Yuzuncu Yil University
Journal of the Institute of Natural & Applied Sciences
https://dergipark.org.tr/en/pub/yyufbed

Research Article

Numerical Solutions of the Singularly Perturbed Semilinear Delay Differential Equations

Hakkı DURU[1], Bahar GÜRBÜZ [*2]

[1]Van Yuzuncu Yil University, Faculty of Science, Department of Mathematics 65080, Van, Turkey
[2] Van Yuzuncu Yil University, Institute of Science, Mathematics Department 65080, Van, Turkey
Hakkı DURU, ORCID No:0000-0002-3179-3758, Bahar GÜRBÜZ, ORCID No: 0000-0002-2851-173X
*Corresponding author e-mail: grbzbaharr@gmail.com

Article Info

Received: 10.03.2022
Accepted: 30.05.2022
Online August 2022
DOI: 10.53433/yyufbed.1085501

Keywords
Bakhvalov mesh,
Difference scheme,
Singular perturbation,
Uniform convergence

Abstract: In this study, the numerical solution of the singularly perturbed semilinear differential equations with constant delay is investigated by the method of integral identities with use of linear basis functions and interpolating quadrature formulas. The finite difference scheme is established on Boglaev-Bakhvalov type mesh. The error approximations are obtained in the discrete maximum norm. A numerical example is solved to clarify the theoretical analysis.

Singüler Pertürbe Özellikli Yarılineer Gecikmeli Diferansiyel Denklemlerin Nümerik Çözümleri

Makale Bilgileri

Geliş: 10.03.2022
Kabul: 30.05.2022
Online Ağustos 2022
DOI: 10.53433/yyufbed.1085501

Anahtar Kelimeler
Bakhvalov şebeke,
Düzgün yakınsaklık,
Fark şeması,
Singüler pertürbasyon

Öz: Bu çalışmada sabit gecikme içeren singüler pertürbe özellikli yarılineer diferansiyel denklemlerin nümerik çözümleri araştırılmıştır. Lineer baz fonksiyonları ve interpolasyon kuadratür kurallarını kullanarak Boglaev-Bakhvalov tipli şebeke üzerinde sonlu fark şeması kurulmuştur. Ayrık maksimum normda hata yaklaşımları elde edilmiştir. Teorik analizi doğrulamak için bir nümerik örnek çözülmüştür.

1. Introduction

Singular perturbation problems occur in applied mathematics and different branches of science. Their modellings are found in fluid flow, electrical networks, chemical reactions, control theory and liquid material process (Miller et al., 1996; Ross et al., 1996; Farell et al., 2000).

In this paper, we consider the following singularly perturbed delay differential equation

$$\varepsilon u''(t) + a(t)u'(t) + f\big(t, u(t), u(t-r)\big) = 0, \quad t \in I = (0, T], \tag{1}$$

$$u(t) = \psi(t), \;\; t \in I_0 = (-r, 0], \tag{2}$$

$$u'(0) = \frac{A}{\varepsilon}. \tag{3}$$

Here, $0 < \varepsilon \leq 1$ is a perturbation parameter, r is a delay parameter, $a(t) \geq \alpha > 0$, $f(t)$ and $\psi(t)$ are sufficiently smooth functions. Also, $I = (0, T] = \cup_{p=1}^{m} \; I_p$, $I_p = \{t: r_{p-1} < t \leq r_p\}$, $1 \leq p \leq m$, $r_s = sr$, $0 \leq s \leq m$, $I_0 = (-r, 0]$ and

$$\left|\frac{\partial f}{\partial u}\right| \leq b^* \text{ ve } \left|\frac{\partial f}{\partial v}\right| \leq c^*. \tag{4}$$

In the literature, singularly perturbed problems have been investigated for many years. In 2003, Amiraliyev and Duru proposed an exponentially fitted difference scheme on uniform mesh for periodic type problem. Cakir & Amiraliyev (2005) considered singularly perturbed boundary value problem with nonlocal boundary condition. First-order accurate difference scheme have been investigated on Shishkin mesh for parametrized problem by (Amiraliyev & Duru, 2005). In 2010, Amiraliyev & Cimen (2010) constructed second-order finite difference scheme for singularly perturbed delay convection-diffusion problem. Except for these studies, many different finite difference schemes have been suggested in the last few years. Nonlinear type singularly perturbed reaction-diffusion problems have been discretized in (Duru & Güneş, 2019; Duru & Gunes, 2020). For semilinear singularly perturbed delay differential equations on piecewise-uniform mesh, a finite difference approach have been used (Erdogan et al., 2020). Also, it can refer to in a series of papers (Amiraliyeva et al., 2010; Erdogan & Amiraliyev, 2012; Kumar, 2014; Gunes et al., 2020; Zheng & Ye, 2020).

The numerical analysis of singular perturbation problems has always been far from trivial because of the layer behavior of the solution. These problems involve the boundary layers in which the solution changes rapidly as $\varepsilon \to 0$. Thus, the classical numerical schemes do not produce stable results. For more details about singularly perturbed differential equations, it can be seen in (Doolan et al., 1980; Miller et al., 1996; Roos et al., 1996; Farrell et al., 2000).

