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ARAŞTIRMA MAKALESİ / **RESEARCH ARTICLE**

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OPTIMAL EMBEDDINGS OF FINITE METRIC SPACES INTO GRAPHS

ABSTRACT

We consider the embedding of a finite metric space into a weighted graph in such a way that the total weight of the edges is minimal. We discuss metric spaces with $n = 3,4,5$ points in detail and show that the already known classification for these cases can be obtained by simple operations on the associated graph of the given metric space.

Keywords: Finite metric spaces, Embeddings, Weighted graphs

SONLU METRİK UZAYLARIN ÇİZGELERE OPTİMAL GÖMÜLMESİ

ÖZ

Bu çalışmada, sonlu bir metrik uzaydan, ağırlıklı bir çizgeye, çizge kenarlarının toplam ağırlığı en az olacak biçimde gerçekleşen gömme dönüşümlerini gözönüne alıyoruz. Nokta sayıları üç, dört ve beş olan metrik uzaylar üzerinde bu tipten gömme dönüşümlerini detaylı olarak inceliyor ve bu durumlar için bilinen sınıflandırmaların, metrik uzaylara karşılık gelen çizgeler üzerinde tanımlanan bazı basit operasyonlar yardımıyla elde edilebileceğini gösteriyoruz.

Anahtar kelimeler: Sonlu metrik uzaylar, Gömme dönüşümleri, Ağırlıklı çizgeler

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1 INTRODUCTION

Embedding of finite metric spaces into Euclidean spaces or normed spaces or even into trees with shortest path metrics has been a century-long adventure which led to some very interesting insights. To name one beautiful result, the Schönberg's theorem, a finite metric space $(\mathrm{X}; \mathrm{d_{ij}}, \mathrm{i}, \mathrm{j} = 1, \dots, \mathrm{n})$ can be embedded into $\mathbb{R}^m$ if and only if the quadratic form

$$F(x_2, \dots, \mathrm{x_n}) = \sum_{i,j=2}^{n} (d_{1i}^2 + d_{1j}^2 - d_{ij}^2) x_i \ x_j$$

is positive semi-definite and of rank m (Blumenthal, 1970), (Matousek, 2010).

It seems that, by the difficulty and the general impossibility of exact isometric embeddings and by demands from computer and informatics sciences, the trend switched to embeddings with distortions and deep theorems resulted from this inquiry as for example the famous theorem of Bourgain which asserts that an n −point metric space can be embedded in ℓ_2 with distortion $O(\log n)$ (Bourgain, 1985).

We consider in this note another interesting version of the embedding question where arbitrary (finite) weighted graphs (with shortest path metrics) are allowed as target spaces. The goal is to find embeddings (also called “realizations”) with the constraint that the total weight of the ambient graph should be as small as possible. There is a rich literature also on this subject. It is proven that any finite metric space has an optimal realization in a graph G in the sense that the total weight of G is minimal among all realizations (Dress, 1984), (Imrich and Simoes-Pereira, 1984). The actual construction of optimal realizations is a rather difficult problem even for metric spaces with a small number of points (Koolen and Lesser, 2009), (Sturmfels and Yu, 2004). A constructive algorithm was given in (Varone, 2006) which works well in many cases. As a finite metric space is itself a complete weighted graph, the question amounts to minimizing the total length of the ``connecting threads" between the nodes. In this note we want to make this approach precise and define a "hands-on" procedure to construct realizations with stepwise decreasing total weights with the help of some simple operations, or "moves", on a given weighted graph. This somewhat naive approach yields nevertheless for metric spaces with few points (up to five) optimal realizations and in any case realizations with considerable reduction of the total weight (for metric spaces with any number of points).

We define a notion of "tightness" for weighted graphs and it seems that with the help of the moves we define one can embed a given finite metric space into a tight graph which might be a candidate an optimal embedding.

2.PRELIMINARIES

Definition 2.1 Let $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ be a finite graph with vertex set V and edge set E. We assume G to be simple in the sense that it is unoriented, there are no loop-edges and there is no more than one edge between any two different vertices. If G is connected there is at least one path joining any two vertices P and Q. If there is an edge between two vertices P and Q, we will denote this edge by $[PQ]$ or ($[QP]$) and say that the vertices P and Q are 1-connected. If there is an edge joining each pair of vertices, G is called a complete graph.

Definition 2.2 A weighted graph $G = (V, E, w)$ is a graph G with a positive-valued function w on the set E of the edges. We will denote the weight $w([PQ])$of the edge $[PQ]$ by w_{PQ}. Given two vertices P and Q of a weighted graph G and a path of edges starting at P and ending at Q, the sum of the weights of these edges is called the weight of the path. The total weight $W(G)$ of a weighted graph G is the sum of the weights of its edges.

Definition 2.3 A finite metric space X_n is a set $\{P_1, \dots, P_n\}$ together with a distance function $\mathrm{d}(P_i, P_j) =: d_{ij}$ $(i, j = 1, \dots, n)$ such that $d_{ij} = d_{ji}$, $d_{ii} = 0$, d_{ij} is positive whenever $i \neq j$ and the d_{ij} 's satisfy the triangle inequality $d_{ij} + d_{ik} \geq d_{jk}$ for each triple of indices i, j, k.

We define the quantity Δ_{ijk} as

$$\Delta_{ijk} = \mathrm{d_{ij}} + \mathrm{d_{ik}} - \mathrm{d_{jk}} \,, \tag{2.1}$$

and call it the ``excess'' of the triangle $[P_iP_jP_k]$ at the vertex P_i.

The triangle inequality is equivalent to the non-negativity of the Δ_{ijk}'s.

Note that a finite metric space X_n can be viewed as a weighted complete graph whose weights satisfy the triangle inequality. We will henceforth identify X_n with the associated complete weighted graph. The total weight of X_n is then

$$W(X_n) = \sum_{i<j}^{n} d_{ij} \ .$$

We recall that the vertex-set of any weighted graph has a natural metric, called the shortest-path metric. Given two vertices P and Q of a weighted graph G and a path of edges starting at P and ending at Q, the sum of the weights of these edges is called the weight of the path; The distance between the vertices P and Q is defined as the minimum of the weights of the paths between these vertices. A path realizing this minimum is called a shortest path between P and Q. Note that, the distance between two 1-connected vertices P and Q might be less than the weight w_{PQ} as there could be a path between P and Q with weight less than the weight of the edge $[PQ]$. We will however assume that this should not be happen, so that in weighted graphs we consider below, the edges should be shortest paths between their endpoints. This is a convenient and not restrictive assumption for our purposes and to our knowledge there is not a separate term for such weighted graphs. To avoid a cumbersome terminology we don't want to introduce one either. In such a weighted graph we can use consistently the notation w_{PQ} for the distance (weight of the shortest path) between P and Q, whether they are 1-connected or not.

