 \\ w_2 & w_3 & 0 & -w_1 & 0 & 0 & 0 & 0 \\ w_3 & -w_2 & w_1 & 0 & 0 & 0 & 0 & 0 \\ w_4 & w_5 & w_6 & w_7 & 0 & -w_1 & -w_2 & -w_3 \\ w_5 & -w_4 & w_7 & -w_6 & w_1 & 0 & w_3 & -w_2 \\ w_6 & -w_7 & -w_4 & w_5 & w_2 & -w_3 & 0 & w_1 \\ w_7 & w_6 & -w_5 & -w_4 & w_3 & w_2 & -w_1 & 0 \end{bmatrix},$$

be a real matrix. One immediately finds $W^2 = -I_8$. We have a natural generalization of Euler's formula for matrix $W$ ;

$$\begin{aligned} e^{W\theta} &= I_8 + W\theta + \frac{(W\theta)^2}{2!} + \frac{(W\theta)^3}{3!} + \frac{(W\theta)^4}{4!} + \dots \\ &= I_8(1 - \frac{\theta^2}{2!} + \frac{\theta 4}{4!} - \dots) + W(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots) . \\ &= I_8 \cos\theta + W.\sin\theta \end{aligned}$$

### 4.3. *n-th R*oots of Matrices of Semi-Octonions

Let $x = \cos\theta + \vec{w}\sin\theta$ be a unit semi-octonion. The matrix associated with this semi-octonion $x$ is of the form (1).
The equation $x^n = A$ has $n$ roots, and they are as follows

$$A_k^{\frac{1}{n}}=\begin{bmatrix}
\cos(\frac{\theta+2k\pi}{n}) & -w_1\sin(\frac{\theta+2k\pi}{n}) & -w_2\sin(\frac{\theta+2k\pi}{n}) & -w_3\sin(\frac{\theta+2k\pi}{n}) & 0 & 0 & 0 & 0 \\
w_1\sin(\frac{\theta+2k\pi}{n}) & \cos(\frac{\theta+2k\pi}{n}) & -w_3\sin(\frac{\theta+2k\pi}{n}) & w_2\sin(\frac{\theta+2k\pi}{n}) & 0 & 0 & 0 & 0 \\
w_2\sin(\frac{\theta+2k\pi}{n}) & w_3\sin(\frac{\theta+2k\pi}{n}) & \cos(\frac{\theta+2k\pi}{n}) & -w_1\sin(\frac{\theta+2k\pi}{n}) & 0 & 0 & 0 & 0 \\
w_3\sin(\frac{\theta+2k\pi}{n}) & -w_2\sin(\frac{\theta+2k\pi}{n}) & w_1\sin(\frac{\theta+2k\pi}{n}) & \cos(\frac{\theta+2k\pi}{n}) & 0 & 0 & 0 & 0 \\
w_4\sin(\frac{\theta+2k\pi}{n}) & w_5\sin(\frac{\theta+2k\pi}{n}) & w_6\sin(\frac{\theta+2k\pi}{n}) & w_7\sin(\frac{\theta+2k\pi}{n}) & \cos(\frac{\theta+2k\pi}{n}) & -w_1\sin(\frac{\theta+2k\pi}{n}) & -w_2\sin(\frac{\theta+2k\pi}{n}) & -w_3\sin(\frac{\theta+2k\pi}{n}) \\
w_5\sin(\frac{\theta+2k\pi}{n}) & -w_4\sin(\frac{\theta+2k\pi}{n}) & w_7\sin(\frac{\theta+2k\pi}{n}) & -w_6\sin(\frac{\theta+2k\pi}{n}) & w_1\sin(\frac{\theta+2k\pi}{n}) & \cos(\frac{\theta+2k\pi}{n}) & w_3\sin(\frac{\theta+2k\pi}{n}) & -w_2\sin(\frac{\theta+2k\pi}{n}) \\
w_6\sin(\frac{\theta+2k\pi}{n}) & -w_7\sin(\frac{\theta+2k\pi}{n}) & -w_4\sin(\frac{\theta+2k\pi}{n}) & w_5\sin(\frac{\theta+2k\pi}{n}) & w_2\sin(\frac{\theta+2k\pi}{n}) & -w_3\sin(\frac{\theta+2k\pi}{n}) & \cos(\frac{\theta+2k\pi}{n}) & w_1\sin(\frac{\theta+2k\pi}{n}) \\
w_7\sin(\frac{\theta+2k\pi}{n}) & w_6\sin(\frac{\theta+2k\pi}{n}) & -w_5\sin(\frac{\theta+2k\pi}{n}) & -w_4\sin(\frac{\theta+2k\pi}{n}) & w_3\sin(\frac{\theta+2k\pi}{n}) & w_2\sin(\frac{\theta+2k\pi}{n}) & -w_1\sin(\frac{\theta+2k\pi}{n}) & \cos(\frac{\theta+2k\pi}{n})
\end{bmatrix}$$