The main purpose of this paper is to present reliable numerical method for solving singularly perturbed semilinear initial-value problems including delay argument.

The outline of this work is as follows: The asymptotic estimations of the solution of the problem (1)-(3) are considered in Section 2. In Section 3, using the linear basis functions, a finite difference scheme is constructed. The error bounds are analyzed in Section 4. In Section 5, the theory is tested on a numerical example. In Section 6, paper ends with "Discussion and Conclusion".

2. Analysis of the Continuous Problem

In this section, we give the some analytical properties of the problem (1)-(3). Furthermore, generation of the numerical method is presented and error analysis is discussed.

2.1. Asymptotic estimates

We can rewrite the equation (1) as follows

$$\varepsilon u''(t) + a(t)u' + b(t)u(t) + c(t)u(t-r) = -f(t, 0, 0), \quad t \in I. \tag{5}$$

where

$$b(t) = \frac{\partial f}{\partial u}(t, \tilde{u}, \tilde{v}), \quad c(t) = \frac{\partial f}{\partial v}(t, \tilde{u}, \tilde{v}),$$

$$f(t, u, v) = f(t, 0,0) + \frac{\partial f(t, \tilde{u}, \tilde{v})}{\partial u} u + \frac{\partial f(t, \tilde{u}, \tilde{v})}{\partial v} v,$$

$$\tilde{u} = \gamma u, \tilde{v} = \gamma u(t-r), (0 < \gamma < 1).$$

Therefore, it can be written that

$$Lu := \varepsilon u''(t) + a(t)u'(t) = F(t), \tag{6}$$

where

$$F(t) = -f(t,0,0) - b(t)u(t) - c(t)u(t-r).$$

Lemma 1. $a, b, c, f \in C^1(\bar{I}), \varphi \in C^1(\bar{I}_0)$ the solution u of the problem (1)-(3) holds that

$$\|u(t)\|_{(\infty,\bar{I})} \leq C, \tag{7}$$

$$|u'(t)| \leq C\left\{1 + \frac{1}{\varepsilon} exp\left(-\frac{\alpha t}{\varepsilon}\right)\right\}, t \in I, \tag{8}$$

$$|u''(t)| \leq C\left\{1 + \frac{(t-r_{p-1})^{p-1}}{\varepsilon^2} exp\left(-\frac{\alpha(t-r_{p-1})}{\varepsilon}\right)\right\}, t \in I_p,\ p = 1,2, \tag{9}$$

$$|u''(t)| \leq C, t \in I_p, 3 \leq p \leq m. \tag{10}$$

Proof. Firstly, from the equation (6), we obtain

$$u'(t) = u'(0)\exp\left(-\frac{1}{\varepsilon}\int_0^t a(s)ds\right) + \frac{1}{\varepsilon}\int_0^t F(\tau)\exp\left(-\frac{1}{\varepsilon}\int_\tau^t a(s)ds\right)d\tau \tag{11}$$

Integrating the equation (11) from 0 to t, we have

$$u(t) = u(0) + A\varepsilon^{-1}\int_0^t \exp\left(-\frac{1}{\varepsilon}\int_0^s a(\tau)d\tau\right)ds$$

$$+\frac{1}{\varepsilon}\int_0^t ds\int_0^s F(\tau)\exp\left(-\frac{1}{\varepsilon}\int_\tau^t a(\zeta)d\zeta\right)d\tau\,.$$

From here, it is found that

$$|u(t)| \leq |u(0)| + \left|A\varepsilon^{-1}\int_0^t \exp\left(-\frac{1}{\varepsilon}\int_0^s a(\tau)d\tau\right)ds\right| + \frac{1}{\varepsilon}\left|\int_0^t d\tau F(\tau)\int_\tau^t \exp\left(-\frac{1}{\varepsilon}\int_\tau^s a(\zeta)d\zeta\right)ds\right|.$$

Thus,

$$|u(t)| \leq |u(0)| + \alpha^{-1}(|A| + \|f\|_1) + \alpha^{-1}\int_0^t (\|b\|_\infty|u(\tau)| + \|c\|_\infty|u(\tau-r)|)d\tau.$$

From here, we arrive at the proof of the relation (7).

Rewriting $u'(t) = w(t)$ in the equation (11) and integrating on the interval $[0, t]$, we get

$$|w(t)| \leq |w(0)|\exp\left(-\frac{1}{\varepsilon}\int_0^t a(s)ds\right) + \frac{1}{\varepsilon}\int_0^t |F(\tau)|\exp\left(-\frac{1}{\varepsilon}\int_\tau^t a(s)ds\right)d\tau$$
$$\leq C\left\{|A|\varepsilon^{-1}\exp\left(-\frac{\alpha t}{\varepsilon}\right) + \alpha^{-1}\left(1 - \exp\left(-\frac{\alpha t}{\varepsilon}\right)\right)\right\},$$

which immediately leads to the relation (8). Thus, the proof of the lemma is completed.

