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Some new equivalents of the Brouwer fixed point theorem

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Abstract

This is to recollect the equivalent formulations of the Brouwer fixed point theorem. We collect a large number of recently known sources of such equivalents. More recently, Jinlu Li obtained two fixed point theorems on newly defined *quasi-point-separable* topological vector spaces. His theorems extend the Tychonoff fixed point theorem on locally convex t.v.s. However, we note that his new theorems are logically equivalent to the Brouwer fixed point theorem. Consequently, we add up our large list of such equivalents.

Keywords: Brower fixed point theorem, Sperner, KKM, Ky Fan, Grand KKM theory.
2010 MSC: 47H10, 49J35, 49J53, 49K35, 54C60, 54H25, 91A10, 91B50.

1. Introduction

The Brouwer fixed point theorem is one of the most well-known and important existence principles in mathematics. The theorem and its many equivalent formulations or extensions are powerful tools in showing the existence of solutions of a lot of problems in pure and applied mathematics. This is why many scholars have been studying its further extensions and applications in thousands of papers.

Extensions of the Brouwer theorem have appeared with related to theory of topological vector spaces in mathematical analysis. The compactness, convexity, single-valuedness, continuity, self-mapness, and finite dimensionality related to the Brouwer theorem are all extended and, moreover, for the case of infinite dimensions, it is known that the domain and range of the map may have different topologies. This is why the Brouwer theorem has so many extensions. Current study of its extensions is concentrated to more general

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class of compact or condensing multimaps defined on convex subsets of more general topological vector spaces or abstract convex spaces. There are perhaps thousands of fixed-point theorems, but Brouwer's had a domino effect, proving other results, not only in topology, but also in other mathematical fields (differential equations, combinatorics, probabilities, game theory, mathematical economics, etc) and in non-mathematical domains such as economics, engineering and philosophy or even poetry. Although Brouwer himself had other valuable results in topology, complex theory, set theory, differential geometry and logic, his name remained tied to the famous theorem. See for example Caraman and Caraman [4].

Along with these developments, there have been found a large number of equivalent formulations of the Brouwer fixed point theorem. One of the earliest ones was a theorem of Knaster, Kuratowski, and Mazurkiewicz, which initiated the so-called KKM theory since 1992. At first, the basic theorems in the KKM theory were established for convex subsets of topological vector spaces, and later, for various generalized abstract convexities. Those basic theorems have many applications to various equilibrium problems and many others.

Other directions of the extensions in topology are studies of spaces having the fixed point property, various degree or index theories, the Lefschetz fixed point theory, the Nielsen fixed point theory, Reidemeister numbers, and the fixed point theorems in the Atiyah-Singer index theory which generalizes the Lefschetz theory. However, we will not follow these lines of study.

This survey article is to recollect the equivalent formulations of the Brouwer fixed point theorem. We collect a large number of recently known sources of such equivalents as a supplement of our previous history articles [24, 31].

This article is organized as follows: Section 2 is to introduce the so-called mathematical trinity; that is, the Brouwer fixed point theorem, the Sperner combinatorial lemma, and the Knaster-Kuratowski-Mazurkiewicz theorem. In Section 3, we list relatively early known equivalents of the Brouwer theorem given in our one hundred year history [31]. Section 4 deals with many further sources of equivalents of the Brouwer theorem. Actually we list thirty four articles old and new, and these are basis of some add-ups in the list in Section 3. In Section 5, we introduce new equivalents of the Brouwer theorem due to Jinlu Li [18, 19]. Finally, Section 6 deals with some conclusion and comments.

2. The Mathematical Trinity

In 1912, the Brouwer theorem appeared:

Theorem (Brouwer) *A continuous map from an n-simplex to itself has a fixed point.*

It is clear that, in this theorem, the n-simplex can be replaced by the unit ball $\mathbb{B}^n$ or any compact convex subset of $\mathbb{R}^n$.

Sperner in 1928 gave the following combinational lemma and its applications:

Lemma (Sperner) *Let K be a simplicial subdivision of an n-simplex $v_0v_1\cdots v_n$. To each vertex of K, let an integer be assigned in such a way that whenever a vertex u of K lies on a face $v_{i_0}v_{i_1}\cdots v_{i_k}$ $(0 \leq k \leq n, 0 \leq i_0 \leq i_1 \leq \cdots \leq i_k \leq n)$, the number assigned to u is one of the integers $i_0, i_1, \cdots, i_k$. Then the total number of those n-simplices of K, whose vertices receive all $n+1$ integers $0, 1, \cdots, n$, is odd. In particular, there is at least one such n-simplex.*

The particular case is usually called the *weak* Sperner lemma. Fifty years after the birth of the lemma, at a conference at Southampton, England in 1979, Sperner himself listed early applications of his lemma; see [24, 31].

