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ARAŞTIRMA MAKALESİ / **RESEARCH ARTICLE**

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A COMPARISON THEOREM FOR SPECTRAL SEQUENCES

ABSTRACT

We give a comparison result for two first quadrant spectral sequences that are isomorphic at every point other than the bottom horizontal line and the one that is the source of the map vanishes on that line. This situation comes out naturally when comparing niveau spectral sequences related to certain homology theories and an example is given here.

Keywords: Homological Algebra

SPEKTRAL DİZİLER İÇİN BİR KARŞILAŞTIRMA TEOREMİ

ÖZ

Bu çalışmada alt yatay doğru hariç eşyapısal, o doğru üzerinde ise fonksiyonun kaynağı olan spektral dizinin sıfır olduğu iki birinci bölge spektral dizisini karşılaştırdık. Bu durum burada da bir örneğinin verildiği gibi bazı homoloji teorilerine ait niveau spektral dizilerini karşılaştırırken doğal bir biçimde ortaya çıkmaktadır.

Anahtar kelimeler: Homolojikal Cebir

1 INTRODUCTION

When showing certain relations between two homology theories related to a scheme X that is of finite type over a ground field k, it is sometimes easier to compare niveau spectral sequences (Bloch, 1974) related to these homology theories since the E_1 terms in niveau spectral sequence consist of homology groups of a point on X.

In this short paper we prove a comparison theorem for two first quadrant spectral sequences where all terms are isomorphic other than the bottom horizontal line and we also assume that one of the spectral sequnce vanishes on that line. This situation arises when we compare higher Chow groups (Bloch, 1984) and étale cohomology. But one might expect that it can also arise in other situations. This result was previously noticed by experts but we are giving here a written proof. It was also proved and used in authors Ph.D thesis (Uzun, 2013).

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2. Comparison Theorem

Theorem 1. *Let $E = \bigoplus E^1_{a,b}$ and $\bar{E} = \bigoplus \bar{E}^1_{a,b}$ be two first quadrant spectral sequences that strongly converge to H_* and $\bar{H}_*$ respectively. Assume $E^1_{a,0} = 0$ for all a and and we have a map $\rho : E \to \bar{E}$ of spectral sequences such that*

$$\rho^1_{a,b} : E^1_{a,b} \to \bar{E}^1_{a,b}$$

is an isomorphism for $b \geq 1$. Then we have a long exact sequence

$$\cdots \to \bar{H}_{n+1} \to \bar{E}^2_{n+1,0} \to H_n \to \bar{H}_n \to \bar{E}^2_{n,0} \to H_{n-1} \to$$

PROOF. Consider the following commutative diagram

$$\begin{array}{ccccc} E^r_{a+r,b-r+1} & \to & E^r_{a,b} & \to & E^r_{a-r,b+r-1} \\ \downarrow \rho^r_{a+r,b-r+1} & & \downarrow \rho^r_{a,b} & & \downarrow \rho^r_{a-r,b+r-1} \\ \bar{E}^r_{a+r,b-r+1} & \to & \bar{E}^r_{a,b} & \to & \bar{E}^r_{a-r,b+r-1} \end{array} \qquad (*)$$

Note that injectivity of $\rho^r_{a,b}$ and the surjectivity of $\rho^r_{a+r,b-r+1}$ implies injectivity of $\rho^{r+1}_{a,b}$. Also surjectivity of $\rho^r_{a,b}$ and injectivity of $\rho^r_{a-r,b+r-1}$ implies surjectivity of $\rho^{r+1}_{a,b}$. Using this and applying induction on r we see that for all $n \in \mathbb{Z}^+$, $\rho^n_{a,b}$ is surjective if $b \geq 1$ and injective if $b \geq n-1$. This implies $\rho^\infty_{a,b} : E^\infty_{a,b} \to \bar{E}^\infty_{a,b}$ is surjective if $b \geq 1$.

