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On Homotopy Theory of Quadratic Modules of Lie Algebras

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Abstract

In this work, we will introduce the homotopy theory of quadratic modules over Lie algebras. We will construct a homotopy connecting one morphism of quadratic modules of Lie algebras to another.

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1. Introduction

Crossed modules of Lie algebras have been introduced by Kassel and Loday in [10], as computational algebraic objects. Algebraic models for homotopy connected 3-types can be thought of as an extended version of the crossed modules which model for 2-types. One of these is the 2-crossed modules introduced by Conduché for groups in [6]. Another model for 3-types defined by Baues is quadratic modules, [4]. In [12], Ulualan and Uslu have adapted this algebraic 3-type model to Lie algebras. Some of related works can be found in [7], [1], [14] and [13].

Akça and Sıdal worked on the homotopy problem for the morphisms of crossed modules of Lie algebras, [3], as well as Brown and Higgins, [5], introduced the concept of homotopy of the morphisms of crossed complexes that are also crossed modules on groups. In this paper, we give the homotopy theory of morphisms between quadratic modules of Lie algebras. Homotopy theory for 2-crossed modules is included in [9], [8] and [2] for group and commutative algebra cases, respectively. We will construct a homotopy relation for the morphism of quadratic modules of Lie algebras, another algebraic 3-type. One sees more in [11].

2. Preliminaries

2.1. Crossed modules of Lie algebras

Let Y and Z be two Lie algebras, a k-bilinear map

$$\begin{array}{ccc} Z\times Y & \longrightarrow & Y \\ (z,y) & \longmapsto & z*y, \end{array}$$

is called a left Lie algebra action of Z on Y if it satisfies the following conditions:

L1) $z*[y,y'] = [z*y,y'] + [y,z*y']$,
L2) $[z,z']*y = z*(z'*y) - z'*(z*y)$

for each $z,z' \in Z$ and each $y,y' \in Y$.

A pre-crossed module over Lie algebras (Y, Z, ∂) is given by a Lie homomorphism $\partial : Y \to Z$, together with a left Lie algebra action of Z on Y such that the condition

XMod$_L$1 $\partial(z*y) = [z,\partial(y)]$ is satisfied for each $z\in Z$ and each $y\in Y$.

A crossed module over Lie algebras (Y, Z, ∂) is a pre-crossed module satisfying, in addition "Peiffer identity" condition:

XMod$_L$2 $\partial(y)*y' = [y,y']$

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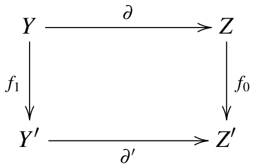
for all $y, y' \in Y$.

Example 2.1. *Let I be any ideal of a Lie algebra Z. We have a crossed module (I, Z, i) where $i : I \longrightarrow Z$ is an inclusion map. Conversely given any crossed module $\partial : I \to Z$, one can easily verify that $\partial(Y) = I$ is an ideal in Z.*

Example 2.2. *Any Z-module Y can be considered as a Lie algebra with zero multiplication, and then $\mathbf{0} : Y \to Z$ is a crossed module.*

A crossed module morphism $f : (Y, Z, \partial) \to (Y', Z', \partial')$ consists of Lie algebra morphisms f_1 and f_0 such that the following diagram is commutative and preserves the action of Z on Y:



2.2. Quadratic modules of Lie algebras

In this section, we recall the definition of quadratic modules over Lie algebras given [12].
Let $\partial : Y \to Z$ be pre-crossed module, $P_1(\partial) = Y$ and $P_2(\partial)$ be the Peiffer Lie ideal of Y generated by the Peiffer elements of the type

$< y_1, y_2 >= \partial(y_1) * y_2 - [y_1, y_2],$

for $y_1, y_2 \in C$. Then the homomorphism

$\partial^{cr} : Y^{cr} = Y/P_2(\partial) \to Z$

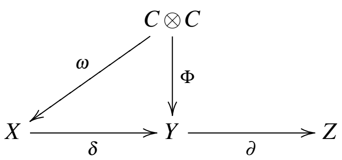
is a crossed module since

$$\begin{aligned} &\partial^{cr}(y_1 + P_2(\partial)) * (y_2 + P_2(\partial)) - [y_1 + P_2(\partial), y_2 + P_2(\partial)] \\ &= \partial^{cr}(y_1 + P_2(\partial)) * (y_2 + P_2(\partial)) - ([y_1, y_2] + P_2(\partial)) \\ &= (\partial(y_1) * y_2 - [y_1, y_2]) + P_2(\partial) \\ &= 0 + P_2(\partial) \quad (\because \quad < y_1, y_2 >\in P_2(\partial)) \end{aligned}$$

for $y_1 + P_2(\partial)$, $y_2 + P_2(\partial) \in Y^{cr}$.
A $nil(2)$-module is a pre-crossed module $\partial : Y \to Z$ with the an additional "nilpotency" condition, $P_3(\partial) = 0$, where $P_3(\partial)$ is the ideal of the Lie algebra C generated by the Peiffer elements $< y_1, y_2, y_3 >$ of length 3.
Similarly, if $\partial : Y \to Z$ s a pre-crossed module, then the homomorphism

