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On computational formulas for parametric type polynomials and its applications

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Abstract

In this paper, many formulas and identities for computing the r-parametric Hermite type polynomials are given with the help of generating functions. Using generating functions and algebraic methods, a relation is also given including these polynomials and the 2-variable Hermite Kampé de Fériet polynomials. Moreover, many relations and formulas containing the two parametric type of Apostol-Bernoulli polynomials of higher order, the two parametric type of Apostol-Euler polynomials of higher order, the two parametric type of Apostol-Genocchi polynomials of higher order and the Dickson polynomials are obtained. Finally, some special values of these polynomials and their applications with trigonometric functions are presented.

***Keywords**: Hermite type polynomials, Apostol type numbers and polynomials, parametric type polynomials, special numbers and polynomials, generating functions.*

Parametrik tip polinomlar için hesaplama formülleri ve uygulamaları

Öz

Bu çalışmada, r-parametreli Hermite tipli polinomların hesaplanması için birçok formüller ve bağıntılar, üreteç fonksiyonları yardımıyla verilmiştir. Üreteç fonksiyonları ve cebirsel yöntemler kullanılarak, bu polinomları ve 2-değişkenli Hermite Kampé de Fériet polinomlarını içeren bir bağıntı da verilmiştir. Ayrıca, yüksek mertebeden iki parametreli Apostol-Bernoulli polinomları, yüksek mertebeden iki parametreli Apostol-Euler polinomları, yüksek mertebeden iki parametreli Apostol-Genocchi polinomları ve

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Dikson polinomlarını içeren birçok bağıntılar ve formüller elde edilmiştir. Son olarak, bu polinomların bazı özel değerleri ve trigonometrik fonksiyonlarla uygulamaları sunulmuştur.

Anahtar kelimeler: *Hermite tipli polinomlar, Apostol tipli sayılar ve polinomlar, parametrik tipli polinomlar, özel sayılar ve polinomlar, üreteç fonksiyonları.*

1. Introduction

Special functions and trigonometric functions have many important applications in mathematics and other applied sciences. Likewise, special numbers and polynomials and their generating functions, such as the Hermite polynomials and the Dickson polynomials, have same effect in many areas. One of the main reasons for the interest in these type numbers and polynomials are their common use in probability, algebra, combinatorics, wavelet transform analysis, physics, random matrix theory in Gaussian, permutations of finite fields. In addition, it is known that the Hermite type polynomials have been examined by many researchers with different methods [1-8]. In [8], Kilar and Simsek introduced r-parametric Hermite type polynomials and some special polynomials by the aid of the Euler's formula. They also give many identities and relations related to these polynomials. The main motivation of this paper is to give many explicit and computational formulas for the r-parametric Hermite type polynomials, Apostol type parametric polynomials of higher order, the Dickson polynomials of the first and second kinds, and special polynomials, using not only generating functions but also their functional equations. Moreover, many applications of these polynomials with some special values are given.

Throughout this paper, we use the following notations, definitions and relations.

Let $\mathbb{N} = \{1,2,3,\dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and also $\mathbb{Z}$ denote the set of integers, $\mathbb{R}$ denote the set of real numbers, $\mathbb{C}$ denote the set of complex numbers. Let

$$e^t = \exp(t),$$

and

$$w = x + iy,$$

where $x = \mathrm{Re}\{w\}$, $y = \mathrm{Im}\{w\}$ and $i^2 = -1$ (see [1-24]).
The Apostol-Bernoulli polynomials of order v, $\mathcal{B}_l^{(v)}(x;\lambda)$, are defined by

$$\left(\frac{t}{\lambda\exp(t) - 1}\right)^v \exp(tx) = \sum_{l=0}^{\infty} \mathcal{B}_l^{(v)}(x;\lambda)\frac{t^l}{l!}, \tag{1}$$

where $\lambda \in \mathbb{C}$ (or $\mathbb{R}$), $|t| < 2\pi$ when $\lambda = 1$ and $|t| < |\log(\lambda)|$ when $\lambda \neq 1$ (see, for detail [5, 8-14]).
Substituting $x = 0$ into (1), we have the Apostol-Bernoulli numbers of order v. That is,

$$\mathcal{B}_l^{(v)}(0;\lambda) = \mathcal{B}_l^{(v)}(\lambda).$$

Moreover $\mathcal{B}_l^{(v)}(x;1) = B_l^{(v)}(x)$ denote the Bernoulli polynomials of order v and $B_l^{(v)}(0) = B_l^{(v)}$ denote the Bernoulli numbers of order v (see [5, 8-14]).
The Apostol-Euler polynomials of order v, $\mathcal{E}_l^{(v)}(x;\lambda)$, are defined by

$$\left(\frac{2}{\lambda \exp(t)+1}\right)^v \exp(tx) = \sum_{l=0}^{\infty} \mathcal{E}_l^{(v)}(x;\lambda)\frac{t^l}{l!}, \tag{2}$$

where $\lambda \in \mathbb{C}$ (or $\mathbb{R}$), $|t| < \pi$ when $\lambda = 1$ and $|t| < |\log(-\lambda)|$ when $\lambda \neq 1$ (see, for detail [5, 8-14]).
Substituting $x = 0$ into (2), we have the Apostol-Euler numbers of order v. That is,

$$\mathcal{E}_l^{(v)}(0;\lambda) = \mathcal{E}_l^{(v)}(\lambda).$$

Moreover $\mathcal{E}_l^{(v)}(x;1) = E_l^{(v)}(x)$ denote the Euler polynomials of order v and $E_l^{(v)}(0) = E_l^{(v)}$ denote the Euler numbers of order v (see [5, 8-14]).
The Apostol-Genocchi polynomials of order v, $\mathcal{G}_l^{(v)}(x;\lambda)$, are defined by