Now we look at the kernel of $\rho^\infty_{a,b}$ for $b \geq 1$. Consider the following commutative diagram

$$\begin{array}{ccccc} 0 & \to & E^{b+1}_{a,b} & \to & E^{b+1}_{a-b-1,2b} \\ \downarrow & & \downarrow \rho^{b+1}_{a,b} & & \downarrow \rho^{b+1}_{a-b-1,2b} \\ \bar{E}^{b+1}_{a+b+1,0} & \to & \bar{E}^{b+1}_{a,b} & \to & \bar{E}^{b+1}_{a-b-1,2b} \end{array}$$

From the above calculations we see that both $\rho^{b+1}_{a,b}$ and $\rho^{b+1}_{a-b-1,2b}$ are isomorphisms. This implies the kernels of the right horizontal maps are isomorphic. Since $E^{b+2}_{a,b} = ker(d^{b+1}_{a,b})$ and $\bar{E}^{b+2}_{a,b} = ker(\bar{d}^{b+1}_{a,b})/Im(\bar{d}^{b+1}_{a+b+1,0})$ we have the following exact sequence

$$\bar{E}^{b+1}_{a+b+1,0} \to E^{b+2}_{a,b} \to \bar{E}^{b+2}_{a,b} \to 0$$

and since the first map is the same as $d^{b+1}_{a+b+1,0}$ its kernel is $\bar{E}^{b+2}_{a+b+1,0}$. This gives the following exact sequence

$$0 \to \bar{E}^{b+2}_{a+b+1,0} \to \bar{E}^{b+1}_{a+b+1,0} \to E^{b+2}_{a,b} \to \bar{E}^{b+2}_{a,b} \to 0$$

If we write the above diagram $(*)$ for the r^{th} sheet where $r \geq b+2$, the left hand side vanishes for both spectral sequences. This implies for ∞ terms

$$0 \to \bar{E}^{b+2}_{a+b+1,0} \to \bar{E}^{b+1}_{a+b+1,0} \to E^{\infty}_{a,b} \to \bar{E}^{\infty}_{a,b} \to 0$$

Finally if we look at the case $a = 0$ we get

$$0 \to \bar{E}^{\infty}_{b+1,0} \to \bar{E}^{b+1}_{b+1,0} \to E^{\infty}_{0,b} \to \bar{E}^{\infty}_{0,b} \to 0$$

since $\bar{E}^{\infty}_{a+b+1,0} = \bar{E}^{a+b+2}_{a+b+1,0}$.

Let F_pH_q and $F_p\bar{H}_q$ denote the corresponding filtrations of H_q and $\bar{H}_q$ respectively. Convergenge gives us the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & F_{p-1}H_q & \to & F_pH_q & \to & E^{\infty}_{p,q-p} & \to & 0 \\ & & \downarrow \rho_q^{p-1} & & \downarrow \rho_q^{p} & & \downarrow \rho^{\infty}_{p,q-p} & & \\ 0 & \to & F_{p-1}\bar{H}_q & \to & F_p\bar{H}_q & \to & \bar{E}^{\infty}_{p,q-p} & \to & 0 \end{array}$$

For $p = 0$, the left hand group vanish. Therefore $ker(\rho_q^0) = ker(\rho^{\infty}_{0,q})$ which is isomorphic to $ker_0 :=$ $\bar{E}^{q+1}_{q+1,0}/\bar{E}^{\infty}_{q+1,0}$. Also since $\rho^{\infty}_{0,q}$ is surjective, we have an isomorphism $F_0H_q/(\bar{E}^{q+1}_{q+1,0}/\bar{E}^{\infty}_{q+1,0}) \cong F_0\bar{H}_q$. Combining this with the above commutative diagram we get

$$\begin{array}{ccccccccc} 0 & \to & F_0H_q/ker_0 & \to & F_1H_q/ker_0 & \to & E^{\infty}_{1,q-1} & \to & 0 \\ & & \downarrow \rho_q^{0} & & \downarrow \rho_q^{1} & & \downarrow \rho^{\infty}_{1,q-1} & & \\ 0 & \to & F_0\bar{H}_q & \to & F_1\bar{H}_q & \to & \bar{E}^{\infty}_{1,q-1} & \to & 0 \end{array}$$