We can consider ``triangles'' $[PQR]$ also in weighted graphs where the ``edges'' of the triangle might be any specified shortest paths between the vertices. We define the excess of such a triangle at P similarly as $\Delta_{PQR}= w_{PQ} + w_{PR} - w_{QR}$. We will use this mostly for cases where Q and R will be $1-$connected to P.

Definition 2.4 Let $X_n = \{P_1, \ldots, P_n\}$ be a finite metric space and let G_m be a weighted graph (not necessarily complete) with $m \geq n$ vertices Q_i, $i = 1, \ldots, m$. An isometric embedding of X_n (or weight-preserving embedding of X_n) into G_m is a map f from the vertex set of X_n into the vertex set of G_m such that the weight of the shortest path between $f(P_i)$ to $f(P_j)$ in G_m equals d_{ij}. i.e. $w_{f(P_i)f(P_j)} = d_{ij}$.

In other words, we have

$$d_{ij} = w([f(P_i)Q_{i_1}]) + w([Q_{i_1}\ Q_{i_2}]) + \cdots + w([Q_{i_k}\ f(P_j)]),$$

where $f(P_i)Q_{i_1}Q_{i_2}\cdots Q_{i_k}\ f(P_j)$ is a shortest path between $f(P_i)$ to $f(P_j)$ in G_m. (The shortest path need not be unique.)

If $f(P_i)$ and $f(P_j)$ are 1-connected in G, then by our general assumption above the edge $[f(P_i)f(P_j)]$ is a shortest path between the vertices $f(P_i)$ and $f(P_j)$ and the weight of the edge $[f(P_i)f(P_j)]$ equals d_{ij}.

We will study the problem of embedding of a finite metric space (or the associated complete weighted graph) into a weighted graph G such that the total weight of G is minimal among all possible ambient weighted graphs into which the given metric space is embeddable. We will call such an embedding an optimal embedding (or realization).

The case for $n = 2$ is trivial. We will first consider $n = 3$ case in detail.

3 OPTIMAL EMBEDDING of X_3

Let X_3 be a metric space with 3 vertices P_1, P_2 and P_3 and let G_4 be the weighted "Y"-space as shown in Figure 1.

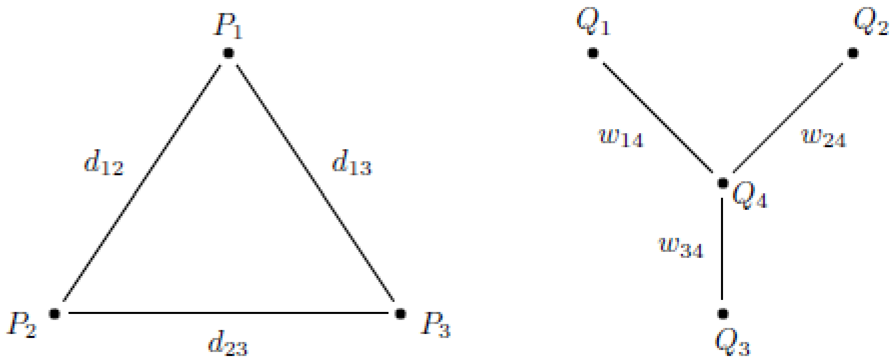

Figure 1

Let $f(P_i) = Q_i$. The weight preservation condition gives

$$\begin{aligned} w_{14} + w_{42} &= d_{12}, \\ w_{14} + w_{43} &= d_{13}, \\ w_{24} + w_{43} &= d_{23}, \end{aligned}$$

i.e,

$$\begin{aligned} w_{14} &= \frac{1}{2}(d_{12} + d_{13} - d_{23}) = \frac{1}{2}\Delta_{123}, \\ w_{24} &= \frac{1}{2}(d_{12} + d_{23} - d_{13}) = \frac{1}{2}\Delta_{213}, \\ w_{34} &= \frac{1}{2}(d_{13} + d_{23} - d_{12}) = \frac{1}{2}\Delta_{312} \end{aligned}$$

The mapping described above is called the "$\Delta - Y$"transform. (In case of a degenerate X_3 the "Y"-space also degenerates.) The total weights of X_3 and G_4 are respectively

$$W(X_3) = d_{12} + d_{13} + d_{23}$$

and

$$W(G_4) = w_{14} + w_{24} + w_{34} = \frac{1}{2}(d_{12} + d_{13} + d_{23})$$

hence

$$\frac{W(G_4)}{W(X_3)} = \frac{1}{2}.$$

We give a direct proof that the $\Delta - Y$ transform is minimal.

Proposition 3.1 The $\Delta - Y$ transform is an optimal embedding.

Proof Let the (non-degenerate) X_3 be embedded into a minimal weighted graph G, with Q_1, Q_2, Q_3 being the images of P_1, P_2, P_3. As the total weight of the previous special Y is $\frac{1}{2}(d_{12} + d_{13} + d_{23})$, the total weight of G can be at most that much. The length of the shortest path between Q_1 and Q_2 has to be equal to d_{12}, which we assume to be realized by $Q_1\, R_1\, R_2 \cdots R_k\, Q_2$. Likewise, let $Q_1\, S_1\, S_2 \cdots S_l\, Q_3$ be a shortest path with length d_{13}. If these paths were edge-disjoint, then the graph G would have a total weight of at least $d_{12} + d_{13}$. But, since $d_{12} + d_{13} > \frac{1}{2}(d_{12} + d_{13} + d_{23})$, this would contradict the minimality of G. Hence, some initial segment of these paths must coincide and let us assume $R_1 = S_1, R_2 = S_2, \ldots, R_t = S_t$ and $R_{t+1} \neq S_{t+1}$. Since G is minimal, the length of the path $S_t\, S_{t+1} \cdots Q_3$ can be at most

$$\frac{1}{2}(\mathrm{d}_{12} + \mathrm{d}_{13} + \mathrm{d}_{23}) - d_{12} = \frac{1}{2}(d_{13} + d_{23} - d_{12}).$$

But then, the length of the path $Q_1\, R_1\, \cdots R_t$ would be at least

$$d_{13} - \frac{1}{2}\ (d_{13} + d_{23} - d_{12}) = \frac{1}{2}\ (d_{12} + d_{13} - d_{23}).$$