3. The Difference Scheme

Let ϖ_{N_0} be a non-uniform mesh on the interval $[0, T]$:

$$\varpi_{N_0} = \{0 = t_0 < t_1 < \cdots < t_{N_0} = T;\ h_i = t_i - t_{i-1}\},$$

$$\omega_{N,p} = \{t_i: (p-1)N + 1 \le i \le pN\}, 1 \le p \le m,$$

$$\omega_{N_0} = \bigcup_{p=1}^{m} \omega_{N,p}.$$

Here, t_i are node points, h_i is the mesh stepsize and $N_0 = mN$. While establishing the difference scheme, we use the following difference rules (Samarskii, 2001):

$$w_{\bar{t},i} = \frac{w_i - w_{i-1}}{h_i},\ w_{t,i} = \frac{w_{i+1} - w_i}{h_{i+1}},\ w_{t,i} = w_{\bar{t},i+1},$$

$$w_{\overset{0}{t}} = \frac{w_{\bar{t},i} + w_{t,i}}{2},\ \ w_{\bar{t}\hat{t},i} = \frac{w_{t,i} - w_{\bar{t},i}}{\hbar_i}, w_{\hat{t},i} = \frac{w_{i+1} - w_i}{\hbar_i},$$

where $\hbar_i = \frac{1}{2}(h_i + h_{i+1})$ and the mesh function $w: \varpi_{N_0} \to \mathbb{R}$. Moreover, the discrete maximum norm is denoted by

$$\|w\|_{\infty,N,p} = \|w\|_{\infty,\omega_{N,p}} \coloneqq \max_{1 \le i \le N} |w_i|.$$

To establish the difference scheme for the equation (1), we use

$$\hbar_i^{-1} \int_{t_{i-1}}^{t_{i+1}} Lu(t)\, \varphi_i(t) dt = 0,\ \ 1 \le i \le N_0 - 1, \tag{12}$$

where the linear basis function

$$\varphi_i(t) = \begin{cases} \varphi_i^{(1)}(t) \equiv \dfrac{t - t_{i-1}}{h_i}, t_{i-1} < t < t_i \\ \varphi_i^{(2)}(t) \equiv \dfrac{t_{i+1} - t}{h_{i+1}}, t_i < t < t_{i+1} \\ 0 \qquad\qquad\quad , t \notin (t_{i-1}, t_{i+1}). \end{cases}$$

Also, $\hbar_i^{-1} \int_{t_{i-1}}^{t_{i+1}} \varphi_i(t) dt = 1$. Using the partial integration for the first term of (12), we find

$$\varepsilon \hbar_i^{-1} \int_{t_{i-1}}^{t_{i+1}} u'' \varphi(t) dt = \varepsilon \hbar_i^{-1} u'(t) \varphi(t) \Big|_{t_{i-1}}^{t_{i+1}} - \varepsilon \hbar_i^{-1} \int_{t_{i-1}}^{t_{i+1}} u' \varphi'(t) dt$$

$$= -\varepsilon \hbar_i^{-1} \int_{t_{i-1}}^{t_i} u' \varphi^{(1)\prime}(t)\, dt - \varepsilon \hbar_i^{-1} \int_{t_i}^{t_{i+1}} u' \varphi^{(2)\prime}(t)\, dt.$$

After, applying the interpolating quadrature rules in (Amiraliyev & Mamedov, 1995), it is obtained that

$$= -\varepsilon \hbar_i^{-1} u_{\bar{t},i} \int_{t_{i-1}}^{t_i} \varphi^{(1)\prime}(t)dt + \varepsilon \hbar_i^{-1} \int_{t_{i-1}}^{t_i} dt \varphi_i^{(1)\prime\prime}(t) \int_{t_{i-1}}^{t_i} u''(\xi)K(\xi)d\xi$$

$$-\varepsilon \hbar_i^{-1} u_{t,i} \int_{t_i}^{t_{i+1}} \varphi_i^{(2)\prime}(t)dt + \varepsilon \hbar_i^{-1} \int_{t_i}^{t_{i+1}} dt \varphi_i^{(2)\prime\prime}(t) \int_{t_i}^{t_{i+1}} u''(\xi)K(\xi)d\xi$$

$$= \varepsilon \hbar_i^{-1}\left[u_{t,i} - u_{\bar{t},i}\right] = \varepsilon u_{\bar{t}\hat{t},\, i}. \tag{13}$$

For the term $\hbar_i^{-1} \int_{t_{i-1}}^{t_{i+1}} a(t)\, u'\varphi_i(t)dt$, again using the interpolating quadrature rules, we have

$$\hbar_i^{-1} \int_{t_{i-1}}^{t_{i+1}} a(t_i)\, u'(t)\varphi_i(t)dt + \hbar_i^{-1} \int_{t_{i-1}}^{t_{i+1}} [a(t) - a(t_i)]\, u'(t)\varphi_i(t)dt$$

$$= \hbar_i^{-1} a_i \left[u_{\bar{t},i} \int_{t_{i-1}}^{t_i} \varphi_i^{(1)} dt + u_{t,i} \int_{t_i}^{t_{i+1}} \varphi_i^{(2)} dt\right] + R_{a,i} + R_1.$$

here

$$R_{a,i} = \hbar_i^{-1} \int_{t_{i-1}}^{t_{i+1}} [a(t) - a(t_i)]\, u'\varphi_i(t)dt$$

and

$$R_1 = \hbar_i^{-1} a_i \int_{t_{i-1}}^{t_i} dt \varphi_i^{(1)\prime}(t) \int_{t_{i-1}}^{t_i} u'(\xi)\, K_0(t,\xi)d\xi$$

$$+\hbar_i^{-1} a_i \int_{t_i}^{t_{i+1}} dt \varphi_i^{(2)\prime}(t) \int_{t_i}^{t_{i+1}} u'(\xi)\, K_0(t,\xi)d\xi. \tag{14}$$