Knaster, Kuratowski, and Mazurkiewicz in 1929 obtained the following so-called KKM theorem from the weak Sperner lemma in 1928:

Theorem (KKM) *Let $A_i\,(0 \leq i \leq n)$ be $n+1$ closed subsets of an n-simplex $p_0p_1\cdots p_n$. If the inclusion relation*

$$p_{i_0}p_{i_1}\cdots p_{i_k} \subset A_{i_0} \cup A_{i_1} \cup \cdots \cup A_{i_k}$$

holds for all faces $p_{i_0}p_{i_1}\cdots p_{i_k}$ $(0 \le k \le n,\ 0 \le i_0 < i_1 < \cdots < i_k \le n)$*, then* $\bigcap_{i=0}^{n} A_i \neq \emptyset$.

A special case or dual form of the KKM theorem is already given by Sperner in 1928. The KKM theorem follows from the weak Sperner lemma and is used to obtain one of the most direct proofs of the Brouwer theorem. Therefore, it was conjectured that those three theorems are mutually equivalent. This was clarified by Yoseloff in 1974. In fact, those three theorems are regarded as a sort of mathematical trinity. All are extremely important and have many applications.

$$
\begin{array}{ccc}
 & \text{Brouwer} & \\
1974 \swarrow & & \nwarrow 1929 \\
\text{weak Sperner} & \underset{1929}{\longrightarrow} & \text{KKM}
\end{array}
$$

De Loera et al. [7] sketched how Brouwer's theorem follows from (weak) Sperner's lemma, and its converse.

Moreover, many important results in nonlinear functional analysis and other fields are known to be equivalent to those three theorems. Only less than a dozen of those results are shown in many text-books on nonlinear analysis.

Ky Fan [8] in 1961 stated: Since Knaster-Kuratowski-Mazurkiewicz's theorem applies only to the finite dimensional case, it is necessary to reformulate it in the following generalized form.

Lemma 1. *Let X be an arbitrary set in a topological vector space Y. To each $x \in X$, let a closed set $F(x)$ in Y be given such that the following two conditions are satisfied:*

(i) *The convex hull of any finite subset $\{x_1, x_2, \dots, x_n\}$ of X is contained in $\bigcup_{i=1}^{n} F(x_i)$.*

(ii) *$F(x)$ is compact for at least one $x \in X$.*

Then $\bigcap_{x\in X} F(x) \neq \emptyset$.

For some later applications of this lemma, see Park [29].

3. Known equivalents of the Brouwer theorem

As the development of the KKM theory, there have appeared many statements equivalent to the Brouwer theorem, especially, in nonlinear functional analysis and mathematical economics. For the classical results, see Granas [10].

Relatively early equivalent forms of the Brouwer theorem are as follows:

1883 Poincaré's theorem.
1904 Bohl's non-retraction theorem.
1912 Brouwer's fixed point theorem.
1928 The weak form of Sperner's combinatorial lemma.
1929 The Knaster-Kuratowski-Mazurkiewicz theorem.
1930 Caccioppoli's fixed point theorem.
1930 Schauder's fixed point theorem.
1935 Tychonoff's fixed point theorem.
1937 von Neumann's intersection lemma.
1941 Intermediate value theorem of Bolzano-Poincaré-Miranda.
1941 Kakutani's fixed point theorem.
1950 Bohnenblust-Karlin's fixed point theorem.
1950 Hukuhara's fixed point theorem.
1950 Nash's equilibrium theorem.
1952 Fan-Glicksberg's fixed point theorem.
1954 Steinhaus' chessboard theorem.
1955 Main theorem of mathematical economics on Walras equilibria of Gale, Nikaido, and Debreu.
1960 Kuhn's cubic Sperner lemma.
1961 Fan's KKM lemma.

1961 Fan's geometric or section property of convex sets.
1964 Debrunner-Flor's variational equality.
1966 Fan's theorem on sets with convex sections.
1966 Hartman-Stampacchia's variational inequality.
1967 Browder's variational inequality.
1967 Scarf's intersection theorem.
1968 Fan-Browder's fixed point theorem.
1969 Fan's best approximation theorems.
1972 Fan's minimax inequality.
1972 Himmelberg's fixed point theorem.
1973 Shapley's generalization of the KKM theorem.
1976 Tuy's generalization of the Walras excess demand theorem.
1983 Yannelis-Prabhakar's existence of maximal elements.
1984 Fan's matching theorems.
1991 Park's generalization of the Brouwer theorem.
1997 Horvath-Lassonde's intersection theorem.
1998 Greco-Moschen's KKM type and minimax theorems for marginally semicontinuous functions.