If the length of the path $Q_1\, R_1 \cdots R_t$ would be more than this value, than the length of the path $R_t \cdots Q_2$ would be less than

$$d_{12} - \tfrac{1}{2}(d_{12} + d_{13} - d_{23}) = \tfrac{1}{2}(d_{12} + d_{23} - d_{13}).$$

This however would create a path $Q_2 \cdots R_t \cdots\ Q_3$ with length less than

$$\frac{1}{2}(d_{12} + d_{23} - d_{13}) + \frac{1}{2}(d_{13} + d_{23} - d_{12}) = d_{23},$$

contradicting the embedding assumption that the shortest path between Q_2 and Q_3 must have length d_{23}. Consequently, the length of the path $Q_1\, R_1 \cdots R_t$ must be exactly $\frac{1}{2}(d_{12} + d_{13} - d_{23})$. In that case, for the path $Q_1 \cdots Q_3$ to have the correct length, the path $S_t \cdots Q_3$ must have (not at most, but exactly) the length $\frac{1}{2}(d_{13} + d_{23} - d_{12})$. This brings us to the "Y" graph (with the middle vertex $R_t = S_t$) and there can't be any other unused edges of G so that we get $G =$ "Y".

4 SOME TOTAL-WEIGHT-DECREASING MOVES

Let $X_n = \{P_1, \ldots, P_n\}$ be a finite metric space and $f: X_n \to G_m$ be an isometric embedding of X_n into a weighted graph G_m with $m \geq n$ vertices. We call the vertices $f(P_i)$ as ``primary nodes" with respect to this embedding while the remaining ones are called "auxiliary nodes".

Note that the ambient graph is itself a metric space hence we can talk of the distances between auxiliary nodes too. By abuse of notation we will use d for both metrics. We will rename these vertices and denote them by P_i $(i = 1, \ldots, n)$ again. Such an embedding can be interpreted as a process of adjoining new vertices to the complete graph X, discarding some edges or adding new edges within the enlarged vertex set and assigning weights to the new edges such that the distances d_{ij} are still preserved as lengths of shortest paths, with the proviso that if an edge $[P_iP_j]$ of X is retained, then it is still a shortest path between P_i and P_j in G.

We will now define two basic operations (say, "moves") on the ambient graph G, which will convert the weighted graph G into another weighted graph G', together with an isometric embedding of X into G' with the aim of reducing the total weight.

i) First move: (Joining edges)

Let Q_i be a vertex of G and let us consider some (or all) of the vertices 1-connected to Q_i, say $R_1, \ldots, R_l$. Let $x = min_{\{1\leq\, j<k\leq\, l\}}\{\frac{1}{2}(w_{Q_iR_j} + w_{Q_iR_k} - w_{R_jR_k})\}$ and assume $x > 0$. (Recall that $w_{R_jR_k}$ is the weight of a shortest path between R_j and R_k.) Now we apply the following process: Delete all the edges from Q_i to $R_1, \ldots, R_l$; introduce a new vertex Q, put an edge between Q_i and Q of weight x, and put edges from Q to $R_1, \dots, R_l$ with weights $w_{Q_iR_j} - x$ for $j = 1, \dots, l$. The new graph G' satisfies our hypothesis and the embedding of X_n into G gives an embedding of X_n into G', preserving the primary nodes (but possibly rendering them no more 1-connected by the presence of the auxiliary node). The total weight of G is decreased by the amount $(l-1)x$.

ii) Second move: (Removing edges)

If an edge of G can be avoided by at least one shortest path between the primary nodes of G simultaneously (i.e. if we still get an embedding of X_n into G', where G' is obtained by deleting an edge from G), then delete it.

The "$\Delta - Y$" transform is a consequence of the above moves and can be applied to any triangle $[Q_i\ Q_j\ Q_k]$ with 1-connected vertices in G : If we apply the first move at the vertex Q_i and delete afterwards the edge $[Q_jQ_k]$, which becomes unnecessary, then we get a $\Delta - Y$ transform. By this move, the total weight of G will be decreased by half the total weight of the triangle $[Q_iQ_jQ_k]$.

We will now exemplify the usefulness of these moves by constructing an isometric embedding of a four-point metric space X_4.

5 AN ISOMETRIC EMBEDDING OF X_4

In this section we want to describe an isometric embedding of a four-point metric space which is known to be optimal among all possible alternatives (Imrich and Simoes-Pereira, 1984).

We want first to propose a definition for being "generic" for a metric space.

Definition 5.1 A finite metric space $X_n = \{P_1, \dots, P_n\}$ is called generic, if the set of the d_{ij}'s are linearly independent over the rationals.

We make this assumption only for convenience and it would be worth to clarify the relationship of this notion with the other genericity notions in the literature. Note that the embeddings of degenerate cases can be obtained by some kind of limiting process. Now, let $X_4 = \{P_1, P_2, P_3, P_4\}$ be a generic 4-point space (see Figure 2).

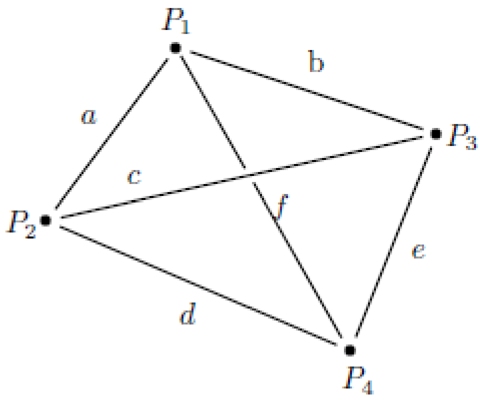

Figure 2

To avoid a mess of indices, we use the abbreviations $d_{12} = a, d_{13} = b, d_{23} = c, d_{24} = d, d_{34} = e, d_{14} = f$. The edge-pairs with lengths $(a, e), (b, d)$ and (c, f) are ``diagonals'' and we assume $a + e < b + d < c + f$. One can always arrange this by renaming the vertices and by genericity.

Let us now start with the complete graph X_4 and apply the first move to the vertex P_1. We have $x = min\{\frac{1}{2}(a + b - c), \frac{1}{2}(a + f - d), \frac{1}{2}(b + f - e)\}$. By our assumption $a + e < b + d < c + f$, this minimum equals $\frac{1}{2}(a + b - c)$. By sticking the ends of edges at P_1, we get the new graph G' with a new auxiliary node Q_1 as shown in Figure 3. (In figures below the primary nodes will be denoted by black vertices.)