$$\left(\frac{2t}{\lambda \exp(t)+1}\right)^v \exp(tx) = \sum_{l=0}^{\infty} \mathcal{G}_l^{(v)}(x;\lambda)\frac{t^l}{l!}, \tag{3}$$

where $\lambda \in \mathbb{C}$ (or $\mathbb{R}$), $|t| < \pi$ when $\lambda = 1$ and $|t| < |\log(-\lambda)|$ when $\lambda \neq 1$ (see, for detail [5, 8-14]).
Substituting $x = 0$ into (3), we have the Apostol-Genocchi numbers of order v. That is,

$$\mathcal{G}_l^{(v)}(0;\lambda) = \mathcal{G}_l^{(v)}(\lambda).$$

Moreover $\mathcal{G}_l^{(v)}(x;1) = G_l^{(v)}(x)$ denote the Genocchi polynomials of order v and $G_l^{(v)}(0) = G_l^{(v)}$ denote the Genocchi numbers of order v (see, for detail [5, 8-14]).
The Dickson polynomials of the first kind, $D_l(x,\alpha)$, are defined by

$$\frac{2-xt}{1-xt+\alpha t^2} = \sum_{l=0}^{\infty} D_l(x,\alpha)t^l \tag{4}$$

(see [15, 16]). By using (4), we get

$$D_l(x,\alpha) = \sum_{k=0}^{\left[\frac{l}{2}\right]} \frac{l}{l-k}\binom{l-k}{k}(-\alpha)^k x^{l-2k}, \tag{5}$$

where $l \in \mathbb{N}$, $D_0(x,\alpha) = 2$, and $[b]$ denotes the greatest integer in b (see [15, 16]).

The Dickson polynomials of the second kind, $\mathfrak{D}_l(x,\alpha)$, are defined by

$$\frac{1}{1-xt+\alpha t^2}=\sum_{l=0}^{\infty}\mathfrak{D}_l(x,\alpha)t^l \tag{6}$$

(see [15, 16]). By using (5), we obtain

$$\mathfrak{D}_l(x,\alpha)=\sum_{k=0}^{\left[\frac{l}{2}\right]}\binom{l-k}{k}(-\alpha)^k x^{l-2k}, \tag{7}$$

where $l\in\mathbb{N}$ and $\mathfrak{D}_0(x,\alpha)=1$ (see [15, 16]).
From (4) and (6), we find that

$$D_l(x,\alpha)=2\mathfrak{D}_l(x,\alpha)-x\mathfrak{D}_{l-1}(x,\alpha), \tag{8}$$

where $l\in\mathbb{N}$ (see [15, 16]).
The Hermite-Kampè de Fèriet (or Gould-Hopper) polynomials, $H_l^{(k)}(x,y)$, are defined by

$$R_{HK}(t,x,y,k)=\exp(tx+yt^k)=\sum_{l=0}^{\infty}H_l^{(k)}(x,y)\frac{t^l}{l!}, \tag{9}$$

where $k\in\mathbb{N}$. Substituting $k=2$ into (9), we have the 2-variable Hermite Kampé de Fériet polynomials, $H_l^{(2)}(x,y)$, which are satisfy solution of the heat equation and partial differential equations that are encountered in physical problems (see [1-9, 17]).
The generalized Hermite-Kampè de Fèriet polynomials, $H_l(\vec{u},r)$, are defined by

$$R_{GH}(t,\vec{u},r)=\exp\left(\sum_{k=1}^{r}u_k t^k\right)=\sum_{l=0}^{\infty}H_l(\vec{u},r)\frac{t^l}{l!}, \tag{10}$$

where $\vec{u}=(u_1,u_2,\dots,u_r)$ (see, for detail [1, 3, 4, 6]).
The polynomials $C_l(x,y)$ and the polynomials $S_l(x,y)$ are defined, respectively by,

$$\exp(tx)\cos(yt)=\sum_{l=0}^{\infty}C_l(x,y)\frac{t^l}{l!} \tag{11}$$

and

$$\exp(tx)\sin(yt)=\sum_{l=0}^{\infty}S_l(x,y)\frac{t^l}{l!} \tag{12}$$

(see [18]; also [8, 9, 14, 19-22]).
By the aid of (11) and (12), the following explicit formulas are derived:

$$C_l(x,y)=\sum_{k=0}^{\left[\frac{l}{2}\right]}(-1)^k\binom{l}{2k}y^{2k}x^{l-2k} \tag{13}$$

and

$$S_l(x,y)=\sum_{k=0}^{\left[\frac{l-1}{2}\right]}(-1)^k\binom{l}{2k+1}y^{2k+1}x^{l-2k-1}, \tag{14}$$

(see [18]; also [8, 9, 14, 19-22]).
The two parametric type of the Apostol-Bernoulli polynomials of order v, $\mathcal{B}_l^{(C,v)}(x,y;\lambda)$ and $\mathcal{B}_l^{(S,v)}(x,y;\lambda)$, are defined, respectively, by

$$\left(\frac{t}{\lambda\exp(t)-1}\right)^v\exp(tx)\cos(yt)=\sum_{l=0}^{\infty}\mathcal{B}_l^{(C,v)}(x,y;\lambda)\frac{t^l}{l!} \tag{15}$$

and

$$\left(\frac{t}{\lambda\exp(t)-1}\right)^v\exp(tx)\sin(yt)=\sum_{l=0}^{\infty}\mathcal{B}_l^{(S,v)}(x,y;\lambda)\frac{t^l}{l!} \tag{16}$$