4. Further sources of equivalents of the Brouwer theorem

Only a few of the equivalencies can be found in standard text-books on nonlinear functional analysis. Many generalizations of theorems and others in the preceding list are also known to be equivalent to the Brouwer theorem. For example, the existence of Walrasian equilibrium in an economy with continuous excess demand functions is proved by the Brouwer theorem.

The following literature are sources of equivalents of the Brouwer theorem:

(I) The mathematical trinity: It is well-known that the Brouwer fixed point theorem, the weak Sperner combinatorial lemma, and the Knaster-Kuratowski-Mazurkiewicz (KKM) theorem are equivalent each other. A special case or dual form of the KKM theorem is already given by Sperner in 1928. The KKM theorem follows from the weak Sperner lemma and is used to obtain one of the most direct proofs of the Brouwer theorem.

(II) Nash in 1950-1951 applied the Brouwer or Kakutani fixed point theorems to the existence of an equilibrium for a finite game. It was followed by several hundreds applications in the theory of games and in economic theory. See [24, 31].

(III) In the 1950's, Kakutani's theorem was extended to Banach spaces by Bohnenblust and Karlin in 1950 and to locally convex Hausdorff t.v.s. by Fan and Glicksberg in 1952. These extensions were mainly used to extend von Neumann's works in the above. See [24, 31].

(IV) Uzawa [38] in 1962 showed (classically) that the existence of a Walrasian equilibrium price vector is equivalent to the Brouwer fixed point theorem, but not constructively.

(V) Vrahatis [39] in 1989: Miranda gave in 1941 an equivalent formulation of the famous Brouwer fixed point theorem. We give a short proof of Miranda's existence theorem and then using the results obtained in this proof we give a generalization of a well-known variant of Bolzano's existence theorem. Finally, we give a generalization of Miranda's theorem.

(VI) Park [23] in 1991: We obtained a generalization of the Brouwer fixed point theorem based on the Fan-Browder fixed point theorem.

(VII) Horvath and Lassonde [12] in 1997 obtained intersection theorems of the KKM-type, Klee-type, and Helly-type, which are all equivalent to the Brouwer theorem.

(VIII) Greco and Moschen [11] in 1998: Equivalent formulations of the minimax theorem for marginally semicontinuous functions and the KKM theorem for marginally closed-valued multimaps are equivalent to the Brouwer theorem; see Park [28].

(IX) Turza'nski [38] in 2000: We present an algorithm for determining on the Euclidean plane the place where the equilibrium points are. For this purpose, we use the Steinhaus chessboard theorem. The existence of market equilibrium is a classical problem in economics (Walras, von Neumann, Nash). The Brouwer fixed point theorem was the main mathematical tool in Nash's paper, for which he won the Nobel prize in economics. The Brouwer Theorem is an easy consequence of Kulpa's Equilibrium Theorem. Hence, an algorithm for determining a fixed point is also given.

(X) In 2000, Z. F. Yang [41]: We introduce a multipermutation-based intersection theorem on the product space of several unit simplices, called the simplotope. This theorem gives a substantial generalization of an intersection result of Scarf on the unit simplex. By using this new result, we also obtain a multipermutation-based generalization of the Brouwer fixed-point theorem on the simplotope. Furthermore, we apply this new result to an economic equilibrium model with indivisibilities and obtain an equilibrium existence theorem.

(XI) Park and Jeong [33] in 2001 showed that the Brouwer theorem is equivalent to a number of results closely related to the Euclidean spaces or n-simplices or n-balls. Among them are the weak Sperner lemma, the KKM theorem, some intersection theorems, various fixed point theorems, an intermediate value theorem, various non-retract theorems, the non-contractibility of spheres, and others.

(XII) Kulpa [15] in 2005 showed new proofs of equivalences of the Kakutani fixed point theorem, the KKMS theorem, and the KKM theorem.

(XIII) Mawhin [20] in 2005: The aim of this note is to use an interesting unpublished result of Tartar to give quite elementary proofs of various fixed point and existence theorems. When written in terms of differential forms, this result (Lemma 2) is an immediate consequence of an elementary computation of exterior calculus.

As an application of Lemma 2, we obtain very simple and seemingly new proofs of Brouwer's fixed point theorem on closed balls and fixed or antipodal point on even-dimensional spheres, of Poincaré-Brouwer's (hairy ball) theorem, of the generalized fundamental theorem of algebra, and of Eilenberg-Niven "fundamental theorem of algebra" for quaternions. The interested reader can consult some references to the many "elementary" analytical proofs of the Brouwer fixed point theorem and of the hairy ball theorem, specially the many ones motivated by Milnor's paper. For history and references on the fundamental theorem of algebra, one can consult some references.