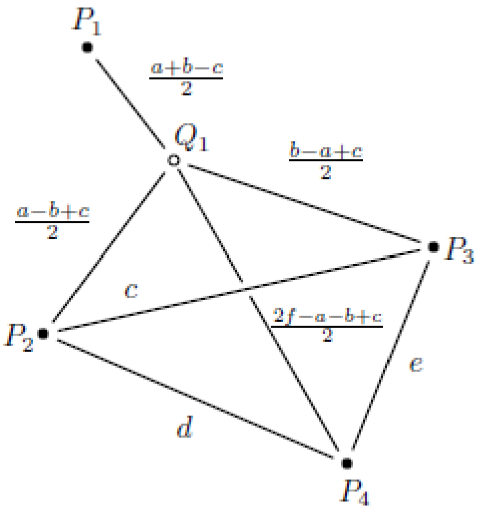

Figure 3 : The graph G

Now we apply the first move at P_2. The excess of the triangle $[P_2Q_1P_4]$ at P_2 is $\frac{1}{2}\left(a-b+\frac{c}{2}+d-\frac{2f-a-b+c}{2}\right)=\frac{1}{2}(a+d-f)$. The excess of $[P_2Q_1P_3]$ at P_2 is $(a-b+c)$ and the excess of $[P_2P_3P_4]$ at P_2 is $\frac{1}{2}(c+d-e)$ so that the sticking length of the edges at P_2 is $min\{\frac{1}{2}(a+d-f),\frac{1}{2}(a-b+c),\frac{1}{2}(c+d-e)\}$. By the assumption $a+e<b+d<c+f$, this minimum is $\frac{1}{2}(a+d-f)$ and we get a new graph G'' with a new auxiliary node Q_2 as shown in Figure 4.

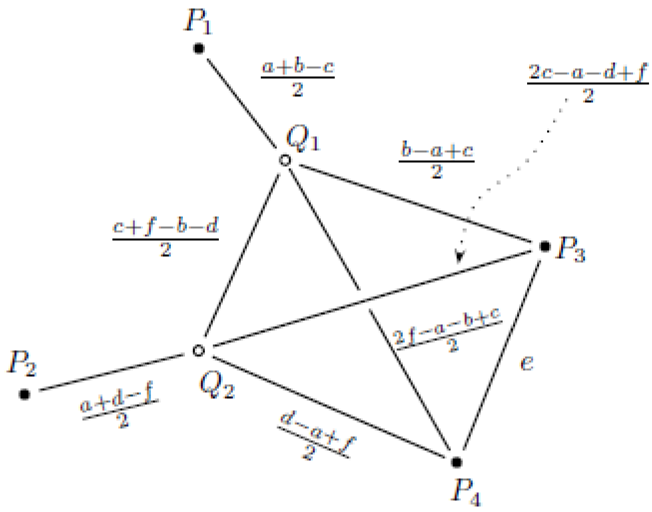

Figure 4 : The graph G''

We now apply the first move at the vertex P_3 of the graph G''. The excesses at P_3 (of the triangles $[P_3Q_1Q_2]$, $[P_3Q_2P_4]$ and $[P_3Q_1P_4]$) are $\frac{1}{2}(b-a+c)$, $\frac{1}{2}(c-d+e)$ and $\frac{1}{2}(b+e-f)$. The minimal excess is $\frac{1}{2}(b+e-f)$ and we get a new graph G''' with a new auxiliary node Q_3 as shown in Figure 5.

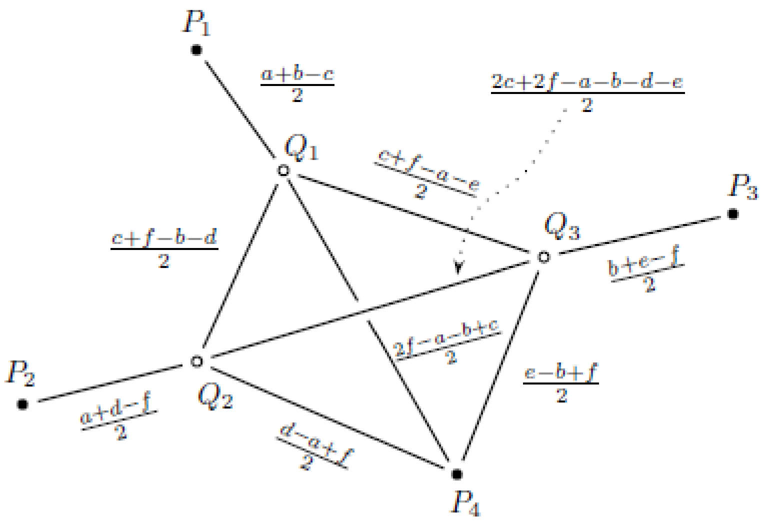

Figure 5 : The graph G'''

As a last application of the first move we stick the ends of the edges at P_4. The excesses are $\frac{1}{2}(d-a+f)$, $\frac{1}{2}(e-b+f)$ and $\frac{1}{2}(d+e-c)$, the last one being the minimum. We now get a graph G'''' with a new auxiliary node Q_4 as shown in Figure 6.

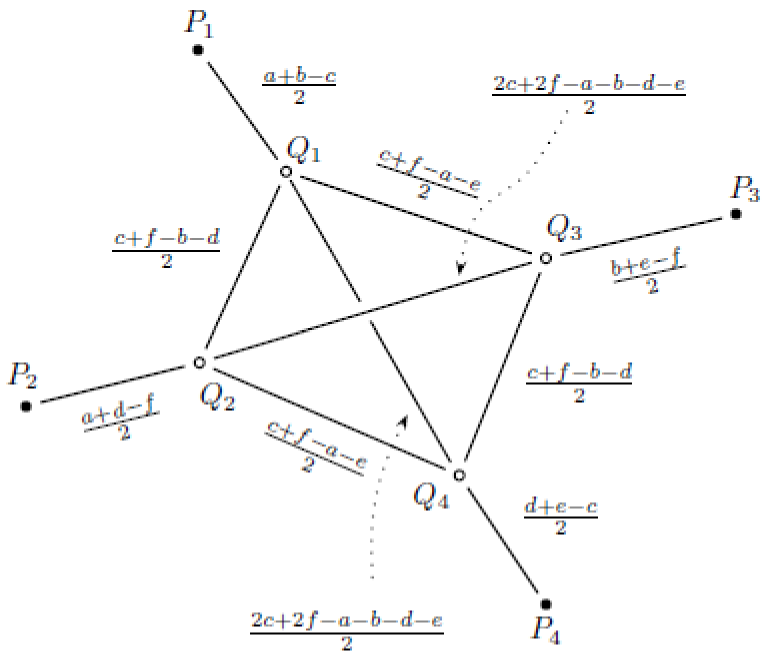

Figure 6 : The graph G''''

We can now apply the second move and delete the edges Q_1Q_4 and Q_2Q_3 in the graph G'''' as they can be avoided by shortest paths between the primary nodes. The resulting graph G with 8 vertices (4 primary and 4 auxiliary) is shown in Figure 7.