(see, for detail [14]).
Combining (15), (16) with (1), (11) and (12), we get

$$\mathcal{B}_l^{(C,v)}(x,y;\lambda)=\sum_{k=0}^{l}\binom{l}{k}\mathcal{B}_{l-k}^{(v)}(\lambda)C_k(x,y) \tag{17}$$

and

$$\mathcal{B}_l^{(S,v)}(x,y;\lambda)=\sum_{k=0}^{l}\binom{l}{k}\mathcal{B}_{l-k}^{(v)}(\lambda)S_k(x,y) \tag{18}$$

(see, for detail [14]).
The two parametric type of the Apostol-Euler polynomials of order v, $\mathcal{E}_l^{(C,v)}(x,y;\lambda)$ and $\mathcal{E}_l^{(S,v)}(x,y;\lambda)$, are defined, respectively, by

$$\left(\frac{2}{\lambda\exp(t)+1}\right)^v\exp(tx)\cos(yt)=\sum_{l=0}^{\infty}\mathcal{E}_l^{(C,v)}(x,y;\lambda)\frac{t^l}{l!} \tag{19}$$

and

$$\left(\frac{2}{\lambda\exp(t)+1}\right)^{v}\exp(tx)\sin(yt)=\sum_{l=0}^{\infty}\mathcal{E}_{l}^{(S,v)}(x,y;\lambda)\frac{t^{l}}{l!} \tag{20}$$

(see, for detail [14]).
Combining (19), (20) with (2), (11) and (12), we obtain

$$\mathcal{E}_{l}^{(C,v)}(x,y;\lambda)=\sum_{k=0}^{l}\binom{l}{k}\mathcal{E}_{l-k}^{(v)}(\lambda)C_{k}(x,y) \tag{21}$$

and

$$\mathcal{E}_{l}^{(S,v)}(x,y;\lambda)=\sum_{k=0}^{l}\binom{l}{k}\mathcal{E}_{l-k}^{(v)}(\lambda)S_{k}(x,y) \tag{22}$$

(see, for detail [14]).
The two parametric type of the Apostol-Genocchi polynomials of order v, $\mathcal{G}_{l}^{(C,v)}(x,y;\lambda)$ and $\mathcal{G}_{l}^{(S,v)}(x,y;\lambda)$, are defined, respectively, by

$$\left(\frac{2t}{\lambda\exp(t)+1}\right)^{v}\exp(tx)\cos(yt)=\sum_{l=0}^{\infty}\mathcal{G}_{l}^{(C,v)}(x,y;\lambda)\frac{t^{l}}{l!} \tag{23}$$

and

$$\left(\frac{2t}{\lambda\exp(t)+1}\right)^{v}\exp(tx)\sin(yt)=\sum_{l=0}^{\infty}\mathcal{G}_{l}^{(S,v)}(x,y;\lambda)\frac{t^{l}}{l!} \tag{24}$$

(see, for detail [14]).
Combining (23), (24) with (3), (11) and (12), we have

$$\mathcal{G}_{l}^{(C,v)}(x,y;\lambda)=\sum_{k=0}^{l}\binom{l}{k}\mathcal{G}_{l-k}^{(v)}(\lambda)C_{k}(x,y) \tag{25}$$

and

$$\mathcal{G}_{l}^{(S,v)}(x,y;\lambda)=\sum_{k=0}^{l}\binom{l}{k}\mathcal{G}_{l-k}^{(v)}(\lambda)S_{k}(x,y) \tag{26}$$

(see, for detail [14]).
The r-parametric Hermite type polynomials, $\mathcal{K}(l;w,\vec{u},r)$, are defined by

$$R_{\mathcal{K}}(t,w,\vec{u},r)=\exp\left(wt+\sum_{k=1}^{r}u_k t^k\right)=\sum_{l=0}^{\infty}\mathcal{K}(l;w,\vec{u},r)\frac{t^l}{l!}, \tag{27}$$

where $r\in\mathbb{N}$ r-tuples $\vec{u}=(u_1,u_2,\dots,u_r)$, $w=x+iy$ and $u_1,u_2,\dots,u_r,x,y\in\mathbb{R}$ (see, for detail [8]).
The polynomials $k_1(l;x,y,\vec{u},r)$ and the polynomials $k_2(l;x,y,\vec{u},r)$ are defined, respectively, by

$$R_{k1}(t,x,y,\vec{u},r)=\exp\left(xt+\sum_{k=1}^{r}u_k t^k\right)\cos(yt)=\sum_{l=0}^{\infty}k_1(l;x,y,\vec{u},r)\frac{t^l}{l!} \tag{28}$$

and

$$R_{k2}(t,x,y,\vec{u},r)=\exp\left(xt+\sum_{k=1}^{r}u_k t^k\right)\sin(yt)=\sum_{l=0}^{\infty}k_2(l;x,y,\vec{u},r)\frac{t^l}{l!} \tag{29}$$

(see, for detail [8]).
Here we note that, by using the Euler formula with decompositions of equation (27), we have (28) and (29). Namely

$$\mathcal{K}(l;w,\vec{u},r)=k_1(l;x,y,\vec{u},r)+ik_2(l;x,y,\vec{u},r)$$

(see, for detail [8]).
The polynomials $C_l(\vec{u},y;r)$ and the polynomials $S_l(\vec{u},y;r)$ are defined, respectively by,

$$\exp\left(\sum_{k=1}^{r}u_k t^k\right)\cos(yt)=\sum_{l=0}^{\infty}C_l(\vec{u},y;r)\frac{t^l}{l!} \tag{30}$$

and

$$\exp\left(\sum_{k=1}^{r}u_k t^k\right)\sin(yt)=\sum_{l=0}^{\infty}S_l(\vec{u},y;r)\frac{t^l}{l!} \tag{31}$$

(see, for detail [8]).
Substituting $r=1$ into above equations, we derive

$$C_l(\vec{u},y;1)=C_l(u_1,y)\qquad\text{and}\qquad S_l(\vec{u},y;1)=S_l(u_1,y).$$

The results of this paper are briefly summarized below:
In section 2, many explicit and computation formulas, and identities for the r-parametric Hermite type polynomials, the 2-variable Hermite Kampé de Fériet polynomials, and some special polynomials are given.
In Section 3, some relations and formulas for the Apostol type parametric polynomials of higher order are derived. In addition, some applications with special values of these

polynomials, the Dickson polynomials of the first and second kinds, and trigonometric functions are presented.