$\partial^{nil} : Y^{nil} = Y/P_3(\partial) \to Z$

is a $nil(2)$-module associated with the pre-crossed module ∂. Clearly, a $nil(1)$-module is a crossed module.
A quadratic module $(\omega, \delta, \partial)$ of Lie algebras consists of Lie algebra homomorphisms as illustrated in the below diagram, satisfying following conditions:



QM$_L$1) The homomorphism $\partial : Y \to Z$ is a $nil(2)$-module and $Y \twoheadrightarrow C = Y^{cr}/[Y^{cr}, Y^{cr}]$ is defined by $y \mapsto [y]$ and Φ is defined by $\Phi([y_1] \otimes [y_2]) = \partial(y_1) *_1 y_2 - [y_1, y_2]$ for $y_1, y_2 \in Y$,
QM$_L$2) The composition of δ and ∂ is the zero map and $\delta\omega = \Phi$,
QM$_L$3) X is a Lie Z-algebra, all of the homomorphisms in the diagram are Z-equivariant, and the action of Z on X also holds the following equality

$\partial(y) *_3 x = \omega([\delta(x)] \otimes [y] + [y] \otimes [\delta(x)])$

for $x \in X$ and $y \in Y$,
QM$_L$4) All $x_1, x_2 \in X$;

$\omega([\delta(x_1)] \otimes [\delta(x_2)]) = [x_2, x_1].$

Remark 2.3. *It should be noted that $X \xrightarrow{\delta} Y$ is a crossed module, with the Lie action defined by*

$y *_2 x = \omega([\delta(x)] \otimes [y]),$

for each $y \in Y$ and $x \in X$. On the other hand generally $Y \xrightarrow{\partial} Z$ only a pre-crossed module.

Remark 2.4. *A Lie algebra action $*_2$ of Y on X with the help of* ***QM_L3*** *axiom, we have:*

$\partial(y) *_3 x - y *_2 x = \omega([y] \otimes [\delta(x)])$.

Let $\mathscr{L} = (\omega, \delta, \partial)$ and $\mathscr{L}' = (\omega', \delta', \partial')$ be two quadratic modules of Lie algebras. A morphism of quadratic modules from $\mathscr{L}$ to $\mathscr{L}'$ is illustrated by the following commutative diagram:

$$\begin{array}{ccccccc}
C \otimes C & \xrightarrow{\omega} & X & \xrightarrow{\delta} & Y & \xrightarrow{\partial} & Z \\
\downarrow{\scriptstyle \varphi\otimes\varphi} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\
C' \otimes C' & \xrightarrow[\omega']{} & X' & \xrightarrow[\delta']{} & Y' & \xrightarrow[\partial']{} & Z'
\end{array}$$

where (f_1, f_0) is a morphism of pre-crossed modules which induces $\varphi : C \to C'$ and also following equations are satisfied:

$$\begin{aligned}
f_1(z *_1 y) &= f_0(z) *'_1 f_1(y), \\
f_2(z *_3 x) &= f_0(z) *'_3 f_2(x), \\
f_2(\omega([y_1] \otimes [y_2])) &= \omega'([f_1(y_1)] \otimes [f_1(y_2)]),
\end{aligned}$$

for all $z \in Z$, $y, y_1, y_2 \in Y$ and $x \in X$.

3. Homotopy of Quadratic Modules of Lie Algebras Maps

We now fix quadratic modules of Lie algebra $\mathscr{L} = (\omega, \delta, \partial)$ and $\mathscr{L}' = (\omega', \delta', \partial')$.
Let $f = (f_2, f_1, f_0) : \mathscr{L} \to \mathscr{L}'$ be a quadratic module morphism. A quadratic f-derivation is a pair (s_0, s_1), where $s_0 : Z \to Y'$ and $s_1 : Y \to X'$ are k-linear maps, satisfying:

$$s_0[z_1, z_2] = f_0(z_1) *'_1 s_0(z_2) - f_0(z_2) *'_1 s_0(z_1) + [s_0(z_1), s_0(z_2)] \tag{3.1}$$

(Which means that $s_0 : Z \to Y'$ is an f_0-derivation) and, for all $z, z_1, z_2 \in Z$ and $y, y_1, y_2 \in Y$.

$$\begin{aligned}
s_1[y_1, y_2] \;=\; & -\omega'([f_1(y_1)] \otimes [s_0(\partial(y_2))]) + \omega'([f_1(y_2)] \otimes [s_0(\partial(y_1))]) \\
& + f_1(y_1) *'_2 s_1(y_2) + s_0(\partial(y_1)) *'_2 s_1(y_2) - f_1(y_2) *'_2 s_1(y_1) \\
& - s_0(\partial(y_2)) *'_2 s_1(y_1) + [s_1(y_1), s_1(y_2)],
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
s_1(z *_1 y) \;=\; & f_0(z) *'_3 s_1(y) + \partial'(s_0(z)) *'_3 s_1(y) + \omega'([s_0(z)] \otimes [f_1(y)]) \\
& + \omega'([f_1(y)] \otimes [s_0(z)]) + \omega'([s_0(z)] \otimes [s_0(\partial(y))]).
\end{aligned} \tag{3.3}$$

Lemma 3.1. *If s_0, s_1 is a quadratic f-derivation, then the following equations are satisfied:*

$$\begin{aligned}
s_1[\delta(x_1), \delta(x_2)] \;=\; & [f_2(x_1), s_1(\delta(x_2))] - [f_2(x_2), s_1(\delta(x_2))] \\
& + [s_1(\delta(x_1)), s_1(\delta(x_2))],
\end{aligned}$$

$$s_1(z *_2 \delta(x)) = f_0(z) *'_3 (s_1\delta)(x) + (\partial' s_0)(z) *'_3 (s_1\delta)(x) + (\partial' s_0)(z) *'_3 f_2(x)$$

for all $x, x_1, x_2 \in X$ and $z \in Z$.