Then, taking into account

$$\int_{t_{i-1}}^{t_i} \frac{t - t_{i-1}}{h_i} dt = \left.\frac{(t - t_{i-1})^2}{2h_i}\right|_{t_{i-1}}^{t_i} = \frac{h_i}{2},$$

$$\int_{t_i}^{t_{i+1}} \frac{t_{i+1} - t}{h_{i+1}} dt = -\left.\frac{(t_{i+1} - t)^2}{2h_{i+1}}\right|_{t_i}^{t_{i+1}} = \frac{h_{i+1}}{2}$$

and

$$\hbar_i = \frac{1}{2}(h_i + h_{i+1}) = \frac{t_{i+1} - t_{i-1}}{2},$$

we obtain that

$$\hbar_i^{-1}a_i\left(u_{\bar{t},i}\int_{t_{i-1}}^{t_i}\varphi_i^{(1)}dt+u_{t,i}\int_{t_i}^{t_{i+1}}\varphi_i^{(2)}dt\right)=\hbar_i^{-1}a_i\left(\frac{u_i-u_{i-1}}{2}+\frac{u_{i+1}-u_i}{2}\right)$$

$$=a_i\left(\frac{u_{i+1}-u_{i-1}}{h_{i+1}+h_i}\right)=a_i\left(\frac{u_{i+1}-u_{i-1}}{t_{i+1}-t_{i-1}}\right)$$

$$=a_i u_{\overset{0}{t}}+R_{a,i}+R_1. \tag{15}$$

For the term $\hbar_i^{-1}\int_{t_{i-1}}^{t_{i+1}} f(t,\ u,\ u(t-r))\varphi_i\,dt$, it is written that

$$\hbar_i^{-1}\int_{t_{i-1}}^{t_{i+1}} f(t,u,u(t-r))\varphi_i\,dt=\hbar_i^{-1}\left\{\int_{t_{i-1}}^{t_{i+1}}\left[f(t,u(t),u(t-r))-f(t_i,u(t),u(t-r))\right]\varphi_i\,dt\right.$$

$$+\int_{t_{i-1}}^{t_{i+1}}[f(t_i,u(t),u(t-r))-f(t_i,u_i,u(t-r))]\varphi_i\,dt$$

$$+\int_{t_{i-1}}^{t_{i+1}}[f(t_i,u_i,u(t-r))-f(t_i,u_i,u(t_i-r))]\varphi_i\,dt$$

$$\left.+\int_{t_{i-1}}^{t_{i+1}} f(t_i,u_i,u(t_i-r))\varphi_i\,dt\right\}$$

$$=f\left(t_i,u_i,u(t_{i-M_0})\right)+R_{f,i}. \tag{16}$$

Taking $h=\frac{T}{N_0}$ and $M_0=r\frac{N_0}{T}$, we get

$$t_i-r=ih-hM^0=(i-M^0)h=t_{i-M_0}.$$

Moreover, it is found that

$$R_{f,i}=\hbar_i^{-1}\left\{\int_{t_{i-1}}^{t_{i+1}}[f(t,u(t),u(t-r))-f(t_i,u(t),u(t-r))]\varphi_i\,dt\right.$$

$$+\int_{t_{i-1}}^{t_{i+1}}[f(t_i,u(t),u(t-r))-f(t_i,u_i,u(t-r))]\varphi_i\,dt$$

$$\left.+\int_{t_{i-1}}^{t_{i+1}}\left[f\left(t_i,u_i,u(t-r)\right)-f\left(t_i,u_i,u(t_i-r)\right)\right]\varphi_i\,dt\right\}. \tag{17}$$

For the condition $u'(0)=\frac{A}{\varepsilon}$, we consider that

$$\int_{t_0}^{t_1}\left[Lu+f\left(t,u,u(t-r)\right)\right]\varphi_0(t)\,dt=0.$$

Here the function $\varphi_0(x)$ is as follows:

$$\varphi_0(t) \equiv \frac{t_1 - t}{h_1}, t_0 < t < t_1.$$

From here, we get

$$\int_{t_0}^{t_1} \left[Lu + f\big(t, u, u(t-r)\big)\right]\varphi_0\,(t)dt = -A + \varepsilon u_{t,0} + r^{(0)} = 0$$

and

$$u_{t,0} = \frac{A}{\varepsilon} - \frac{r^{(0)}}{\varepsilon}, \tag{18}$$

where

$$r^{(0)} = \int_{t_0}^{t_1} \left[a(t)u'(t) + f\big(t, u, u(t-r)\big)\right]\varphi_0(t)dt.$$