The upper row is still exact. Also since ker_0 injects into $ker(\rho_q^1)$, we still have commutativity. The left hand map is an isomorphism which implies as before $ker(\rho_q^1) \cong ker(\rho^{\infty}_{1,q-1})$ which is isomorphic to $ker_1 := \bar{E}^{q}_{q+1,0}/\bar{E}^{q+1}_{q+1,0}$. Noting $ker_1 \cong (\bar{E}^{q}_{q+1,0}/\bar{E}^{\infty}_{q,0})/(\bar{E}^{q+1}_{q+1,0}/\bar{E}^{\infty}_{q,0})$, we see that

$$(F_1H_q/ker_0)/ker_1 \cong F_1H_q/(\bar{E}^{q}_{q+1,0}/\bar{E}^{\infty}_{q+1,0}) \cong F_1\bar{H}_q$$

Applying the same method gives $ker(\rho_q^{q-1}: F_{q-1}H_q \to F_{q-1}\bar{H}_q) \cong \bar{E}^{2}_{q+1,0}/\bar{E}^{\infty}_{q+1,0}$. Finally we look at the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & F_{q-1}H_q & \to & H_q & \to & 0 & & \\ & & \downarrow \rho_q^{q-1} & & \downarrow \rho_q & & \downarrow & & \\ 0 & \to & F_{q-1}\bar{H}_q & \to & \bar{H}_q & \to & \bar{E}_{q,0}^{\infty} & \to & 0 \end{array}$$

By snake lemma we find $ker(\rho_q) \cong ker(\rho_q^{q-1}) \cong \bar{E}_{q+1,0}^2/\bar{E}_{q+1,0}^{\infty}$ and $coker(\rho_q) \cong \bar{E}_{q,0}^{\infty}$. This gives us exact sequnces

$$0 \to \bar{E}_{q+1,0}^{\infty} \to \bar{E}_{q+1,0}^2 \to H_q \to \bar{H}_q \to \bar{E}_{q,0}^{\infty} \to 0$$

for all $q \geq 0$. Since $\bar{E}_{q,0}^{\infty}$ inject into $\bar{E}_{q,0}^2$ we can combine these to get the desired long exact sequences.

3. An Application

Let X be a smooth variety of pure dimension d over a perfect field k and n be invertible in k. The main application of the above result is comparing higher Chow groups and étale cohomology using cycle class map (Geisser and Levine, 2001) with the following indices.

$$\rho_X^{d+c,j}: CH^{d+c}(X, j; \mathbb{Z}/n) \to H_{\text{ét}}^{2d-j+2c}(X, \mathbb{Z}/n\,(d+c))$$

Here j is and integer and $c = cd(k) - 1$ where $cd(k)$ denotes the cohomological dimension of k. The reason of interest for these indices is that when $j = c$ the Poincaré duality identifies the right hand group with the étale fundamental group modulo n. Hence calculating the kernel and cokernel of this map is of interest for understanding the class field theory of X. We have the following result using Thm. 1 and Beilinson-Lichtenbaum conjecture.

Theorem 2. *Let the notation be as above. We have a long exact sequence*

$$\cdots \to KH_{j-c+2}^{(c)} \to CH^{d+c}(X, j; \mathbb{Z}/n) \to H_{\text{ét}}^{2d-j+2c}\big(X, \mathbb{Z}/n\,(d+c)\big)$$
$$\to KH_{j-c+1}^{(c)} \to CH^{d+c}(X, j-1; \mathbb{Z}/n) \to H_{\text{ét}}^{2d-j+2c+1}\big(X, \mathbb{Z}/n\,(d+c)\big) \to \cdots$$

PROOF. See Theorem 8(b) in (Uzun, 2016).

Here $KH_a^{(c)}$ denotes the Kato homology (Kato, 1986) and using known vanishing results on Kato homology (Jannsen and Saito, 2003; Jannsen and Saito, 2009; Saito and Kerz, 2012), one can show that the cycle class map is an isomorphism under certain assumptions as in Prop. 13 and Thm. 14 in (Uzun, 2016).

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