Proof.

$$\begin{aligned}
s_1[\delta(x_1), \delta(x_2)] \;=\; & -\omega'([f_1(\delta(x_1))] \otimes [s_0(\partial(\delta(x_2)))]) + \omega'([f_1(\delta(x_2))] \otimes [s_0(\partial(\delta(x_1)))]) \\
& + f_1(\delta(x_1)) *'_2 s_1(\delta(x_2)) + s_0(\partial(\delta(x_1))) *'_2 s_1(\delta(x_2)) \\
& - f_1(\delta(x_2)) *'_2 s_1(\delta(x_1)) - s_0(\partial(\delta(x_2))) *'_2 s_1(\delta(x_1)) \\
& + [s_1(\delta(x_1)), s_1(\delta(x_2))] \\
=\; & -\omega'([(f_1\delta)(x_1)] \otimes [s_0((\partial\delta)(x_2))]) + \omega'([(f_1\delta)(x_2)] \otimes [s_0((\partial\delta)(x_1))]) \\
& + (f_1\delta)(x_1) *'_2 (s_1\delta)(x_2) + s_0((\partial\delta)(x_1)) *'_2 (s_1\delta)(x_2) \\
& - (f_1\delta)(x_2) *'_2 (s_1\delta)(x_1) - s_0((\partial\delta)(x_2)) *'_2 (s_1\delta)(x_1) \\
& + [s_1(\delta(x_1)), s_1(\delta(x_2))] \\
=\; & -\omega'([(\delta' f_2)(x_1)] \otimes [s_0(0_Z)]) + \omega'([(\delta' f_2)(x_2)] \otimes [s_0(0_Z)]) \\
& + (\delta' f_2)(x_1) *'_2 (s_1\delta)(x_2) + s_0(0_Z) *'_2 (s_1\delta)(x_2) \\
& - (\delta' f_2)(x_2) *'_2 (s_1\delta)(x_1) - s_0(0_Z) *'_2 (s_1\delta)(x_1) \\
& + [s_1(\delta(x_1)), s_1(\delta(x_2))] \\
=\; & (\delta' f_2)(x_1) *'_2 (s_1\delta)(x_2) + s_0(0_Z) *'_2 (s_1\delta)(x_2) \\
& - (\delta' f_2)(x_2) *'_2 (s_1\delta)(x_1) - s_0(0_Z) *'_2 (s_1\delta)(x_1) \\
& + [s_1(\delta(x_1)), s_1(\delta(x_2))] \\
=\; & \big(((\delta' f_2)(x_1) + 0_{Y'}) *'_2 (s_1\delta)(x_2)\big) - \big(((\delta' f_2)(x_2) + 0_{Y'}) *'_2 (s_1\delta)(x_1)\big) \\
& + [s_1(\delta(x_1)), s_1(\delta(x_2))] \\
=\; & \omega'([\delta'(s_1\delta)(x_2)] \otimes [(\delta' f_2)(x_1)]) - \omega'([\delta'(s_1\delta)(x_1)] \otimes [(\delta' f_2)(x_2)]) \\
& + [s_1(\delta(x_1)), s_1(\delta(x_2))] \\
=\; & [f_2(x_1), (s_1\delta)(x_2)] - [f_2(x_2), (s_1\delta)(x_2)] + [s_1(\delta(x_1)), s_1(\delta(x_2))]
\end{aligned}$$

for all $x_1, x_2 \in X$. Also:

$$\begin{aligned}
s_1(z *_3 \delta(x)) &= f_0(z) *'_3 s_1(\delta(x)) + \partial'(s_0(z)) *'_3 s_1(\delta(x)) + \omega'([s_0(z)] \otimes [f_1(\delta(x))]) \\
&\quad + \omega'([f_1(\delta(x))] \otimes [s_0(z)]) + \omega'([s_0(z)] \otimes [s_0(\partial(\delta(x)))]) \\
&= f_0(z) *'_3 (s_1\delta)(x) + (\partial' s_0)(z) *'_3 (s_1\delta)(x) + \omega'([s_0(z)] \otimes [(f_1\delta)(x)]) \\
&\quad + \omega'([(f_1\delta)(x)] \otimes [s_0(z)]) + \omega'([s_0(z)] \otimes [s_0((\partial\delta)(x))]) \\
&= f_0(z) *'_3 (s_1\delta)(x) + (\partial' s_0)(z) *'_3 (s_1\delta)(x) + \omega'([s_0(z)] \otimes [(f_1\delta)(x)]) \\
&\quad + \omega'([(f_1\delta)(x)] \otimes [s_0(z)]) + \omega'([s_0(z)] \otimes [s_0((\partial\delta)(x))]) \\
&= f_0(z) *'_3 (s_1\delta)(x) + (\partial' s_0)(z) *'_3 (s_1\delta)(x) + \omega'([s_0(z)] \otimes [(\delta' f_2)(x)]) \\
&\quad + \omega'([(\delta' f_2)(x)] \otimes [s_0(z)]) + \omega'([s_0(z)] \otimes [s_0(0_z)]) \\
&= f_0(z) *'_3 (s_1\delta)(x) + (\partial' s_0)(z) *'_3 (s_1\delta)(x) + (\partial' s_0)(z) *'_3 f_2(x)
\end{aligned}$$