Combining (13), (15), (16) and (18), we obtain the following problem

$$\varepsilon u_{\bar{t}\hat{t},\,i} + a_i u_{\overset{0}{t},i} + f\big(t_i, u_i, u_{i-M_0}\big) + R_i = 0,\ \ i = 1,2,\dots,N_0 - 1, \tag{19}$$

$$u(t_i) = \psi(t_i),\ \ -M_0 \le i \le 0, \tag{20}$$

$$u_{t,0} = \frac{A}{\varepsilon} - \frac{r^{(0)}}{\varepsilon}, \tag{21}$$

where

$$R_i = R_{a,i} + R_1 + R_{f,i}. \tag{22}$$

By neglecting the remainder term R_i, the following difference scheme is presented for the approximate solution:

$$\varepsilon y_{\bar{t}\hat{t},\,i} + a_i y_{\overset{0}{t},i} + f\big(t_i, y_i, y_{i-M_0}\big) = 0,\ \ i = 1,2,\dots,N_0 - 1, \tag{23}$$

$$y(t) = \psi(t),\ \ -M_0 \le i \le 0, \tag{24}$$

$$y_{t,0} = \frac{A}{\varepsilon}. \tag{25}$$

3.1. Bakhvalov mesh

Let $\varpi_{N,1} = \{0 = t_0 < t_1 < t_2 < \ldots < t_{N-1} < t_N = r\}$ be a non-uniform mesh. The transition points is taken as

$$\sigma = min\left\{\frac{r}{2}, \alpha^{-1}\varepsilon ln\varepsilon\right\}$$

We divide into the interval $I_1=[0,r]$ subintervals of $[0,\sigma]$ and $[\sigma,r]$. The corresponding t_i nodes are as follow:

$$t_i=\begin{cases}-\alpha^{-1}\varepsilon ln\left(1-(1-\varepsilon)\frac{2i}{N}\right), i=0,1,\dots,\frac{N}{2}, t_i\in[0,\sigma], if\ \sigma<\frac{r}{2};\\ -\alpha^{-1}\varepsilon ln\left(1-\left(1-e^{\frac{\alpha r}{2\varepsilon}}\right)\frac{2i}{N}\right), i=0,1,\dots,\frac{N}{2}, t_i\in[0,\sigma], if\ \sigma=\frac{r}{2};\\ \sigma+\left(i-\frac{N}{2}\right)h^{(1)}, i=\frac{N}{2}+1,\dots,N, t_i\in[0,\sigma], if\ h^{(1)}=\frac{2(r-\sigma)}{N};\end{cases}$$

It can be determined by similarly on the other I_p intervals (Boglaev, 1984).

4. Error Analysis

Let u be the solution of the problem (19)-(21) and y be the solution of the problem (23)-(25). The error function $z_i=y_i-u_i$ satisfies the following discrete problem:

$$\varepsilon z_{\bar{t}\hat{t},\,i}+a_i z_{\overset{0}{t},i}+f\left(t_i,y_i,y_{i-M_0}\right)-f\left(t_i,u_i,u_{i-M_0}\right)=R_i,\ \ i=1,2,\dots,N_0-1, \tag{26}$$

$$z(t_i)=0,\quad -M_0\le i\le 0, \tag{27}$$

$$z_{t,0}=\frac{1}{\varepsilon}r^{(0)}. \tag{28}$$

Lemma 2. For the error function z, the following relation is held:

$$\|z\|_{\infty,N,p}\le\gamma\left|r^{(0)}\right|+C\sum_{k=1}^{p}\|R\|_{\infty,\omega_{N,k}}, 1\le p\le m.$$

Proof. Rewriting $z_{t,i}=v_i$, we obtain that

$$\varepsilon\hbar_i^{-1}(v_i+v_{i-1})+\frac{a_i}{2}(v_i+v_{i-1})=F_i,$$

where

$$F_i=R_i-f\left(t_i,y_i,y_{i-M_0}\right)+f\left(t_i,u_i,u_{i-M_0}\right).$$

From here, we can write that

$$\varepsilon h_i^{-1}(v_i-v_{i-1})+\frac{a_i}{2}(v_i+v_{i-1})=F_i,$$

$$\left(\varepsilon h_i^{-1}+\frac{a_i}{2}\right)v_i=\left(\varepsilon h_i^{-1}-\frac{a_i}{2}\right)v_{i-1}+F_i,$$

$$v_i=\frac{\left(\varepsilon h_i^{-1}-\frac{a_i}{2}\right)}{\left(\varepsilon h_i^{-1}+\frac{a_i}{2}\right)}v_{i-1}+\frac{F_i}{\left(\varepsilon h_i^{-1}+\frac{a_i}{2}\right)}.$$

Therefore, we get

$$v_i=v_0Q_i+\sum_{k=1}^{i}\frac{F_k}{\varepsilon h_k^{-1}+\frac{a_k}{2}}Q_{i-k}, \tag{29}$$

here

$$Q_{i-k}=\begin{cases} 1, & k=i, \\ \prod_{j=k+1}^{i}\left(\dfrac{\varepsilon h_k^{-1}-\frac{a_j}{2}}{\varepsilon h_k^{-1}+\frac{a_j}{2}}\right), & 0\le k\le i-1 \end{cases}$$