2. Explicit formulas for Hermite type parametric polynomials

In this section, by using generating function methods, some explicit formulas and computational identities for the r-parametric Hermite type polynomials, the polynomials $k_1(l;x,y,\vec{u},r)$ and the polynomials $k_2(l;x,y,\vec{u},r)$ are obtained. Moreover, an identity including the 2-variable Hermite Kampé de Fériet polynomials is given.

In order to give these formulas, firstly we present the following series product:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(n,k)=\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{m}\right]}A(k,n-km), \tag{32}$$

where $m\in\mathbb{N}$ (see [13, 23, 24]).

The algebraic interpretation for the polynomials $\mathcal{K}(l;w,\vec{u},r)$, including the 2-variable Hermite Kampé de Fériet polynomials, is given by following theorem:

Theorem 2.1 Let $\vec{u}=(u_1,u_2,\dots,u_r)$, $w=x+iy$, $l\in\mathbb{N}_0$ and $r\geq 3$. Then we have

$$\mathcal{K}(l;w,\vec{u},r)=l!\sum_{j_{r-2}=0}^{\left[\frac{l}{r}\right]}\sum_{j_{r-3}=0}^{\left[\frac{l-rj_{r-2}}{r-1}\right]}\sum_{j_{r-4}=0}^{\left[\frac{l-rj_{r-2}-(r-1)j_{r-3}}{r-2}\right]}\cdots$$

$$\times\sum_{j_1=0}^{\left[\frac{l-rj_{r-2}-\cdots-5j_3-4j_2}{3}\right]}\frac{H^{(2)}_{l-rj_{r-2}-\cdots-3j_1}(w+u_1,u_2)u_3{}^{j_1}u_4^{j_2}u_5^{j_3}\dots u_r^{j_{r-2}}}{j_{r-2}!\cdots j_2!\,j_1!\,(l-rj_{r-2}-\cdots-4j_2-3j_1)!}.$$

Proof. We setting

$$F_1(t)=\exp(wt+u_1t),$$

$$F_2(t)=\exp(wt+u_1t)\exp(u_2t^2)=F_1(t)\exp(u_2t^2),$$

$$F_3(t)=F_2(t)\exp(u_3t^3),$$

$$\vdots$$

$$F_{r-1}(t)=F_{r-2}(t)\exp(u_{r-1}t^{r-1}),$$

$F_r(t) = F_{r-1}(t)\exp(u_r t^r).$

From the above sequences and using (9) in which $k = 2$ and (32), we have

$$F_3(t) = \sum_{l=0}^{\infty} H_l^{(2)}(w+u_1,u_2)\frac{t^l}{l!}\sum_{l=0}^{\infty} u_3^l \frac{t^{3l}}{l!} = \sum_{l=0}^{\infty}\sum_{j_1=0}^{\left[\frac{l}{3}\right]} \frac{H_{l-3j_1}^{(2)}(w+u_1,u_2)u_3^{j_1}}{(l-3j_1)!\,j_1!}t^l,$$

$$F_4(t) = \sum_{l=0}^{\infty}\sum_{j_2=0}^{\left[\frac{l}{4}\right]}\sum_{j_1=0}^{\left[\frac{l-4j_2}{3}\right]} \frac{H_{l-4j_2-3j_1}^{(2)}(w+u_1,u_2)u_3^{j_1}u_4^{j_2}}{(l-4j_2-3j_1)!\,j_1!\,j_2!}t^l,$$

$\vdots$

$$F_r(t) = \sum_{l=0}^{\infty}\mathcal{K}(l;w,\vec{u},r)\frac{t^l}{l!} = \sum_{l=0}^{\infty}\sum_{j_{r-2}=0}^{\left[\frac{l}{r}\right]}\sum_{j_{r-3}=0}^{\left[\frac{l-rj_{r-2}}{r-1}\right]}\sum_{j_{r-4}=0}^{\left[\frac{l-rj_{r-2}-(r-1)j_{r-3}}{r-2}\right]} \cdots$$

$$\times \sum_{j_1=0}^{\left[\frac{l-rj_{r-2}-\cdots-5j_3-4j_2}{3}\right]} \frac{H_{l-rj_{r-2}-\cdots-3j_1}^{(2)}(w+u_1,u_2)u_3{}^{j_1}u_4^{j_2}u_5^{j_3}\ldots u_r^{j_{r-2}}}{j_{r-2}!\cdots j_2!\,j_1!\,(l-rj_{r-2}-\cdots-4j_2-3j_1)!}t^l.$$

Comparing coefficient of t^l on both sides of the above equation, we arrive the desired result.