for all $z \in Z$ and $x \in X$. □

Theorem 3.2. *(Pointed homotopy of quadratic module morphisms) $f = (f_2, f_1, f_0)$ be a quadratic module morphism $\mathscr{L} \to \mathscr{L}'$. In the condition of previous definition, if (s_0, s_1) is a quadratic f-derivation, and if we define $g = (g_2, g_1, g_0)$ as:*

$$\begin{aligned}
g_0(z) &= f_0(z) + (\partial' \circ s_0)(z) \\
g_1(y) &= f_1(y) + (s_0 \circ \partial)(y) + (\delta' \circ s_1)(y) \\
g_2(x) &= f_2(x) + (s_1 \circ \delta)(x)
\end{aligned} \tag{3.4}$$

where $z \in Z$, $y \in Y$, and $x \in X$, then g also defines a quadratic module morphism $g : \mathscr{L} \to \mathscr{L}'$. In such case, $((f, s_0, s_1)$ is called by a homotopy (or quadratic derivation), connecting f to g and it is denoted by

$$f \xrightarrow{(f,s_0,s_1)} g.$$

Proof. Primarily we show that g_0, g_1 and g_2 Lie algera morphisms. For this purpose, we need to show that $g_0(z_1 + z_2) = g_0(z_1) + g_0(z_2)$ and $g_0(kz) = kg_0(z)$, follows form k-linearity, and similarly for g_1 and g_2. Additionally:

$$\begin{aligned}
g_0[z_1, z_2] &= f_0([z_1, z_2]) + (\partial' s_0)([z_1, z_2])) \\
&= [f_0(z_1), f_0(z_2)] + \partial'(s_0[z_1, z_2]) \\
&= [f_0(z_1), f_0(z_2)] \\
&\quad + \partial'\left(f_0(z_1) *'_1 s_0(z_2) - f_0(z_2) *'_1 s_0(z_1) + [s_0(z_1), s_0(z_2)]\right) \\
&= [f_0(z_1), f_0(z_2)] + \partial'\left(f_0(z_1) *'_1 s_0(z_2)\right) \\
&\quad - \partial'\left(f_0(z_2) *'_1 s_0(z_1)\right) + \partial'([s_0(z_1), s_0(z_2)]) \\
&= [f_0(z_1), f_0(z_2)] + [f_0(z_1), (\partial' s_0)(z_2)] - [f_0(z_2), (\partial' s_0)(z_1)] \\
&\quad + [(\partial' s_0)(z_1), (\partial' s_0)(z_2)] \\
&= [f_0(z_1), f_0(z_2)] + [f_0(z_1), (\partial' s_0)(z_2)] + [(\partial' s_0)(z_1), f_0(z_2)] \\
&\quad + [(\partial' s_0)(z_1), (\partial' s_0)(z_2)] \\
&= [f_0(z_1) + (\partial' s_0)(z_1)], [f_0(z_2) + (\partial' s_0)(z_2)] \\
&= [g_0(z_1), g_0(z_2)]
\end{aligned}$$

for all $z_1, z_2 \in Z$ and $k \in k$, which means g_0 is a Lie algebra morphisims. Similarly:

$$\begin{aligned}
g_1[y_1, y_2] &= f_1([y_1, y_2]) + (s_0\partial)([y_1, y_2]) + (\delta' s_1)([y_1, y_2]) \\
&= [f_1(y_1), f_1(y_2)] + s_0[\partial(y_1), \partial(y_2)] + \delta'(s_1[y_1, y_2]) \\
&= [f_1(y_1), f_1(y_2)] + (f_0\partial)(y_1) *'_1 (s_0\partial)(y_2) - (f_0\partial)(y_2) *'_1 (s_0\partial)(y_1) \\
&\quad + [(s_0\partial)(y_1), (s_0\partial)(y_2)] + \delta'(-\omega'([f_1(y_1)] \otimes [(s_0\partial)(y_2)]) \\
&\quad + \omega'([f_1(y_2)] \otimes [(s_0\partial)(y_1)]) + f_1(y_1) *'_2 s_1(y_2) + (s_0\partial)(y_1) *'_2 s_1(y_2) \\
&\quad - f_1(y_2) *'_2 s_1(y_1) - (s_0\partial)(y_2) *'_2 s_1(y_1) + [s_1(y_1), s_1(y_2)]) \\
&= [f_1(y_1), f_1(y_2)] + (\partial' f_1)(y_1) *'_1 (s_0\partial)(y_2) - (\partial' f_1)(y_2) *'_1 (s_0\partial)(y_1) \\
&\quad + [(s_0\partial)(y_1), (s_0\partial)(y_2)] - \delta'\omega'([f_1(y_1)] \otimes [(s_0\partial)(y_2)]) \\
&\quad + \delta'\omega'([f_1(y_2)] \otimes [(s_0\partial)(y_1)]) + \delta'\left(f_1(y_1) *'_2 s_1(y_2)\right) \\
&\quad + \delta'\left((s_0\partial)(y_1) *'_2 s_1(y_2)\right) - \delta'\left(f_1(y_2) *'_2 s_1(y_1)\right) \\
&\quad - \delta'\left((s_0\partial)(y_2) *'_2 s_1(y_1)\right) + \delta'([s_1(y_1), s_1(y_2)]) \\
&= [f_1(y_1), f_1(y_2)] + \partial'(f_1(y_1)) *'_1 (s_0\partial)(y_2) - \partial'(f_1(y_2)) *'_1 (s_0\partial)(y_1) \\
&\quad + [(s_0\partial)(y_1), (s_0\partial)(y_2)] - \partial'(f_1(y_1)) *'_1 (s_0\partial)(y_2) \\
&\quad + [f_1(y_1), (s_0\partial)(y_2)] + \partial'(f_1(y_2)) *'_1 (s_0\partial)(y_1) \\
&\quad - [f_1(y_2), (s_0\partial)(y_1)] + [(f_1(y_1)), (\delta' s_1)(y_2)] \\
&\quad + [(s_0\partial)(y_1), (\delta' s_1)(y_2)] - [f_1(y_2), (\delta' s_1)(y_1)] \\
&\quad - [(s_0\partial)(y_2), (\delta' s_1)(y_1)] + [(\delta' s_1)(y_1), (\delta' s_1)(y_2)] \\
&= [f_1(y_1), f_1(y_2)] + [(s_0\partial)(y_1), (s_0\partial)(y_2)] + [f_1(y_1), (s_0\partial)(y_2)] \\
&\quad + [(s_0\partial)(y_1), f_1(y_2)] + [(f_1(y_1)), (\delta' s_1)(y_2)] \\
&\quad + [(s_0\partial)(y_1), (\delta' s_1)(y_2)] + [(\delta' s_1)(y_1), f_1(y_2)] \\
&\quad + [(\delta' s_1)(y_1), f_1(y_2)] + [(\delta' s_1)(y_1), (\delta' s_1)(y_2)] \\
&= [f_1(y_1), f_1(y_2)] + [f_1(y_1), (s_0\partial)(y_2)] + [(f_1(y_1)), (\delta' s_1)(y_2)] \\
&\quad + [(s_0\partial)(y_1), f_1(y_2)] + [(s_0\partial)(y_1), (s_0\partial)(y_2)] \\
&\quad + [(s_0\partial)(y_1), (\delta' s_1)(y_2)] + [(\delta' s_1)(y_1), f_1(y_2)] \\
&\quad + [(\delta' s_1)(y_1), f_1(y_2)] + [(\delta' s_1)(y_1), (\delta' s_1)(y_2)] \\
&= [f_1(y_1) + (s_0\partial)(y_1) + (\delta' s_1)(y_1), f_1(y_2) + (s_0\partial)(y_2) + (\delta' s_1)(y_2)] \\
&= [g_1(y_1), g_1(y_1)]
\end{aligned}$$

for all $y_1, y_2 \in Y$, thus g_1 is a Lie algebra morphism. Also, we have:

$$\begin{array}{rcl}
g_2[x_1,x_2] & = & f_2([x_1,x_2]) + (s_1\delta)([x_1,x_2]) \\
& = & [f_2(x_1), f_2(x_2)] + s_1(\delta[x_1,x_2]) \\
& = & [f_2(x_1), f_2(x_2)] + s_1[\delta(x_1), \delta(x_2)] \\
& = & [f_2(x_1), f_2(x_2)] + [f_2(x_1), s_1(\delta(x_2))] \\
& & -[f_2(x_2), s_1(\delta(x_2))] + [s_1(\delta(x_1)), s_1(\delta(x_2))] \\
& = & [f_2(x_1), f_2(x_2)] + [f_2(x_1), (s_1\delta)(x_2)] \\
& & +[(s_1\delta)(x_2), f_2(x_2)] + [(s_1\delta)(x_1), (s_1\delta)(x_2)] \\
& = & [f_2(x_1) + (s_1\delta)(x_1), f_2(x_2) + (s_1\delta)(x_2)] \\
& = & [g_2(x_1), g_2(x_2)]
\end{array}$$

for all $x_1, x_2 \in X$, thus g_2 is a Lie algebra morphism. Also:

$$\begin{array}{rcl}
g_0(\partial(y)) & = & f_0(\partial(y)) + (\partial' s_0)(\partial(y)) \\
& = & f_0(\partial(y)) + \partial'((s_0\partial)(y)) + 0_Z \\
& = & (\partial' f_1)(y) + \partial'((s_0\partial)(y)) + \partial'\delta'(s_1(y)) \\
& = & \partial'(f_1(y) + (s_0\partial)(y) + (\delta' s_1)(y)) \\
& = & \partial'(g_1(y))
\end{array}$$