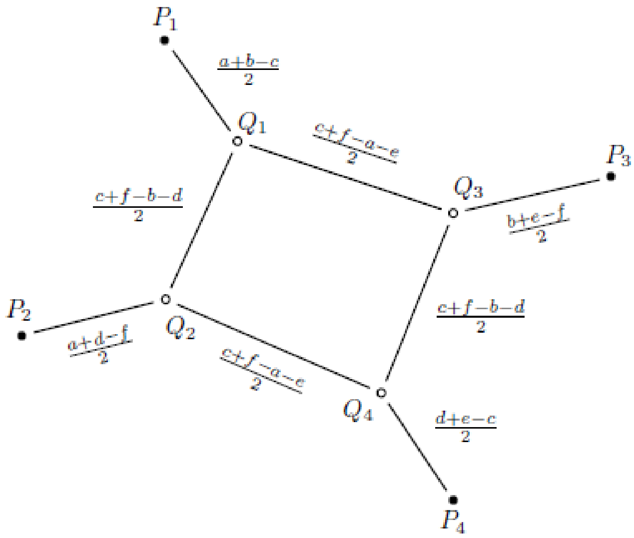

Figure 7: The graph G

Its total weight $W(G_8)$ equals $c+f$, the sum of the long ``diagonals" of X_4. The ratio $\frac{W(G_8)}{W(X_4)} = \frac{c+f}{a} + b + c + d + e + f$ is less than $\frac{1}{2}$ since $a+b+d+e > c+f$ for the generic case (and $\leq \frac{1}{2}$ generally). Non-generic cases where

i) $a+e = b+d < c+f$,

ii) $a+e < b+d = c+f$ and

iii) $a+e = b+d = c+f$

are depicted in Figures 8, 9 and 10.

All of these isometric embeddings of X_4 are optimal.

6 SIMULTANEOUS MOVES

In the example of the four-point metric space above, we sequentially applied several times the first move (of joining the edges); but this could have been done also simultaneously.

Proposition 6.1

Let $X_n = \{P_1, \ldots, P_n\}$ be a generic finite metric space, regarded as a complete weighted graph G. Let G' be the graph obtained from G by applying the first move at the vertex P_1, G'' the graph obtained from G' by applying the first move at P_2, and $G^{(n)}$ the graph obtained from $G^{(n-1)}$ by applying the first move at P_n. Let, on the other hand, G^* be the graph obtained from G by applying the first move at all vertices $P_1, \ldots, P_n$ simultaneously (in the obviously understood sense, creating the auxiliary points Q_i simultaneously and defining the weight $w_{Q_iQ_j}$ to be $w_{P_iP_j} - x_i - x_j$, where x_i is the sticking length at P_i, i.e. $x_i = min_{k\neq i, l\neq i, k\neq l} \{\frac{1}{2}(w_{P_iP_k} + w_{P_iP_l} - w_{P_kP_l})\}$. Then, the graphs $G^{(n)}$ and G^* are the same (isometric) weighted graphs.

This proposition can be proven by a straightforward (but somewhat tedious) check.

Note that after applying the simultaneous first move, there will be n, or possibly fewer, edges to be deleted by the second move.

Using this property we could obtain the optimal representation of X_4 instantly: We would get from Figure 2 by simultaneous moves directly Figure 6 by virtue of

$$min\{\frac{1}{2}(a+b-c), \frac{1}{2}(a+f-d), \frac{1}{2}(b+f-e)\} = \frac{1}{2}(a+b-c),$$

$$min\{\frac{1}{2}(a+c-b), \frac{1}{2}(a+d-f), \frac{1}{2}(c+d-e)\} = \frac{1}{2}(a+d-f),$$

etc. by our assumption $a+e < b+d < c+f$. Then, applying the second move (removing the edges), we would get the Figure 7, the optimal representation.

7 CLASSIFICATION OF GENERIC 5 −POINT METRIC SPACES

In this section we will classify 5 −point metric spaces by the possible sets of triangle excess to obtain the 3 types of optimal graphs given in (Koolen and Lesser, 2009).

Let Δ_{abc} be the minimal excess at node a and i be a node different from a, b and c. The following relations among triangle excess can be checked easily by using the definition.

$$\Delta_{\mathrm{abi}} - \Delta_{abc} = \Delta_{\mathrm{cbi}} - \Delta_{cai} = \Delta_{\mathrm{iac}} - \Delta_{ibc} = \Delta_{\mathrm{bac}} - \Delta_{bai} \qquad (7.1a)$$

$$\Delta_{\mathrm{aci}} - \Delta_{abc} = \Delta_{\mathrm{bci}} - \Delta_{bai} = \Delta_{\mathrm{iab}} - \Delta_{ibc} = \Delta_{\mathrm{cab}} - \Delta_{cai} \qquad (7.1b)$$

Thus if Δ_{abc} is the minimal excess at node a then, Δ_{cbi} and Δ_{cab} cannot be minimal at node c, Δ_{bci} and Δ_{bac} cannot be minimal at node b and Δ_{iac} and Δ_{iab} cannot be minimal at node i. Putting $a = 1$, $b = 2$, $c = 5$ and $i = 3{,}4$ we can see that the sets of possible minimal excess' at nodes 2, 3, 4 and 5 are

Node 2: $\{\Delta_{213}, \Delta_{214}, \Delta_{234}\}$,
Node 3: $\{\Delta_{314}, \Delta_{324}, \Delta_{325}, \Delta_{345}\}$,
Node 4: $\{\Delta_{413}, \Delta_{423}, \Delta_{425}, \Delta_{435}\}$,
Node 5: $\{\Delta_{513}, \Delta_{514}, \Delta_{534}\}$.