Theorem 2.2 Let $\vec{u} = (u_1, u_2, \ldots, u_r)$, $w = x + iy$ and $l \in \mathbb{N}_0$. Then we have

$$\mathcal{K}(l;w,\vec{u},r) = l!\sum_{j_r=0}^{\left[\frac{l}{r}\right]}\sum_{j_{r-1}=0}^{\left[\frac{l-rj_r}{r-1}\right]}\sum_{j_{r-2}=0}^{\left[\frac{l-rj_r-(r-1)j_{r-1}}{r-2}\right]} \cdots \tag{33}$$

$$\times \sum_{j_2=0}^{\left[\frac{l-rj_r-\cdots-3j_3}{2}\right]} \frac{(u_1+w)^{l-rj_r-\cdots-3j_3-2j_2}u_2^{j_2}u_3^{j_3}\ldots u_r^{j_r}}{j_r!\,j_{r-1}!\cdots j_3!\,j_2!\,(l-rj_r-\cdots-3j_3-2j_2)!}.$$

Proof. By using (27), we get

$$\sum_{l=0}^{\infty}\mathcal{K}(l;w,\vec{u},r)\frac{t^l}{l!} = \sum_{l=0}^{\infty}(u_1+w)^l\frac{t^l}{l!}\sum_{l=0}^{\infty}u_2^l\frac{t^{2l}}{l!}\sum_{l=0}^{\infty}u_3^l\frac{t^{3l}}{l!}\cdots\sum_{l=0}^{\infty}u_r^l\frac{t^{rl}}{l!}.$$

By applying (32) to the above equation, we obtain

$$\sum_{l=0}^{\infty}\mathcal{K}(l;w,\vec{u},r)\frac{t^l}{l!}=\sum_{l=0}^{\infty}\sum_{j_r=0}^{\left[\frac{l}{r}\right]}\sum_{j_{r-1}=0}^{\left[\frac{l-rj_r}{r-1}\right]}\sum_{j_{r-2}=0}^{\left[\frac{l-rj_r-(r-1)j_{r-1}}{r-2}\right]}\cdots$$

$$\times\sum_{j_2=0}^{\left[\frac{l-rj_r-\cdots-3j_3}{2}\right]}\frac{(u_1+w)^{l-rj_r-\cdots-3j_3-2j_2}u_2^{j_2}u_3^{j_3}\ldots u_r^{j_r}}{j_r!\,j_{r-1}!\cdots j_3!\,j_2!\,(l-rj_r-\cdots-3j_3-2j_2)!}t^l.$$

Comparing coefficient of t^l on both sides of the above equation, we arrive the desired result.

By using binomial theorem and (33), we have the following corollary:

Corollary 2.3 Let $\vec{u}=(u_1,u_2,\ldots,u_r)$, $w=x+iy$ and $l\in\mathbb{N}_0$. Then we have

$$\mathcal{K}(l;w,\vec{u},r)=l!\sum_{j_r=0}^{\left[\frac{l}{r}\right]}\sum_{j_{r-1}=0}^{\left[\frac{l-rj_r}{r-1}\right]}\sum_{j_{r-2}=0}^{\left[\frac{l-rj_r-(r-1)j_{r-1}}{r-2}\right]}\cdots\sum_{j_2=0}^{\left[\frac{l-rj_r-\cdots-3j_3}{2}\right]} \tag{34}$$

$$\times\sum_{j_1=0}^{l-rj_r-\cdots-3j_3-2j_2}\binom{l-rj_r-\cdots-3j_3-2j_2}{j_1}\frac{w^{l-rj_r-\cdots-3j_3-2j_2-j_1}{u_1}^{j_1}u_2^{j_2}u_3^{j_3}\ldots u_r^{j_r}}{j_r!\,j_{r-1}!\cdots j_3!\,j_2!\,(l-rj_r-\cdots-3j_3-2j_2)!}.$$

Now, we give some applications the equation (27). Substituting $r=2$ into (34), we have

$$\sum_{l=0}^{\infty}\mathcal{K}(l;w,\vec{u},2)\frac{t^l}{l!}=\sum_{l=0}^{\infty}(u_1+w)^l\frac{t^l}{l!}\sum_{l=0}^{\infty}u_2^l\frac{t^{2l}}{l!}.$$

Therefore

$$\sum_{l=0}^{\infty}\mathcal{K}(l;w,\vec{u},2)\frac{t^l}{l!}=\sum_{l=0}^{\infty}\sum_{j_2=0}^{\left[\frac{l}{2}\right]}\sum_{j_1=0}^{l-2j_2}\binom{l-2j_2}{j_1}\frac{w^{l-2j_2-j_1}{u_1}^{j_1}u_2^{j_2}}{j_2!\,(l-2j_2)!}t^l.$$

Comparing coefficient of t^l on both sides of the above equation, one has

$$\mathcal{K}(l;w,\vec{u},2)=l!\sum_{j_2=0}^{\left[\frac{l}{2}\right]}\sum_{j_1=0}^{l-2j_2}\binom{l-2j_2}{j_1}\frac{w^{l-2j_2-j_1}{u_1}^{j_1}u_2^{j_2}}{j_2!\,(l-2j_2)!}.$$

From the above computation formula, some values of the polynomials $\mathcal{K}(l;w,\vec{u},2)$ are given as follows:

$$\mathcal{K}(0; w, \vec{u}, 2) = 1,$$

$$\mathcal{K}(1; w, \vec{u}, 2) = x + u_1 + iy,$$

$$\mathcal{K}(2; w, \vec{u}, 2) = (x + u_1)^2 - y^2 + 2u_2 + 2iy(x + u_1).$$

Substituting $r = 3$ into (34), we obtain the following result:

$$\mathcal{K}(l; w, \vec{u}, 3) = l! \sum_{j_3=0}^{\left[\frac{l}{3}\right]} \sum_{j_2=0}^{\left[\frac{l-3j_3}{2}\right]} \sum_{j_1=0}^{l-3j_3-2j_2} \binom{l-3j_3-2j_2}{j_1} \frac{w^{l-3j_3-2j_2-j_1} {u_1}^{j_1} u_2^{j_2} u_3^{j_3}}{j_3!\, j_2!\, (l-3j_3-2j_2)!}.$$