For $k = 0$, the first root is

$$A_0^{\frac{1}{n}}=\begin{bmatrix} \cos\frac{\theta}{n} & -w_1\sin\frac{\theta}{n} & -w_2\sin\frac{\theta}{n} & -w_3\sin\frac{\theta}{n} & 0 & 0 & 0 & 0 \\ w_1\sin\frac{\theta}{n} & \cos\frac{\theta}{n} & -w_3\sin\frac{\theta}{n} & w_2\sin\frac{\theta}{n} & 0 & 0 & 0 & 0 \\ w_2\sin\frac{\theta}{n} & w_3\sin\frac{\theta}{n} & \cos\frac{\theta}{n} & -w_1\sin\frac{\theta}{n} & 0 & 0 & 0 & 0 \\ w_3\sin\frac{\theta}{n} & -w_2\sin\frac{\theta}{n} & w_1\sin\frac{\theta}{n} & \cos\frac{\theta}{n} & 0 & 0 & 0 & 0 \\ w_4\sin\frac{\theta}{n} & w_5\sin\frac{\theta}{n} & w_6\sin\frac{\theta}{n} & w_7\sin\frac{\theta}{n} & \cos\frac{\theta}{n} & -w_1\sin\frac{\theta}{n} & -w_2\sin\frac{\theta}{n} & -w_3\sin\frac{\theta}{n} \\ w_5\sin\frac{\theta}{n} & -w_4\sin\frac{\theta}{n} & w_7\sin\frac{\theta}{n} & -w_6\sin\frac{\theta}{n} & w_1\sin\frac{\theta}{n} & \cos\frac{\theta}{n} & w_3\sin\frac{\theta}{n} & -w_2\sin\frac{\theta}{n} \\ w_6\sin\frac{\theta}{n} & -w_7\sin\frac{\theta}{n} & -w_4\sin\frac{\theta}{n} & w_5\sin\frac{\theta}{n} & w_2\sin\frac{\theta}{n} & -w_3\sin\frac{\theta}{n} & \cos\frac{\theta}{n} & w_1\sin\frac{\theta}{n} \\ w_7\sin\frac{\theta}{n} & w_6\sin\frac{\theta}{n} & -w_5\sin\frac{\theta}{n} & -w_4\sin\frac{\theta}{n} & w_3\sin\frac{\theta}{n} & w_2\sin\frac{\theta}{n} & -w_1\sin\frac{\theta}{n} & \cos\frac{\theta}{n} \end{bmatrix}.$$

Similarly, for $k=n-1$, we obtain the *n-th* root.

### 4.4 Relations Between Powers of Matrices

Some relations between the powers of matrices associated with a real octonion is sketched in the following theorem.

**Teorem 12.** Let $x$ be a unit semi-octonion with the polar form $x=\cos\theta+\vec{u}\sin\theta.$ And let

$$p=\frac{2\pi}{\theta}\in \mathbb{Z}^{+}-\{1\}$$

and the matrix $A$ correspond to $x$. Then $n\equiv m(\text{mod } p)$ is true if and only if $A^n=A^m$.