Without loss of generality, we may assume that the nodes are labeled so that $\Delta_{123} \geq \Delta_{124}$. Then, by using the relations

$$\Delta_{abi} - \Delta_{abj} = \Delta_{baj} - \Delta_{bai} = \Delta_{iaj} - \Delta_{ibj} = \Delta_{jbi} - \Delta_{jai}, \qquad (7.2)$$

with $a = 1, b = 2, i = 3, j = 4$, we can see that Δ_{214}, Δ_{314}, Δ_{423} cannot be minimal. Thus the set of possible minimal excess's is reduced to

Node 2: $\{\Delta_{213}, \Delta_{234}\}$
Node 3: $\{\Delta_{324}, \Delta_{325}, \Delta_{345}\}$
Node 4: $\{\Delta_{413}, \Delta_{425}, \Delta_{435}\}$
Node 5: $\{\Delta_{513}, \Delta_{514}, \Delta_{534}\}$

First, assume that Δ_{234} is minimal at node 2. Applying (7.1a) with $a = 2, b = 3, c = 4$ and $i = 1{,}5$ we can see that $\Delta_{324}, \Delta_{345}, \Delta_{413}$ and Δ_{435} cannot be minimal but there is no further restriction at node 5. The 3 alternatives at node 5 give the types E_1, E_2 and E_3 of Table 1.

E_2: $\Delta_{125} \rightarrow \Delta_{234} \rightarrow \Delta_{325} \rightarrow \Delta_{425} \rightarrow \Delta_{513}$.

E_1: $\Delta_{125} \rightarrow \Delta_{234} \rightarrow \Delta_{325} \rightarrow \Delta_{425} \rightarrow \Delta_{514}$.

E_3: $\Delta_{125} \rightarrow \Delta_{234} \rightarrow \Delta_{325} \rightarrow \Delta_{425} \rightarrow \Delta_{534}$.

Now, let Δ_{213} be minimal at node 2. This condition gives no restriction on the set of minimal excess's at node 3. If Δ_{324} is minimal, then applying (7.1a) with $a = 3, b = 2, c = 4$ and $i = 1{,}5$ we can see that Δ_{425} cannot be minimal. If Δ_{413} is minimal then Δ_{534} and Δ_{514} cannot be minimal, while if Δ_{435} is minimal then Δ_{513} and Δ_{534} cannot be minimal. These two alternative give respectively the cases C_1 and A_1 of Table 1.

$$C_1: \quad \Delta_{125} \to \Delta_{213} \to \Delta_{324} \to \Delta_{413} \to \Delta_{513},$$

$$A_1: \quad \Delta_{125} \to \Delta_{213} \to \Delta_{324} \to \Delta_{435} \to \Delta_{514}.$$

If Δ_{325} is minimal, then applying (7.1a) with $a = 3$, $b = 2$, $c = 5$ and $i = 1,4$ we can see that Δ_{435} can not be minimal. Then, if Δ_{413} is minimal, it can be seen that at node 5, the only possibility is the minimality of Δ_{513}. On the other hand, if Δ_{425} is minimal there is no further restriction at node 5. This gives the cases C_2, B_1, A_2 and D_1.

$$C_2: \quad \Delta_{125} \to \Delta_{213} \to \Delta_{325} \to \Delta_{413} \to \Delta_{513},$$

$$B_1: \quad \Delta_{125} \to \Delta_{213} \to \Delta_{325} \to \Delta_{425} \to \Delta_{513},$$

$$A_2: \quad \Delta_{125} \to \Delta_{213} \to \Delta_{325} \to \Delta_{425} \to \Delta_{514},$$

$$D_1: \quad \Delta_{125} \to \Delta_{213} \to \Delta_{325} \to \Delta_{425} \to \Delta_{534}.$$

Finally, if Δ_{345} is minimal, then applying (7.1a), with we can see that Δ_{413} is the only alternative at node 4 and Δ_{513} is the only alternative at node 5. This gives the case C_3 of Table 1.

$$C_3: \quad \Delta_{125} \to \Delta_{213} \to \Delta_{345} \to \Delta_{413} \to \Delta_{513}.$$

Table 1: Minimal Excesses at node i

Type	1	2	3	4	5	
A_1	Δ_{125}	Δ_{213}	Δ_{324}	Δ_{435}	Δ_{514}	5 edges removed
A_2	Δ_{125}	Δ_{213}	Δ_{325}	Δ_{425}	Δ_{514}	3 edges removed
B_1	Δ_{125}	Δ_{213}	Δ_{325}	Δ_{425}	Δ_{513}	2 edges removed
C_1	Δ_{125}	Δ_{213}	Δ_{324}	Δ_{413}	Δ_{513}	3 edges removed
C_2	Δ_{125}	Δ_{213}	Δ_{325}	Δ_{413}	Δ_{513}	2 edges removed
C_3	Δ_{125}	Δ_{213}	Δ_{345}	Δ_{413}	Δ_{513}	3 edges removed
D_1	Δ_{125}	Δ_{213}	Δ_{325}	Δ_{425}	Δ_{534}	3 edges removed
E_1	Δ_{125}	Δ_{234}	Δ_{325}	Δ_{425}	Δ_{514}	3 edges removed
E_2	Δ_{125}	Δ_{234}	Δ_{325}	Δ_{425}	Δ_{513}	3 edges removed
E_3	Δ_{125}	Δ_{234}	Δ_{325}	Δ_{425}	Δ_{534}	2 edges removed

We will now show how to arrive at the three classes of generic $5-$point spaces of (Koolean and Lesser, 2009). The table above is a guide how to do the joining and removing operations on a given $5-$point graph. We will now illustrate this on the types A_1, A_2 and B_1. These will yield the types (a), (c) and (b) of (Koolean and Lesser, 2009). In a similar vein, all the remaining cases in our table can easily be seen to result in one of the three types of [5].

Type A_1:

Given a generic $5-$point metric space $X_5 = \{P_1, P_2, P_3, P_4, P_5\}$ of type A_1, applying first the simultaneous joining move we get the graph G in Fig.8. The information on the excesses given in the first row of the table enables us to remove all the "diagonals" of the "pentagon". (Notice that the excesses at the auxiliary nodes Q_i of G vanish and at each Q_i we remove the diagonal causing this vanishing.) We thus get the graph G' in Fig.9 which is of class (a) of (Koolean and Lesser, 2009).