For $h=\max_{0\le i\le N} h_i$, we have

$$z_p=h\sum_{i=0}^{p-1}v_i=h\sum_{i=1}^{p}v_{i-1}.$$

Substituting this relation in the equation (29), it is found that

$$|z_p|\le 4\alpha^{-1}\left(\varepsilon|z_{t,0}|+h\sum_{i=1}^{p-1}\left(|R_i|+b^*|z_i|+c^*|z_{i-M_0}|\right)\right), 1\le p\le N_0-1,$$

where

$$h\sum_{i=1}^{p}|z_{i-M_0}|=h\sum_{j=1-M_0}^{p-M_0}|z_j|\le h\sum_{j=1-M_0}^{0}|z_j|+h\sum_{j=1}^{p-1}|z_j|+\|\psi\|_{L_1(\omega_{N,0})}+h\sum_{j=1}^{p-1}|z_j|.$$

Thus, we obtain

$$|z_p|\le C\left(|r^{(0)}|+\|\psi\|_1+\sum_{j=1}^{p}(|R_i|+b^*z_i+c^*|z_{i-1}|)\right), \tag{30}$$

which shows the proof of the lemma.

Lemma 3. For the remainder term R_i, it can be written the following estimate:

$$\|R\|_{\infty,\omega_{N,p}}\le CN^{-1},\quad 1\le p\le m$$

and

$$|r^{(0)}|\le CN^{-1}.$$

Proof. Combining all the remainder terms, we can write that

$$|R_i|\le \hbar_i^{-1}\int_{t_{i-1}}^{t_{i+1}}\left|\left(a(t)-a(t_i)\right)\right||u'(t)|\varphi_i(t)dt$$
$$+\hbar_i^{-1}\int_{t_{i-1}}^{t_{i+1}}(t-t_i)\left\{\left|\frac{\partial}{\partial t}f\left(t,u(t),u(t-r)\right)\right|\right.$$
$$+\left|\frac{\partial}{\partial u}f(t,u(t),u(t-r))\right||u'(t)|$$

$$+\left|\frac{\partial}{\partial v}f(t,u(t),u(t-r))\right||u'(t-r)|\Big\}\varphi_i(t)dt$$

$$+\hbar_i^{-1}\int_{t_{i-1}}^{t_i}dt\,\varphi_i^{(1)}{}'(t)\int_{t_{i-1}}^{t_i}|u'(\xi)|\,K_0(t,\xi)d\xi$$

$$+\hbar_i^{-1}\int_{t_i}^{t_{i+1}}dt\varphi_i^{(2)}{}'(t)\int_{t_i}^{t_{i+1}}|u'(\xi)|\,K_0(t,\xi)d\xi,\quad 1\le i\le N_0,$$

since partial derivatives are bounded and using the estimate (8), we get

$$|R_i|\le C\left\{\hbar_i+\int_{t_{i-1}}^{t_{i+1}}[|u'(t)|+|u'(t-r)|]dt\right\}$$

$$\le C\left\{\hbar_i+\frac{1}{\varepsilon}\int_{t_{i-1}}^{t_{i+1}}e^{-\frac{\alpha t}{\varepsilon}}dt\right\},1\le i\le N_0.$$

At the each submesh $\omega_{N,p}$, we estimate the truncation error R as follows. We consider first the case $\sigma_p=r_{p-1}+r/2$ and so $r/2\le\alpha^{-1}\varepsilon ln\varepsilon$, as on the interval I_p. Thus, we find that

$$h_i=\begin{cases}h_p^{(1)}=2(\sigma_p-r_{p-1})N^{-1},(p-1)N\le i\le(p-1/2)N\\ h_p^{(2)}=2(r_p-\sigma_p)N^{-1},\qquad(p-1/2)N+1\le i\le pN.\end{cases}$$

For the $h_p^{(1)}=h_p^{(2)}$, if $\sigma_p<\frac{r}{N}$

$$h_p^{(1)}=-\alpha^{-1}\varepsilon\ln\left(1-(1-\varepsilon)\frac{2i}{N}\right)+\alpha^{-1}\varepsilon\ln\left(1-(1-\varepsilon)\frac{2(i-1)}{N}\right)$$

is obtained. Applying the mean value theorem according to i, we have

$$h_p^{(1)}=\alpha^{-1}\varepsilon\frac{-(1-\varepsilon)2N^{-1}}{1-(1-\varepsilon)2i_*N^{-1}}(1-\varepsilon)2N^{-1}\le2\alpha^{-1}(1-\varepsilon)N^{-1}.$$