**Proof:** The proof is similar proof of Theorem 5.

**Example 8.** Let $x=-\frac{1}{2}+(\frac{\sqrt{3}}{4},\frac{\sqrt{3}}{4},\frac{\sqrt{6}}{4},1,2,-1,1)=\cos\frac{2\pi}{3}+\vec{w}\sin\frac{2\pi}{3}$ be a unit semi-octonion. The matrix corresponding to this semi-octonion is

$$A=\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{\sqrt{6}}{4} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{4} & -\frac{1}{2} & -\frac{\sqrt{6}}{4} & \frac{\sqrt{3}}{4} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{4} & \frac{\sqrt{6}}{4} & -\frac{1}{2} & -\frac{\sqrt{3}}{4} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{6}}{4} & -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 1 & 2 & -1 & 1 & -\frac{1}{2} & -\frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{\sqrt{6}}{4} \\ 2 & -1 & 1 & 1 & \frac{\sqrt{3}}{4} & -\frac{1}{2} & \frac{\sqrt{6}}{4} & -\frac{\sqrt{3}}{4} \\ -1 & -1 & -1 & 2 & \frac{\sqrt{3}}{4} & -\frac{\sqrt{6}}{4} & -\frac{1}{2} & \frac{\sqrt{3}}{4} \\ 1 & -1 & -2 & -1 & \frac{\sqrt{6}}{4} & \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{2} \end{bmatrix}$$

The square roots of the matrix $A$ are:

i. The first root for $k=0$ is

$$A_0^{\frac{1}{2}} == \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{\sqrt{6}}{4} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{4} & \frac{1}{2} & -\frac{\sqrt{6}}{4} & \frac{\sqrt{3}}{4} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{4} & \frac{\sqrt{6}}{4} & \frac{1}{2} & -\frac{\sqrt{3}}{4} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{6}}{4} & -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 1 & 2 & -1 & 1 & \frac{1}{2} & -\frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{\sqrt{6}}{4} \\ 2 & -1 & 1 & 1 & \frac{\sqrt{3}}{4} & \frac{1}{2} & \frac{\sqrt{6}}{4} & -\frac{\sqrt{3}}{4} \\ -1 & -1 & -1 & 2 & \frac{\sqrt{3}}{4} & -\frac{\sqrt{6}}{4} & \frac{1}{2} & \frac{\sqrt{3}}{4} \\ 1 & -1 & -2 & -1 & \frac{\sqrt{6}}{4} & \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & \frac{1}{2} \end{bmatrix}$$

ii. and the second one for $k=1$ is

$$A_1^{\frac{1}{2}} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & \frac{\sqrt{6}}{4} & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{4} & -\frac{1}{2} & \frac{\sqrt{6}}{4} & -\frac{\sqrt{3}}{4} & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{4} & -\frac{\sqrt{6}}{4} & -\frac{1}{2} & \frac{\sqrt{3}}{4} & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{6}}{4} & \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ -1 & -2 & 1 & -1 & -\frac{1}{2} & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & \frac{\sqrt{6}}{4} \\ -2 & 1 & -1 & -1 & -\frac{\sqrt{3}}{4} & -\frac{1}{2} & -\frac{\sqrt{6}}{4} & \frac{\sqrt{3}}{4} \\ 1 & 1 & 1 & -2 & \frac{\sqrt{3}}{4} & -\frac{\sqrt{6}}{4} & -\frac{1}{2} & -\frac{\sqrt{3}}{4} \\ -1 & 1 & 2 & 1 & -\frac{\sqrt{6}}{4} & -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & -\frac{1}{2} \end{bmatrix}$$

See appendix for more clarification. Also, it is easy to see that $A_0^{\frac{1}{2}} + A_1^{\frac{1}{2}} = 0$.

From the Theorem 12, with $p = \frac{2\pi}{2\pi/3} = 3$, we get

$$A = A^4 = A^7 = A^{10} = \dots$$
$$A^2 = A^5 = A^8 = A^{11} = \dots$$
$$A^3 = A^6 = A^9 = A^{12} = \dots = I_8.$$