for all $y \in Y$, and

$$\begin{array}{rcl}
g_1(\delta(x)) & = & f_1(\delta(x)) + (s_0\partial)(\delta(x)) + (\delta' s_1)(\delta(x)) \\
& = & (f_1\delta)(x) + s_0((\partial\delta)(x)) + \delta'((s_1\delta)(x)) \\
& = & (\delta' f_2)(x) + s_0(0_Z) + \delta'((s_1\delta)(x)) \\
& = & (\delta' f_2)(x) + 0_{Y'} + \delta'((s_1\delta)(x)) \\
& = & \delta'(f_2(x) + (s_1\delta)(x)) \\
& = & \delta'(g_2(x))
\end{array}$$

for all $x \in X$. Since the diagram below commutes:

$$\begin{array}{ccccccc}
C\otimes C & \xrightarrow{\omega} & X & \xrightarrow{\delta} & Y & \xrightarrow{\partial} & Z \\
\downarrow{\scriptstyle \varphi\otimes\varphi} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} \\
C'\otimes C' & \xrightarrow[\omega']{} & X' & \xrightarrow[\delta']{} & Y' & \xrightarrow[\partial']{} & Z'
\end{array}$$

Finally, we should show that these morphisms satisfy Lie algebra action and the ω,ω' quadratic liftings:

$$\begin{array}{rcl}
g_1(z *_1 y) & = & f_1(z *_1 y) + (s_0\partial)(z *_1 y) + (\delta' s_1)(z *_1 y) \\
& = & f_0(z) *'_1 f_1(y) + s_0(\partial(z *_1 y)) + \delta'(s_1(z *_1 y)) \\
& = & f_0(z) *'_1 f_1(y) + s_0[z, \partial(y)] + \delta'(s_1(z *_1 y)) \\
& = & f_0(z) *'_1 f_1(y) + f_0(z) *'_1 (s_0\partial)(y) - (f_0\partial)(y) *'_1 s_0(z) \\
& & +[s_0(z), (s_0\partial)(y)] + \delta'((f_0(z) *'_3 s_1(y) \\
& & +(\partial' s_0)(z) *'_3 s_1(y) + \omega'([s_0(z)] \otimes [f_1(y)]) + \omega'([f_1(y)] \otimes [s_0(z)]) \\
& & +\omega'([s_0(z)] \otimes [(s_0\partial)(y)])) \\
& = & f_0(z) *'_1 f_1(y) + f_0(z) *'_1 (s_0\partial)(y) - \partial'(f_1(y)) *'_1 s_0(z) \\
& & +[s_0(z), (s_0\partial)(y)] + \delta'\left(f_0(z) *'_3 s_1(y)\right) \\
& & +\delta'\left((\partial' s_0)(z) *'_3 s_1(y)\right) + \delta'\omega'([s_0(z)] \otimes [f_1(y)]) \\
& & +\delta'\omega'([f_1(y)] \otimes [s_0(z)]) + \delta'\omega'([s_0(z)] \otimes [(s_0\partial)(y)]) \\
& = & f_0(z) *'_1 f_1(y) + f_0(z) *'_1 (s_0\partial)(y) - \partial'(f_1(y)) *'_1 s_0(z) \\
& & +[s_0(z), (s_0\partial)(y)] + f_0(z) *'_1 (\delta' s_1)(y) \\
& & +(\partial' s_0)(z) *'_1 (\delta' s_1)(y) + (\partial' s_0)(z) *'_1 f_1(y) \\
& & -[s_0(z), f_1(y)] + (\partial' f_1)(y) *'_1 s_0(z) - [f_1(y), s_0(z)] \\
& & +(\partial' s_0)(z) *'_1 (s_0\partial)(y) - [s_0(z), (s_0\partial)(y)] \\
& = & f_0(z) *'_1 f_1(y) + f_0(z) *'_1 (s_0\partial)(y) + f_0(z) *'_1 (\delta' s_1)(y) \\
& & +(\partial' s_0)(z) *'_1 f_1(y) + (\partial' s_0)(z) *'_1 (s_0\partial)(y) \\
& & +(\partial' s_0)(z) *'_1 (\delta' s_1)(y) \\
& = & (f_0(z) + (\partial' s_0)(z)) *'_1 (f_1(y) + (s_0\partial)(y) + (\delta' s_1)(y)) \\
& = & g_0(z) *'_1 g_1(y)
\end{array}$$

and we have:

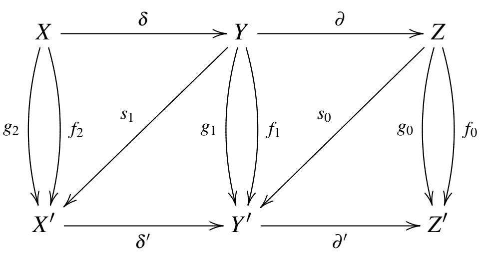
$$\begin{array}{rcl}
g_2(z *_3 x) & = & f_2(z *_3 x) + (s_1\delta)(z *_3 x) \\
& = & f_0(z) *'_3 f_2(x) + s_1(\delta(z *_3 x)) \\
& = & f_0(z) *'_3 f_2(x) + s_1(z *_2 \delta(x)) \\
& = & f_0(z) *'_3 f_2(x) + f_0(z) *'_3 (s_1\delta)(x) + (\partial' s_0)(z) *'_3 (s_1\delta)(x) \\
& & +(\partial' s_0)(z) *'_3 f_2(x) \\
& = & f_0(z) *'_3 f_2(x) + f_0(z) *'_3 (s_1\delta)(x) + (\partial' s_0)(z) *'_3 f_2(x) \\
& & +(\partial' s_0)(z) *'_3 (s_1\delta)(x) \\
& = & (f_0(z) + (\partial' s_0)(z)) *'_3 (f_2(x) + (s_1\delta)(x)) \\
& = & g_0(z) *'_3 g_2(x)
\end{array}$$