From here, it is written that $|h_p^{(1)}|\le CN^{-1}$. Also, if $\sigma_p=\frac{r}{N}$, since $e^{-\frac{\alpha t_i}{\varepsilon}}-e^{-\frac{\alpha t_{i-1}}{\varepsilon}}\le2(1-\varepsilon)N^{-1}\le CN^{-1}$, we get

$$h_i=-\alpha^{-1}\varepsilon\ln\left(1-\left(1-e^{-\frac{\alpha r}{\varepsilon}}\right)2iN^{-1}\right)$$
$$+\alpha^{-1}\varepsilon\ln(1-(1-e^{-\frac{\alpha r}{2\varepsilon}})\frac{2(i-1)}{N}$$

and

$$h_i=\alpha^{-1}\varepsilon\frac{-(1-(1-e^{-\frac{\alpha r}{\varepsilon}})2N^{-1}}{1-(1-(1-e^{-\frac{\alpha r}{\varepsilon}})2i_*N^{-1}}(1-e^{-\frac{\alpha r}{\varepsilon}})2N^{-1}\le2\alpha^{-1}(1-e^{-\frac{\alpha r}{\varepsilon}})2N^{-1}\le CN^{-1}.$$

Here $e^{-\frac{\alpha t_i}{\varepsilon}} - e^{-\frac{\alpha t_{i-1}}{\varepsilon}} \le 2(1-e^{-\frac{\alpha r}{2\varepsilon}})N^{-1} \le CN^{-1}$. For the interval $[\sigma_p, r_p]$, it is found that $h_p^{(2)} = \frac{2(r_p-\sigma_p)}{N} = \frac{2(r_p-\alpha^{-1}\varepsilon\ln\varepsilon)}{N} \le CN^{-1}$. Substituting these results in term R_i, we obtain

$$h_i^{-1}\varepsilon^{-1}\int_{t_{i-1}}^{t_{i+1}} e^{-\frac{\alpha t}{\varepsilon}}dt \le \varepsilon^{-1}h_p \le \varepsilon^{-1}\frac{2\alpha^{-1}\varepsilon}{r}\frac{r}{N} = 2\alpha^{-1}(1-\varepsilon)N^{-1} \le CN^{-1},$$

$$|R_i| \le CN^{-1}, (p-1)N \le i \le pN, 1 \le p \le m-1.$$

$$|R_i| \le C(1+\varepsilon^{-1})h_p^{(1)} = C(2(1+\varepsilon^{-1}))\frac{\alpha^{-1}}{N}, \qquad (p-1)N \le i \le (p-1/2)N,$$

$$1 \le p \le m-1.$$

$$|R_i| \le CN^{-1}, (p-1)N \le i \le (p-1/2)N, 1 \le p \le m-1.$$

$$|R_i| \le C\left\{h_p^{(2)} + \alpha^{-1}\left(e^{-\frac{\alpha(t_{i-1})}{\varepsilon}} - e^{-\frac{\alpha(t_i)}{\varepsilon}}\right)\right\}$$

$$|R_i| \le CN^{-1}.$$

Similarly, for the remainder term $r^{(0)}$, we have $\left|r^{(0)}\right| \le CN^{-1}$.

Theorem 1. Let u be the solution of the problem (19)-(21) and y be the solution of the problem (23)-(25). Then the following estimate is satisfied:

$$|y_i - u_i| \le CN^{-1}, \qquad 0 \le i \le N_0.$$

5. Numerical Results

In this section, the presented method is tested on a numerical example and the obtained results will be discussed. For this, we consider the following problem:

$$\varepsilon u''(t) + u'(t) + u(t-1) = 1, \;\; t \in (0,\infty)$$

$$u(t) = \psi(t) = 1+t, \;\; -1 \le t \le 0.$$

$$u'(t) = -\frac{1}{\varepsilon}.$$

The exact solution of this problems is as follow:

$$u(t) = \begin{cases} e^{-t/\varepsilon}, & t \in [0,1] \\ -1-2\varepsilon+t+e^{-t/\varepsilon}+(-1+2\varepsilon+t)e^{-(t-1)/\varepsilon}, & t \in (1,2]. \end{cases}$$

Now, the difference scheme is written as the form

$$B_i y_{i-1}^{(k)} - C_i y_i^{(k)} + A_i y_{i+1}^{(k)} = -F_i \tag{31}$$

Using the quasilinearization technique, we can write

$$f(t_i, y_i, y_{i-N}) = f\left(t_i, y_i^{(k-1)}, y_{i-M_0}^{(k-1)}\right) + \frac{\partial f\left(t_i, y_i^{(k-1)}, y_{i-M_0}^{(k-1)}\right)}{\partial u}\left(y_i^{(k)} - y_i^{(k-1)}\right)$$
$$+ \frac{\partial f\left(t_i, y_i^{(k-1)}, y_{i-M_0}^{(k-1)}\right)}{\partial v}\left(y_{i-M_0}^{(k)} - y_{i-M_0}^{(k-1)}\right).$$

Thus, we obtain

$$\varepsilon \hbar_i^{-1}\left(\frac{y_{i+1}^{(k)} - y_i^{(k)}}{h_{i+1}} - \frac{y_i^{(k)} - y_{i-1}^{(k)}}{h_i}\right) + a_i\left(\frac{y_{i+1}^{(k)} - y_{i-1}^{(k)}}{2\hbar_i}\right) + \frac{\partial f\left(t_i, y_i^{(k-1)}, y_{i-M_0}^{(k-1)}\right)}{\partial u} y_i^{(k)}$$
$$= \frac{\partial f\left(t_i, y_i^{(k-1)}, y_{i-M_0}^{(k-1)}\right)}{\partial u} y_i^{(k-1)} + \frac{\partial f\left(t_i, y_i^{(k-1)}, y_{i-M_0}^{(k-1)}\right)}{\partial v}\left(y_{i-M_0}^{(k-1)} - y_{i-M_0}^{(k)}\right)$$
$$-f\left(t_i, y_i^{(k-1)}, y_{i-M_0}^{(k-1)}\right).$$