**APPENDIX**

In example 8, the square roots of the matrix $A$ can be calculated as follows:

$$A_k^{\frac{1}{2}}=\begin{bmatrix} \cos(\frac{2k\pi+2\pi/3}{2}) & -w_1\sin(\frac{2k\pi+2\pi/3}{2}) & -w_2\sin(\frac{2k\pi+2\pi/3}{2}) & -w_3\sin(\frac{2k\pi+2\pi/3}{2}) & 0 & 0 & 0 & 0 \\ w_1\sin(\frac{2k\pi+2\pi/3}{2}) & \cos(\frac{2k\pi+2\pi/3}{2}) & -w_3\sin(\frac{2k\pi+2\pi/3}{2}) & w_2\sin(\frac{2k\pi+2\pi/3}{2}) & 0 & 0 & 0 & 0 \\ w_2\sin(\frac{2k\pi+2\pi/3}{2}) & w_3\sin(\frac{2k\pi+2\pi/3}{2}) & \cos(\frac{2k\pi+2\pi/3}{2}) & -w_1\sin(\frac{2k\pi+2\pi/3}{2}) & 0 & 0 & 0 & 0 \\ w_3\sin(\frac{2k\pi+2\pi/3}{2}) & -w_2\sin(\frac{2k\pi+2\pi/3}{2}) & w_1\sin(\frac{2k\pi+2\pi/3}{2}) & \cos(\frac{2k\pi+2\pi/3}{2}) & 0 & 0 & 0 & 0 \\ w_4\sin(\frac{2k\pi+2\pi/3}{2}) & w_5\sin(\frac{2k\pi+2\pi/3}{2}) & w_6\sin(\frac{2k\pi+2\pi/3}{2}) & w_7\sin(\frac{2k\pi+2\pi/3}{2}) & \cos(\frac{2k\pi+2\pi/3}{2}) & -w_1\sin(\frac{2k\pi+2\pi/3}{2}) & -w_2\sin(\frac{2k\pi+2\pi/3}{2}) & -w_3\sin(\frac{2k\pi+2\pi/3}{2}) \\ w_5\sin(\frac{2k\pi+2\pi/3}{2}) & -w_4\sin(\frac{2k\pi+2\pi/3}{2}) & w_7\sin(\frac{2k\pi+2\pi/3}{2}) & -w_6\sin(\frac{2k\pi+2\pi/3}{2}) & w_1\sin(\frac{2k\pi+2\pi/3}{2}) & \cos(\frac{2k\pi+2\pi/3}{2}) & w_3\sin(\frac{2k\pi+2\pi/3}{2}) & -w_2\sin(\frac{2k\pi+2\pi/3}{2}) \\ w_6\sin(\frac{2k\pi+2\pi/3}{2}) & -w_7\sin(\frac{2k\pi+2\pi/3}{2}) & -w_4\sin(\frac{2k\pi+2\pi/3}{2}) & w_5\sin(\frac{2k\pi+2\pi/3}{2}) & w_2\sin(\frac{2k\pi+2\pi/3}{2}) & -w_3\sin(\frac{2k\pi+2\pi/3}{2}) & \cos(\frac{2k\pi+2\pi/3}{2}) & w_1\sin(\frac{2k\pi+2\pi/3}{2}) \\ w_7\sin(\frac{2k\pi+2\pi/3}{2}) & w_6\sin(\frac{2k\pi+2\pi/3}{2}) & -w_5\sin(\frac{2k\pi+2\pi/3}{2}) & -w_4\sin(\frac{2k\pi+2\pi/3}{2}) & w_3\sin(\frac{2k\pi+2\pi/3}{2}) & w_2\sin(\frac{2k\pi+2\pi/3}{2}) & -w_1\sin(\frac{2k\pi+2\pi/3}{2}) & \cos(\frac{2k\pi+2\pi/3}{2}) \end{bmatrix}$$