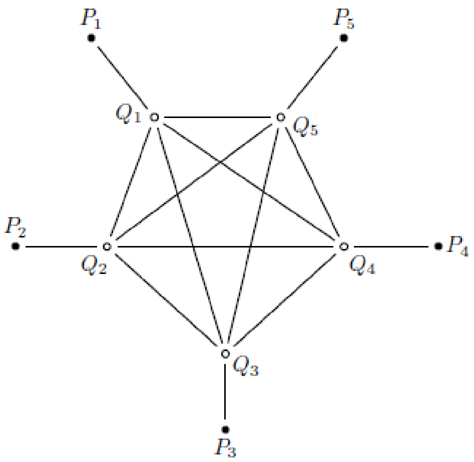

Figure 8 : The graph G

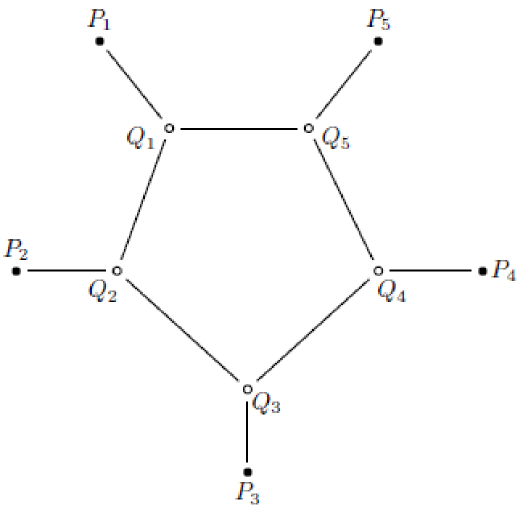

Figure 9 : The graph G'

Type A_2:

Given a generic 5-point metric space X_5 of type A_2, applying first the simultaneous joining move we get again the graph G in Fig.8. The information on the excesses given in the second row of the table enables us to remove the "diagonals" Q_1Q_3, Q_1Q_4, Q_2Q_5, giving the graph G' in Fig.10.

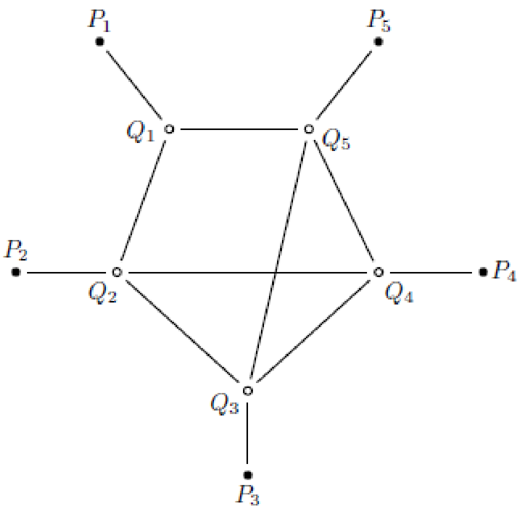

Figure 10: The graph G'

We now apply the $\Delta - Y$ transform to the triangle $[Q_2Q_3Q_4]$ and get the graph G'', shown in Fig.11, which can also be drawn as in Fig.12.

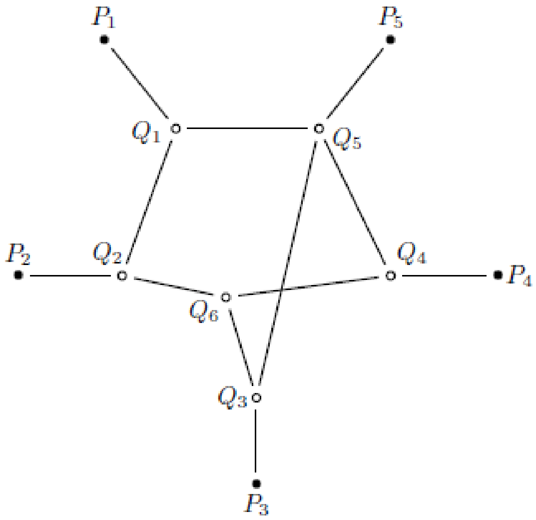

Figure 11: The graph G''

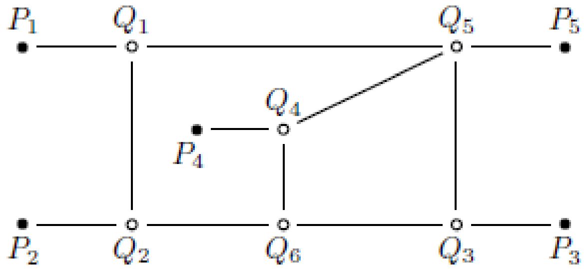

Figure 12

One can easily compute that $\Delta_{Q_5Q_3Q_4} = 2(\Delta_{P_5P_4P_3} - \Delta_{P_5P_1P_4})$, which is positive by the minimality of $\Delta_{P_5P_1P_4}$ and the genericity assumption, so that we can apply the joining move at $[Q_5Q_3Q_4]$ to obtain the final graph G''', shown in Fig.13. This graph belongs to class (c) of (Koolean and Lesser, 2009).

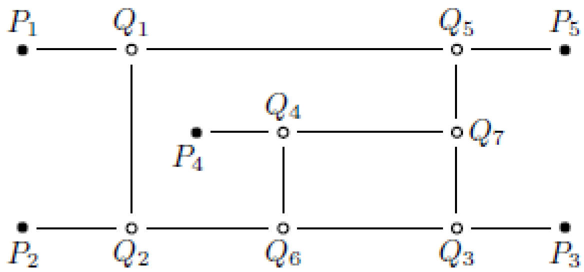

Figure 13: The graph G'''

Type B_1:

Given a generic $5-$point metric space X_5 of type B_1, applying first the simultaneous joining move we get again the graph G in Fig.8. The information on the excesses given in the third row of the table enables us to remove the "diagonals" Q_1Q_3 and Q_2Q_5, giving the graph G' in Fig.14.

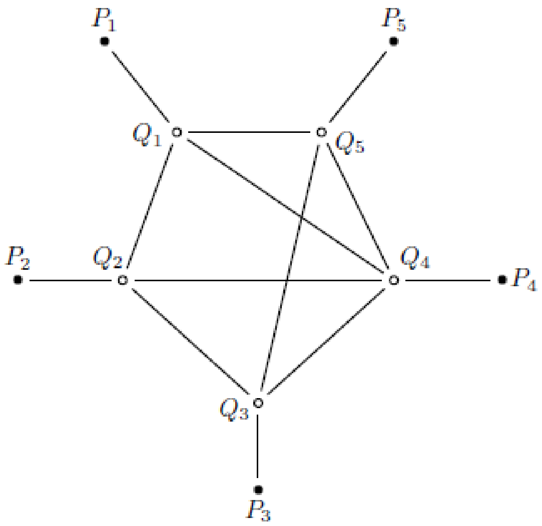

Figure 14: The graph G'

Now we apply the $\Delta - Y$ transform to the triangles $[Q_1Q_4Q_5]$ and $[Q_2Q_3Q_4]$, obtaining the graph G'' in Fig.15. This graph can also be drawn as in Fig.16, which belongs to the class (b) of (Koolean and Lesser, 2009).