Here the coefficients of the equation (31) are as follow:

$$A_i = \varepsilon \hbar_i^{-1} h_{i+1}^{-1} + \frac{a_i}{2\hbar_i},$$

$$B_i = \varepsilon \hbar_i^{-1} h_i^{-1} - \frac{a_i}{2\hbar_i},$$

$$C_i = \varepsilon \hbar_i^{-1}\left(h_{i+1}^{-1} + h_i^{-1}\right) + \frac{\partial f\left(t_i, y_i^{(k-1)}, y_{i-M_0}^{(k-1)}\right)}{\partial u},$$

$$F_i = f\left(t_i, y_i^{(k-1)}, y_{i-M_0}^{(k-1)}\right) - \frac{\partial f\left(t_i, y_i^{(k-1)}, y_{i-M_0}^{(k-1)}\right)}{\partial u} y_i^{(k-1)}$$
$$+ \frac{\partial f\left(t_i, y_i^{(k-1)}, y_{i-M_0}^{(k-1)}\right)}{\partial v}\left(y_{i-M_0}^{(k)} - y_{i-M_0}^{(k-1)}\right).$$

Then, we consider the following iteration:

$$A_i y_{i-1}^{(k)} - C_i y_i^{(k)} + B_i y_{i+1}^{(k)} = -F, i = 1, \dots, N-1; k = 0, 1, \dots$$

$$y_i^{(0)} = 0,$$

$$y_{i-M_0} = 1 + t_{i-M_0},$$

$$y_1 = \psi(0) + h_1 A/\varepsilon.$$

Also, for the elimination method in (Samarskii, 2001), we have

$$\alpha_{i+1} = \frac{B_i}{C_i - A_i a_i}, \alpha_1 = -1, i = 2,3, \dots, N,$$

$$\beta_{i+1} = \frac{A_i \beta_i + F_i}{C_i - A_i a_i}, \beta_1 = -h_1 A/\varepsilon, i = 2,3, \dots, N,$$

$$y_i^{(k)} = \alpha_{i+1} y_{i+1}^{(k)} + \beta_{i+1}, i = N-1, N-2, \dots ,1,$$

Also, the maximum errors are denoted by

$$e^N = max|y_i{}^N - y_i{}^{2N}|,$$

and the convergence rates are calculated as

$$p = \frac{ln\left(e^N / e^{2N} \right)}{ln2}.$$

According to these, the computed results are presented in Table 1.

Table 1. Exact errors and order of convergence on Bakhvalov mesh

ε	N=64	N=128	N=256	N=512
2^{-1}	e^N=0.01349102 e^{2N}=0.00823915 p=0.71143235	e^N=0.00823915 e^{2N}=0.00450459 p=0.87109910	e^N=0.00450459 e^{2N}=0.00235012 p=0.93865950	e^N=0.00235012 e^{2N}=0.00119972 p=0.97003589
2^{-2}	e^N=0.01410116 e^{2N}=0.00716200 p=0.97737798	e^N=0.00716200 e^{2N}=0.00358096 p=1.00001598	e^N=0.00358096 e^{2N}=0.00178738 p=1.00250216	e^N=0.00178774 e^{2N}=0.00089275 p=1.00180251
2^{-3}	e^N=0.00773744 e^{2N}=0.00233993 p=1.72539127	e^N=0.00233993 e^{2N}=0.00105512 p=1.14905641	e^N=0.00105512 e^{2N}=0.00050214 p=1.07123342	e^N=0.00050239 e^{2N}=0.00024490 p=1.03659430
2^{-4}	e^N=0.00547751 e^{2N}=0.00191764 p=1.51418468	e^N=0.00191764 e^{2N}=0.00056934 p=1.75196790	e^N=0.00056959 e^{2N}=0.00017388 p=1.71183973	e^N=0.00017400 e^{2N}=0.00005810 p=1.58240113
2^{-5}	e^N=0.00589368 e^{2N}=0.00937462 p=-0.66959124	e^N=0.00937462 e^{2N}=0.00280064 p=1.74300168	e^N=0.00280064 e^{2N}=0.00135697 p=1.04536938	e^N=0.00135739 e^{2N}=0.00126656 p=0.09992271

In Table 1, exact errors and convergence rates are demonstrated for different values ε of and *N*. According to the obtained results, the scheme is almost first-order convergent.

6. Discussion and Conclusion

In this paper, we presented a finite difference scheme on Boglaev-Bakhvalov type mesh for solving singularly perturbed semilinear delay differential equations. The uniform convergence of the presented scheme was proven in the discrete maximum norm and first-order convergence rate was obtained. A numerical example was solved and the computational results were summarized.

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