Theorem 2.4 Let $\vec{u} = (u_1, u_2, \ldots, u_r)$ and $l \in \mathbb{N}_0$. Then we have

$$k_1(l; x, y, \vec{u}, r) = l! \sum_{v=0}^{\left[\frac{l}{2}\right]} \sum_{j_r=0}^{\left[\frac{l-2v}{r}\right]} \sum_{j_{r-1}=0}^{\left[\frac{l-2v-rj_r}{r-1}\right]} \cdots \tag{35}$$

$$\times \sum_{j_2=0}^{\left[\frac{l-2v-rj_r-\cdots-3j_3}{2}\right]} \frac{(u_1+x)^{l-2v-rj_r-\cdots-3j_3-2j_2} u_2^{j_2} u_3^{j_3} \ldots u_r^{j_r} (-1)^v y^{2v}}{j_r!\, j_{r-1}! \cdots j_3!\, j_2!\, (l-2v-rj_r-\cdots-3j_3-2j_2)!\, (2v)!}.$$

Proof. By using (28), we get

$$R_{k1}(t, x, y, \vec{u}, r) = \exp(xt + u_1 t)\exp(u_2 t^2) \ldots \exp(u_r t^r) \cos(yt).$$

From the above equation, we have

$$\sum_{l=0}^{\infty} k_1(l; x, y, \vec{u}, r) \frac{t^l}{l!} = \sum_{l=0}^{\infty} (u_1 + x)^l \frac{t^l}{l!} \sum_{l=0}^{\infty} u_2^l \frac{t^{2l}}{l!} \sum_{l=0}^{\infty} u_3^l \frac{t^{3l}}{l!} \cdots$$

$$\times \sum_{l=0}^{\infty} u_r^l \frac{t^{rl}}{l!} \sum_{l=0}^{\infty} (-1)^l \frac{(yt)^{2l}}{(2l)!}.$$

Thus

$$\sum_{l=0}^{\infty} k_1(l; x, y, \vec{u}, r) \frac{t^l}{l!} = \sum_{l=0}^{\infty} \sum_{v=0}^{\left[\frac{l}{2}\right]} \sum_{j_r=0}^{\left[\frac{l-2v}{r}\right]} \sum_{j_{r-1}=0}^{\left[\frac{l-2v-rj_r}{r-1}\right]} \cdots$$

$$\times \sum_{j_2=0}^{\left[\frac{l-2v-rj_r-\cdots-3j_3}{2}\right]} \frac{(u_1+x)^{l-2v-rj_r-\cdots-3j_3-2j_2} u_2^{j_2} u_3^{j_3} \ldots u_r^{j_r} (-1)^v y^{2v}}{j_r!\, j_{r-1}! \cdots j_3!\, j_2!\, (l-2v-rj_r-\cdots-3j_3-2j_2)!\, (2v)!} t^l.$$

Comparing coefficient of t^l on both sides of the above equation, we arrive the desired result.

Substituting $r = 2$ into (35), we obtain the following result:

$$k_1(l; x, y, \vec{u}, 2) = l! \sum_{v=0}^{\left[\frac{l}{2}\right]} \sum_{j_2=0}^{\left[\frac{l-2v}{2}\right]} \frac{(u_1 + x)^{l-2v-2j_2} u_2^{j_2} (-1)^v y^{2v}}{j_2! \, (l - 2v - 2j_2)! \, (2v)!}.$$

Using the above formula, some values of the polynomials $k_1(l; x, y, \vec{u}, 2)$ are given as follows:

$$k_1(0; x, y, \vec{u}, 2) = 1,$$

$$k_1(1; x, y, \vec{u}, 2) = x + u_1,$$

$$k_1(2; x, y, \vec{u}, 2) = (x + u_1)^2 + 2u_2 - y^2.$$

Substituting $x = 0$ into (35), we have the following theorem:

Theorem 2.5 Let $\vec{u} = (u_1, u_2, \dots, u_r)$ and $l \in \mathbb{N}_0$. Then we have

$$C_l(\vec{u}, y; r) = l! \sum_{v=0}^{\left[\frac{l}{2}\right]} \sum_{j_r=0}^{\left[\frac{l-2v}{r}\right]} \sum_{j_{r-1}=0}^{\left[\frac{l-2v-rj_r}{r-1}\right]} \cdots \tag{36}$$

$$\times \sum_{j_2=0}^{\left[\frac{l-2v-rj_r-\cdots-3j_3}{2}\right]} \frac{u_1{}^{l-2v-rj_r-\cdots-3j_3-2j_2} u_2^{j_2} u_3^{j_3} \dots u_r^{j_r} (-1)^v y^{2v}}{j_r! \, j_{r-1}! \cdots j_3! \, j_2! \, (l - 2v - rj_r - \cdots - 3j_3 - 2j_2)! \, (2v)!}.$$

Remark 2.6 When $r = 1$, (36) is reduced to the equation (11).

Theorem 2.7 Let $\vec{u} = (u_1, u_2, \dots, u_r)$ and $l \in \mathbb{N}$. Then we have

$$k_2(l; x, y, \vec{u}, r) = l! \sum_{v=0}^{\left[\frac{l-1}{2}\right]} \sum_{j_r=0}^{\left[\frac{l-1-2v}{r}\right]} \sum_{j_{r-1}=0}^{\left[\frac{l-1-2v-rj_r}{r-1}\right]} \cdots \tag{37}$$

$$\times \sum_{j_2=0}^{\left[\frac{l-1-2v-rj_r-\cdots-3j_3}{2}\right]} \frac{(u_1 + x)^{l-1-2v-rj_r-\cdots-3j_3-2j_2} u_2^{j_2} u_3^{j_3} \dots u_r^{j_r} (-1)^v y^{2v+1}}{j_r! \, j_{r-1}! \cdots j_3! \, j_2! \, (l - 1 - 2v - rj_r - \cdots - 3j_3 - 2j_2)! \, (2v + 1)!}.$$