We also get

$$
\begin{aligned}
g_2(\omega([y_1]\otimes[y_2])) &= f_2\left(\omega\left([y_1]\otimes[y_2]\right)\right)+\left(s_1\delta\right)\left(\omega\left([y_1]\otimes[y_2]\right)\right)\\
&= \omega'\left(\left[f_1\left(y_1\right)\right]\otimes\left[f_1\left(y_2\right)\right]\right)+s_1\left(\delta\left(\omega\left([y_1]\otimes[y_2]\right)\right)\right)\\
&= \omega'\left(\left[f_1\left(y_1\right)\right]\otimes\left[f_1\left(y_2\right)\right]\right)+s_1\left(\partial\left(y_1\right)*_1 y_2-[y_1,y_2]\right)\\
&= \omega'\left(\left[f_1\left(y_1\right)\right]\otimes\left[f_1\left(y_2\right)\right]\right)+s_1\left(\partial\left(y_1\right)*_1 y_2\right)-s_1\left([y_1,y_2]\right)\\
&= \omega'\left(\left[f_1\left(y_1\right)\right]\otimes\left[f_1\left(y_2\right)\right]\right)+\left(f_0\partial\right)\left(y_1\right)*_3' s_1\left(y_2\right)+\partial'\left(\left(s_0\partial\right)\left(y_1\right)\right)*_3' s_1\left(y_2\right)\\
&\quad+\omega'\left(\left[\left(s_0\partial\right)\left(y_1\right)\right]\otimes\left[f_1\left(y_2\right)\right]\right)+\omega'\left(\left[\left(s_0\partial\right)\left(y_1\right)\right]\otimes\left[\left(s_0\partial\right)\left(y_2\right)\right]\right)\\
&\quad+\omega'\left(\left[f_1\left(y_1\right)\right]\otimes\left[\left(s_0\partial\right)\left(y_2\right)\right]\right)-\omega'\left(\left[f_1\left(y_2\right)\right]\otimes\left[\left(s_0\partial\right)\left(y_1\right)\right]\right)\\
&\quad-f_1\left(y_1\right)*_2' s_1\left(y_2\right)-\left(s_0\partial\right)\left(y_1\right)*_2' s_1\left(y_2\right)+f_1\left(y_2\right)*_2' s_1\left(y_1\right)\\
&\quad+\left(s_0\partial\right)\left(y_2\right)*_2' s_1\left(y_1\right)-\left[s_1\left(y_1\right),s_1\left(y_2\right)\right]\\
&= \omega'\left(\left[f_1\left(y_1\right)\right]\otimes\left[f_1\left(y_2\right)\right]\right)+\partial'\left(f_1\left(y_1\right)\right)*_3' s_1\left(y_2\right)+\partial'\left(\left(s_0\partial\right)\left(y_1\right)\right)*_3' s_1\left(y_2\right)\\
&\quad+\omega'\left(\left[\left(s_0\partial\right)\left(y_1\right)\right]\otimes\left[f_1\left(y_2\right)\right]\right)+\omega'\left(\left[\left(s_0\partial\right)\left(y_1\right)\right]\otimes\left[\left(s_0\partial\right)\left(y_2\right)\right]\right)\\
&\quad+\omega'\left(\left[f_1\left(y_1\right)\right]\otimes\left[\left(s_0\partial\right)\left(y_2\right)\right]\right)-\omega'\left(\left[f_1\left(y_2\right)\right]\otimes\left[\left(s_0\partial\right)\left(y_1\right)\right]\right)\\
&\quad-f_1\left(y_1\right)*_2' s_1\left(y_2\right)-\left(s_0\partial\right)\left(y_1\right)*_2' s_1\left(y_2\right)+f_1\left(y_2\right)*_2' s_1\left(y_1\right)\\
&\quad+\left(s_0\partial\right)\left(y_2\right)*_2' s_1\left(y_1\right)-\left[s_1\left(y_1\right),s_1\left(y_2\right)\right]\\
&= \omega'\left(\left[f_1\left(y_1\right)\right]\otimes\left[f_1\left(y_2\right)\right]\right)+\omega'\left(\left[\left(\delta s_1\right)\left(y_2\right)\right]\otimes\left[f_1\left(y_1\right)\right]+\left[\left(\delta s_1\right)\left(y_2\right)\right]\otimes\left[f_1\left(y_1\right)\right]\right)\\
&\quad+\omega'\left(\left[\left(\delta s_1\right)\left(y_2\right)\right]\otimes\left[\left(s_0\partial\right)\left(y_1\right)\right]+\left[\left(s_0\partial\right)\left(y_1\right)\right]\otimes\left[\left(\delta s_1\right)\left(y_2\right)\right]\right)\\
&\quad+\omega'\left(\left[\left(s_0\partial\right)\left(y_1\right)\right]\otimes\left[f_1\left(y_2\right)\right]\right)+\omega'\left(\left[f_1\left(y_2\right)\right]\otimes\left[\left(s_0\partial\right)\left(y_1\right)\right]\right)\\