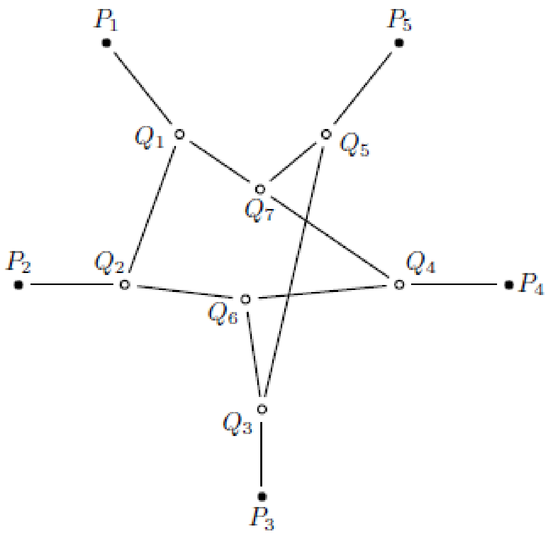

Figure 15: The graph G''

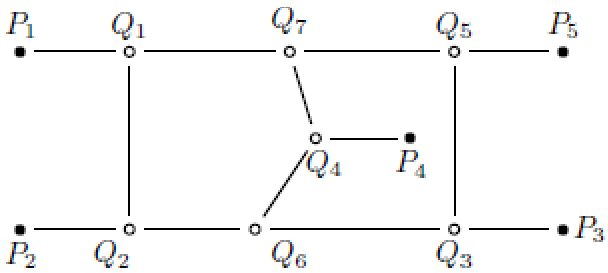

Figure 16

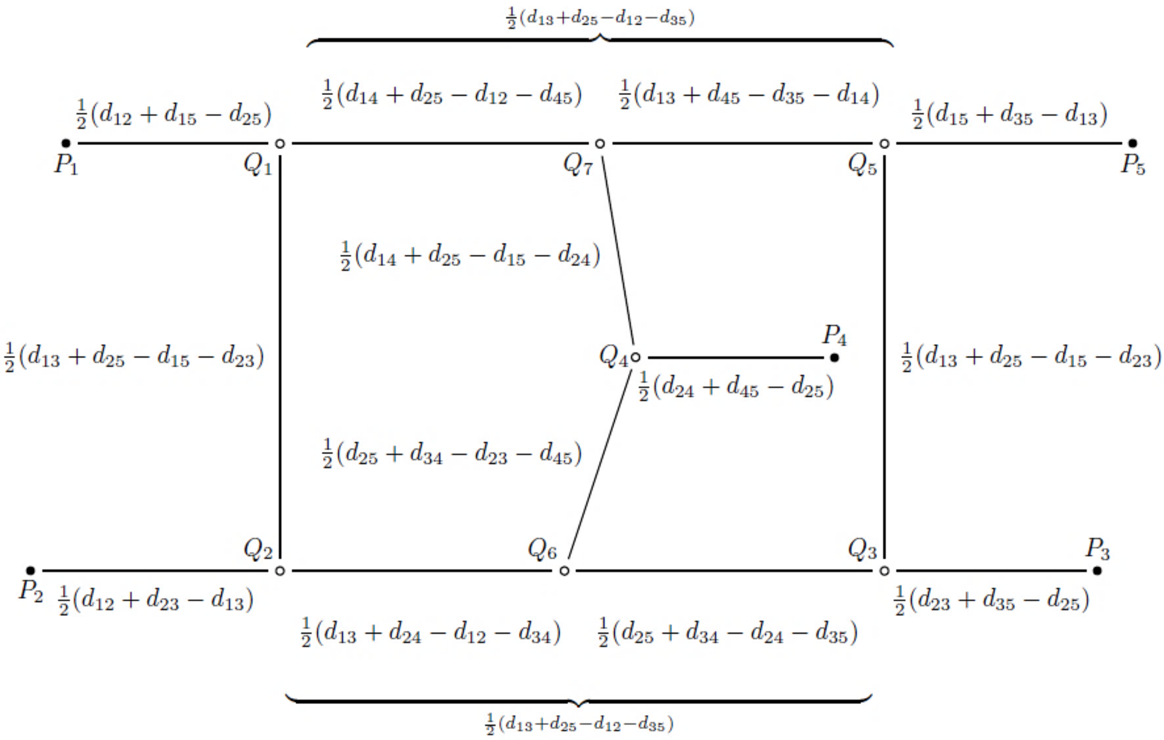

Figure 17: The graph G'' with weights

An advantage of our approach is that we can explicitly give the weights of the ambient graph by stepwise applying the shortening rules. As an example we show them on Fig.16. (In Figs.13 and 16, the quadrangles are "parallelograms" with respect to weights.)

REFERENCES

Blumenthal, L.M. (1970). Theory and Applications of Distance Geometry. Chelsea Pub.Comp. New York.

Bourgain, J. (1985). On Lipschitz Embedding of Finite Metric Spaces in Hilbert Space *Israel J. Math.* 52, no. 1-2, 46–52.

Dress, A. (1984). Trees, Tight Extensions of Metric Spaces and the Cohomological Dimension of Certain Groups: A Note on Combinatorial Properties of Metric Spaces *Advances in Mathematics* 53, 321-402.

Imrich, W., J. Simoes-Pereira, C.M.Zamfirescu (1984). On Optimal Embeddings of Metrics in Graphs *J. of Combin. Th. Ser.B* 36, 1-15.

Koolen, J., A. Lesser, V. Moulton (2009). Optimal Realizations of Generic Five-Point Metrics *European J. of Combin.* 30, 1164-1171.

Matousek, J. (2010). Thirty-Three Miniatures (Mathematical and Algorithmic Applications of Linear Algebra). AMS.

Sturmfels, B., J. Yu (2004). Classification of Six-Point Metrics *Electron. J. of Combin.* 11, 2004, R44.

Varone, S.C. (2006). A constructive Algorithm for Realizing a Distance Matrix *European J. of Oper. Research*, 174, 102-111.