Proof. By using (29), we have

$$R_{k2}(t, x, y, \vec{u}, r) = \exp(xt + u_1 t) \exp(u_2 t^2) \dots \exp(u_r t^r) \sin(yt).$$

From the above equation, we obtain

$$\sum_{l=0}^{\infty} k_2(l;x,y,\vec{u},r)\frac{t^l}{l!} = \sum_{l=0}^{\infty}(u_1+x)^l\frac{t^l}{l!}\sum_{l=0}^{\infty}u_2^l\frac{t^{2l}}{l!}\sum_{l=0}^{\infty}u_3^l\frac{t^{3l}}{l!}\cdots$$

$$\times\sum_{l=0}^{\infty}u_r^l\frac{t^{rl}}{l!}\sum_{l=0}^{\infty}(-1)^l\frac{(yt)^{2l+1}}{(2l+1)!}.$$

Thus

$$\sum_{l=0}^{\infty} k_2(l;x,y,\vec{u},r)\frac{t^l}{l!} = \sum_{l=0}^{\infty}\sum_{v=0}^{\left[\frac{l-1}{2}\right]}\sum_{j_r=0}^{\left[\frac{l-1-2v}{r}\right]}\sum_{j_{r-1}=0}^{\left[\frac{l-1-2v-rj_r}{r-1}\right]}\cdots$$

$$\times\sum_{j_2=0}^{\left[\frac{l-1-2v-rj_r-\cdots-3j_3}{2}\right]}\frac{(u_1+x)^{l-1-2v-rj_r-\cdots-3j_3-2j_2}u_2^{j_2}u_3^{j_3}\ldots u_r^{j_r}(-1)^v y^{2v+1}}{j_r!\,j_{r-1}!\cdots j_3!\,j_2!\,(l-1-2v-rj_r-\cdots-3j_3-2j_2)!\,(2v+1)!}t^l.$$

Comparing coefficient of t^l on both sides of the above equation, we arrive the desired result.
Substituting $r=2$ into (37), we obtain the following result:

$$k_2(l;x,y,\vec{u},2) = l!\sum_{v=0}^{\left[\frac{l-1}{2}\right]}\sum_{j_2=0}^{\left[\frac{l-1-2v}{2}\right]}\frac{(u_1+x)^{l-1-2v-2j_2}u_2^{j_2}(-1)^v y^{2v+1}}{j_2!\,(l-1-2v-2j_2)!\,(2v+1)!}.$$

Using the above formula, some values of the polynomials $k_2(l;x,y,\vec{u},2)$ are given as follows:

$$k_2(0;x,y,\vec{u},2) = 0,$$

$$k_2(1;x,y,\vec{u},2) = y,$$

$$k_2(2;x,y,\vec{u},2) = 2y(x+u_1).$$

Here we note that for more values of the polynomials $k_1(l;x,y,\vec{u},r)$, the polynomials $k_2(l;x,y,\vec{u},r)$ and the polynomials $\mathcal{K}(l;w,\vec{u},r)$, see works of Kilar and Simsek [8] and Kilar [9].

Substituting $x=0$ into (37), we have the following theorem:

Theorem 2.8 Let $\vec{u}=(u_1,u_2,\ldots,u_r)$ and $l\in\mathbb{N}$. Then we have

$$S_l(\vec{u},y;r) = l! \sum_{v=0}^{\left[\frac{l-1}{2}\right]} \sum_{j_r=0}^{\left[\frac{l-1-2v}{r}\right]} \sum_{j_{r-1}=0}^{\left[\frac{l-1-2v-rj_r}{r-1}\right]} \cdots \tag{38}$$

$$\times \sum_{j_2=0}^{\left[\frac{l-1-2v-rj_r-\cdots-3j_3}{2}\right]} \frac{{u_1}^{l-1-2v-rj_r-\cdots-3j_3-2j_2} u_2^{j_2} u_3^{j_3} \ldots u_r^{j_r} (-1)^v y^{2v+1}}{j_r!\, j_{r-1}! \cdots j_3!\, j_2!\, (l-1-2v-rj_r-\cdots-3j_3-2j_2)!\, (2v+1)!}.$$

Remark 2.9 When $r = 1$, (38) is reduced to the equation (12).

3. Some applications of Apostol type parametric polynomials

In this section, many formulas for the two parametric type of Apostol-Bernoulli polynomials of higher order, the two parametric type of Apostol-Euler polynomials of higher order, the two parametric type of Apostol-Genocchi polynomials of higher order, the Dickson polynomials of the first and second kinds, and special polynomials are obtained. Further some applications with trigonometric functions are given.

Setting $y = \sqrt{1-x^2}$ in (11) and (12), and combining the final equations with (5) and (7), after some elementary calculations, we have the following corollary:

Corollary 3.1 (see [9]). For $l \in \mathbb{N}_0$ and $|x| \leq 1$, we have

$$D_l(2x,1) = 2C_l\left(x,\sqrt{1-x^2}\right) \tag{39}$$

and, for $l \in \mathbb{N}$ and $|x| < 1$,

$$\mathfrak{D}_{l-1}(2x,1) = \frac{S_l\left(x,\sqrt{1-x^2}\right)}{\sqrt{1-x^2}}. \tag{40}$$

Since

$$D_{2l}(x,\alpha) = \left(D_l(x,\alpha)\right)^2 - 2\alpha^l,$$

(see [16, P. 375]), and using (39), we obtain the following result:

Corollary 3.2 For $l \in \mathbb{N}_0$ and $|x| \leq 1$, we find that

$$C_{2l}\left(x,\sqrt{1-x^2}\right) = 2\left(C_l\left(x,\sqrt{1-x^2}\right)\right)^2 - 1.$$