&\quad+\omega'\left(\left[\left(s_0\partial\right)\left(y_1\right)\right]\otimes\left[\left(s_0\partial\right)\left(y_2\right)\right]\right)+\omega'\left(\left[f_1\left(y_1\right)\right]\otimes\left[\left(s_0\partial\right)\left(y_2\right)\right]\right)\\
&\quad-\omega'\left(\left[f_1\left(y_2\right)\right]\otimes\left[\left(s_0\partial\right)\left(y_1\right)\right]\right)-\omega'\left(\left[\left(\delta s_1\right)\left(y_2\right)\right]\otimes\left[f_1\left(y_1\right)\right]\right)\\
&\quad-\omega'\left(\left[\left(\delta s_1\right)\left(y_2\right)\right]\otimes\left[\left(s_0\partial\right)\left(y_1\right)\right]\right)+\omega'\left(\left[\left(\delta s_1\right)\left(y_1\right)\right]\otimes\left[f_1\left(y_2\right)\right]\right)\\
&\quad+\omega'\left(\left[\left(\delta s_1\right)\left(y_1\right)\right]\otimes\left[\left(s_{0\partial}\right)\left(y_2\right)\right]\right)+\omega'\left(\left[\left(\delta s_1\right)\left(y_1\right)\right]\otimes\left[\left(\delta s_1\right)\left(y_2\right)\right]\right)\\
&= \omega'(\left[f_1\left(y_1\right)\right]\otimes\left[f_1\left(y_2\right)\right]+\left[f_1\left(y_1\right)\right]\otimes\left[\left(\delta s_1\right)\left(y_2\right)\right]\\
&\quad+\left[\left(s_0\partial\right)\left(y_1\right)\right]\otimes\left[\left(\delta s_1\right)\left(y_2\right)\right]+\left[\left(s_0\partial\right)\left(y_1\right)\right]\otimes\left[f_1\left(y_2\right)\right]\\
&\quad+\left[\left(s_0\partial\right)\left(y_1\right)\right]\otimes\left[\left(s_0\partial\right)\left(y_2\right)\right]+\left[f_1\left(y_1\right)\right]\otimes\left[\left(s_0\partial\right)\left(y_2\right)\right]\\
&\quad+\left[\left(\delta s_1\right)\left(y_1\right)\right]\otimes\left[f_1\left(y_2\right)\right]+\left[\left(\delta s_1\right)\left(y_1\right)\right]\otimes\left[\left(s_0\partial\right)\left(y_2\right)\right]+\left[\left(\delta s_1\right)\left(y_1\right)\right]\otimes\left[\left(\delta s_1\right)\left(y_2\right)\right])\\
&= \omega'(\left[f_1\left(y_1\right)\right]\otimes\left[f_1\left(y_2\right)\right]+\left[f_1\left(y_1\right)\right]\otimes\left[\left(s_0\partial\right)\left(y_2\right)\right]+\left[f_1\left(y_1\right)\right]\otimes\left[\left(\delta s_1\right)\left(y_2\right)\right]\\
&\quad+\left[\left(s_0\partial\right)\left(y_1\right)\right]\otimes\left[f_1\left(y_2\right)\right]+\left[\left(s_0\partial\right)\left(y_1\right)\right]\otimes\left[\left(s_0\partial\right)\left(y_2\right)\right]+\left[\left(s_0\partial\right)\left(y_1\right)\right]\otimes\left[\left(\delta s_1\right)\left(y_2\right)\right]\\
&\quad+\left[\left(\delta s_1\right)\left(y_1\right)\right]\otimes\left[f_1\left(y_2\right)\right]+\left[\left(\delta s_1\right)\left(y_1\right)\right]\otimes\left[\left(s_0\partial\right)\left(y_2\right)\right]+\left[\left(\delta s_1\right)\left(y_1\right)\right]\otimes\left[\left(\delta s_1\right)\left(y_2\right)\right])\\
&= \omega'(\left[f_1\left(y_1\right)+\left(s_0\partial\right)\left(y_1\right)+\left(\delta s_1\right)\left(y_1\right)\right]\otimes\left[f_1\left(y_2\right)+\left(s_0\partial\right)\left(y_2\right)+\left(\delta s_1\right)\left(y_2\right)\right])\\
&= \omega'\left(\left[g_1\left(y_1\right)\right]\otimes\left[g_1\left(y_2\right)\right]\right)
\end{aligned}
$$

for all $z\in Z$, $y_1,y_2\in Y$ and $x\in X$ which completes the proof. □

All these data are summarized diagrammatically as follows:



4. Conclusion

In this paper, the homotopy of morphisms of quadratic modules of Lie algebras has been characterized, and it is concluded that the homotopy notion can also be extended to this context, which is two-dimensional analogous to crossed modules.

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