If $k=0$, then

$$A_0^{\frac{1}{2}}=\begin{bmatrix} \cos\frac{\pi}{3} & -w_1\sin\frac{\pi}{3} & -w_2\sin\frac{\pi}{3} & -w_3\sin\frac{\pi}{3} & 0 & 0 & 0 & 0 \\ w_1\sin\frac{\pi}{3} & \cos\frac{\pi}{3} & -w_3\sin\frac{\pi}{3} & w_2\sin\frac{\pi}{3} & 0 & 0 & 0 & 0 \\ w_2\sin\frac{\pi}{3} & w_3\sin\frac{\pi}{3} & \cos\frac{\pi}{3} & -w_1\sin\frac{\pi}{3} & 0 & 0 & 0 & 0 \\ w_3\sin\frac{\pi}{3} & -w_2\sin\frac{\pi}{3} & w_1\sin\frac{\pi}{3} & \cos\frac{\pi}{3} & 0 & 0 & 0 & 0 \\ w_4\sin\frac{\pi}{3} & w_5\sin\frac{\pi}{3} & w_6\sin\frac{\pi}{3} & w_7\sin\frac{\pi}{3} & \cos\frac{\pi}{3} & -w_1\sin\frac{\pi}{3} & -w_2\sin\frac{\pi}{3} & -w_3\sin\frac{\pi}{3} \\ w_5\sin\frac{\pi}{3} & -w_4\sin\frac{\pi}{3} & w_7\sin\frac{\pi}{3} & -w_6\sin\frac{\pi}{3} & w_1\sin\frac{\pi}{3} & \cos\frac{\pi}{3} & w_3\sin\frac{\pi}{3} & -w_2\sin\frac{\pi}{3} \\ w_6\sin\frac{\pi}{3} & -w_7\sin\frac{\pi}{3} & -w_4\sin\frac{\pi}{3} & w_5\sin\frac{\pi}{3} & w_2\sin\frac{\pi}{3} & -w_3\sin\frac{\pi}{3} & \cos\frac{\pi}{3} & w_1\sin\frac{\pi}{3} \\ w_7\sin\frac{\pi}{3} & w_6\sin\frac{\pi}{3} & -w_5\sin\frac{\pi}{3} & -w_4\sin\frac{\pi}{3} & w_3\sin\frac{\pi}{3} & w_2\sin\frac{\pi}{3} & -w_1\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix},$$

If $k=1$, then

$$A_1^{\frac{1}{2}}=\begin{bmatrix} \cos\frac{4\pi}{3} & -w_1\sin\frac{4\pi}{3} & -w_2\sin\frac{4\pi}{3} & -w_3\sin\frac{4\pi}{3} & 0 & 0 & 0 & 0 \\ w_1\sin\frac{4\pi}{3} & \cos\frac{4\pi}{3} & -w_3\sin\frac{4\pi}{3} & w_2\sin\frac{4\pi}{3} & 0 & 0 & 0 & 0 \\ w_2\sin\frac{4\pi}{3} & w_3\sin\frac{4\pi}{3} & \cos\frac{4\pi}{3} & -w_1\sin\frac{4\pi}{3} & 0 & 0 & 0 & 0 \\ w_3\sin\frac{4\pi}{3} & -w_2\sin\frac{4\pi}{3} & w_1\sin\frac{4\pi}{3} & \cos\frac{4\pi}{3} & 0 & 0 & 0 & 0 \\ w_4\sin\frac{4\pi}{3} & w_5\sin\frac{4\pi}{3} & w_6\sin\frac{4\pi}{3} & w_7\sin\frac{4\pi}{3} & \cos\frac{4\pi}{3} & -w_1\sin\frac{4\pi}{3} & -w_2\sin\frac{4\pi}{3} & -w_3\sin\frac{4\pi}{3} \\ w_5\sin\frac{4\pi}{3} & -w_4\sin\frac{4\pi}{3} & w_7\sin\frac{4\pi}{3} & -w_6\sin\frac{4\pi}{3} & w_1\sin\frac{4\pi}{3} & \cos\frac{4\pi}{3} & w_3\sin\frac{4\pi}{3} & -w_2\sin\frac{4\pi}{3} \\ w_6\sin\frac{4\pi}{3} & -w_7\sin\frac{4\pi}{3} & -w_4\sin\frac{4\pi}{3} & w_5\sin\frac{4\pi}{3} & w_2\sin\frac{4\pi}{3} & -w_3\sin\frac{4\pi}{3} & \cos\frac{4\pi}{3} & w_1\sin\frac{4\pi}{3} \\ w_7\sin\frac{4\pi}{3} & w_6\sin\frac{4\pi}{3} & -w_5\sin\frac{4\pi}{3} & -w_4\sin\frac{4\pi}{3} & w_3\sin\frac{4\pi}{3} & w_2\sin\frac{4\pi}{3} & -w_1\sin\frac{4\pi}{3} & \cos\frac{4\pi}{3} \end{bmatrix}.$$

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