Combining (39) and (40) with (17) and (18), we arrive at the following results:

Corollary 3.3 (see [9]). Let $|x| \leq 1$. For $l \in \mathbb{N}_0$

$$\mathcal{B}_l^{(C,v)}\left(x, \sqrt{1-x^2}; \lambda\right) = \frac{1}{2}\sum_{k=0}^{l} \binom{l}{k} \mathcal{B}_{l-k}^{(v)}(\lambda) D_k(2x, 1) \tag{41}$$

and, for $l \in \mathbb{N}$

$$\mathcal{B}_l^{(S,v)}\left(x, \sqrt{1-x^2}; \lambda\right) = \sqrt{1-x^2}\sum_{k=1}^{l} \binom{l}{k} \mathcal{B}_{l-k}^{(v)}(\lambda) \mathfrak{D}_{k-1}(2x, 1). \tag{42}$$

Combining (39) and (40) with (21) and (22), we arrive at the following results:

Corollary 3.4 (see [9]). Let $|x| \leq 1$. For $l \in \mathbb{N}_0$

$$\mathcal{E}_l^{(C,v)}\left(x, \sqrt{1-x^2}; \lambda\right) = \frac{1}{2}\sum_{k=0}^{l} \binom{l}{k} \mathcal{E}_{l-k}^{(v)}(\lambda) D_k(2x, 1) \tag{43}$$

and, for $l \in \mathbb{N}$

$$\mathcal{E}_l^{(S,v)}\left(x, \sqrt{1-x^2}; \lambda\right) = \sqrt{1-x^2}\sum_{k=1}^{l} \binom{l}{k} \mathcal{E}_{l-k}^{(v)}(\lambda) \mathfrak{D}_{k-1}(2x, 1). \tag{44}$$

Combining (39) and (40) with (25) and (26), we arrive at the following results:

Corollary 3.5 (see [9]). Let $|x| \leq 1$. For $l \in \mathbb{N}_0$

$$\mathcal{G}_l^{(C,v)}\left(x, \sqrt{1-x^2}; \lambda\right) = \frac{1}{2}\sum_{k=0}^{l} \binom{l}{k} \mathcal{G}_{l-k}^{(v)}(\lambda) D_k(2x, 1) \tag{45}$$

and, for $l \in \mathbb{N}$

$$\mathcal{G}_l^{(S,v)}\left(x, \sqrt{1-x^2}; \lambda\right) = \sqrt{1-x^2}\sum_{k=1}^{l} \binom{l}{k} \mathcal{G}_{l-k}^{(v)}(\lambda) \mathfrak{D}_{k-1}(2x, 1). \tag{46}$$

Substituting $x = \frac{1}{\sqrt{2}}$ into (39) and (40), we have the following presumably known relations for the Dickson polynomials of the first and second kinds and trigonometric functions:

Corollary 3.6 For $l \in \mathbb{N}_0$, we have

$$D_l\left(\frac{2}{\sqrt{2}}, 1\right) = 2\cos\left(\frac{l\pi}{4}\right) \tag{47}$$

and, for $l \in \mathbb{N}$,

$$\mathfrak{D}_{l-1}\left(\frac{2}{\sqrt{2}},1\right)=\sqrt{2}\sin\left(\frac{l\pi}{4}\right). \tag{48}$$

Combining (8) with (47) and (48), after some calculations, we have the following well-known trigonometric identity:

$$\sqrt{2}\sin\left(\frac{(l+1)\pi}{4}\right)=\cos\left(\frac{l\pi}{4}\right)+\sin\left(\frac{l\pi}{4}\right).$$

Substituting $x=\frac{1}{\sqrt{2}}$ into (41), (43) and (45), then combining final equations with (47), we have the following results:

Corollary 3.7 Let $l \in \mathbb{N}_0$. Then we have

$$\mathcal{B}_l^{(C,v)}\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}};\lambda\right)=\sum_{k=0}^{l}\binom{l}{k}\mathcal{B}_{l-k}^{(v)}(\lambda)\cos\left(\frac{k\pi}{4}\right),$$

$$\mathcal{E}_l^{(C,v)}\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}};\lambda\right)=\sum_{k=0}^{l}\binom{l}{k}\mathcal{E}_{l-k}^{(v)}(\lambda)\cos\left(\frac{k\pi}{4}\right)$$

and

$$\mathcal{G}_l^{(C,v)}\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}};\lambda\right)=\sum_{k=0}^{l}\binom{l}{k}\mathcal{G}_{l-k}^{(v)}(\lambda)\cos\left(\frac{k\pi}{4}\right).$$

Substituting $x=\frac{1}{\sqrt{2}}$ into (42), (44) and (46), then combining final equations with (48), we arrive at the following results:

Corollary 3.8 Let $l \in \mathbb{N}$. Then we have

$$\mathcal{B}_l^{(S,v)}\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}};\lambda\right)=\sum_{k=1}^{l}\binom{l}{k}\mathcal{B}_{l-k}^{(v)}(\lambda)\sin\left(\frac{k\pi}{4}\right),$$

$$\mathcal{E}_l^{(S,v)}\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}};\lambda\right)=\sum_{k=1}^{l}\binom{l}{k}\mathcal{E}_{l-k}^{(v)}(\lambda)\sin\left(\frac{k\pi}{4}\right)$$

and

$$\mathcal{G}_l^{(S,v)}\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}};\lambda\right)=\sum_{k=1}^{l}\binom{l}{k}\mathcal{G}_{l-k}^{(v)}(\lambda)\sin\left(\frac{k\pi}{4}\right).$$

Remark 3.9 In [14], Srivastava et al. showed that illustrative examples for the Apostol type parametric polynomials with some other special values.

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