(XIV) In 2006, Cobzas [5]: In the appendix to the book by F. F. Bonsal, Lectures on Some Fixed Point Theorems of Functional Analysis (Tata Institute, Bombay, 1962) a proof by Singbal of the Schauder-Tychonoff fixed point theorem, based on a locally convex variant of Schauder mapping method, is included. The aim of this note is to show that this method can be adapted to yield a proof of Kakutani fixed point theorem in the locally convex case. For the sake of completeness we include also the proof of Schauder-Tychonoff theorem based on this method. As applications, one proves a theorem of von Neumann and a minimax result in game theory.

(XV) For Steinhaus' chessboard theorem, see Tkacz and Turzanski [37] in 2008.

(XVI) Balaj [1] in 2010: Using the Brouwer fixed point theorem, we establish a common fixed point theorem for a family of set-valued mappings. As applications of this result we obtain existence theorems for the solutions of two types of vector equilibrium problems, a Ky Fan-type minimax inequality, and a generalization of a known result due to Iohvidov.

(XVII) Park [25] in 2010: The partial KKM principle for an abstract convex space is an abstract form of the classical KKM theorem. A KKM space is an abstract convex space satisfying the partial KKM principle and its "open" version. In this paper, we clearly derive a sequence of a dozen statements which characterize the KKM spaces and equivalent formulations of the partial KKM principle. As their applications, we add more than a dozen statements including generalized formulations of von Neumann minimax theorem, von Neumann intersection lemma, the Nash equilibrium theorem, and the Fan type minimax inequalities for any

KKM spaces. Consequently, this paper unifies and enlarges previously known several proper examples of such statements for particular types of KKM spaces.

Moreover, a large number of equivalent formulations to the KKM theorem in their abstract versions can be seen in Park [25].

(XVIII) In 2011, Park [26]: In this review, we introduce a new KKM-type theorem for intersectionally closed-valued KKM map on abstract convex spaces and its direct consequences such as a Fan-Browder-type fixed point theorem and maximal element theorems. For these basic theorems of the KKM theory, we review previously obtained particular consequences of them mainly due to the author and their recent applications obtained by other authors. Therefore, those applications might be improved by following our new results.

(XIX) In 2011, Park [27]: It is well-known that the Brouwer fixed point theorem, the Sperner combinatorial lemma, the Knaster-Kuratowski-Mazurkiewicz (for short, KKM), the Kakutani fixed point theorem, the Nash equilibrium theorem, Ky Fan's theorem on sets with convex sections, the Fan minimax inequality, the Fan-Browder fixed point theorem and many others, mainly in the KKM theory, are mutually equivalent.

We show that in an abstract convex space $(E, D; \Gamma)$, the partial KKM principle implies the Ky Fan minimax inequality, from which we deduce a generalization of the Nash equilibrium theorem.

(XX) Tanaka [36] in 2012 constructively proved Kakutani's fixed point theorem for multimaps with sequentially at most one fixed point and uniformly closed graph, and applied this result to prove the minimax theorem for two-person zero-sum games with finite strategies. He followed the Bishop style constructive mathematics.

(XXI) Idzik et al. [13] in 2014 collected three sets of equivalents of the Brouwer fixed point theorem. The first set covers some classic results connected to surjectivity property of continuous functions under proper assumptions on their boundary behavior. The second covers some results related to the Himmelberg fixed point theorem. The third loop involves equivalence of the existence of economic equilibrium and the Brouwer theorem.

(XXII) Li, Jinlu [17] in 2016: We introduce the concept of split Nash equilibrium problems associated with two related noncooperative strategic games. Then we apply the Fan-KKM theorem to prove the existence of solutions to split Nash equilibrium problems of related noncooperative strategic games, in which the strategy sets of the players are nonempty closed and convex subsets in Banach spaces. As application of this existence to economics, an example is provided to study the existence of split Nash equilibrium of utilities of two related economies.

(XXIII) Yu, Wang, and Yang [42] in 2016: We show that the Kakutani and Brouwer fixed point theorems can be obtained by directly using the Nash equilibrium theorem. The corresponding set-valued problems, such as the Kakutani fixed point theorem, Walras equilibrium theorem (set-valued excess demand function), and generalized variational inequality, can be derived from the Nash equilibrium theorem, with the aid of an inverse of the Berge maximum theorem. For the single-valued situation, we derive the Brouwer fixed point theorem, Walras equilibrium theorem (single-valued excess demand function), KKM lemma, and variational inequality from the Nash equilibrium theorem directly, without any recourse.

(XXIV) Musin [21] in 2017: We consider generalizations of Gale's colored KKM lemma and Shapley's KKMS theorem. It is shown that spaces and covers can be much more general and the boundary KKM rules can be substituted by more weaker boundary assumptions.

(XXV) Petri and Voorneveld [34] in 2018: We give an elementary proof of Brouwer's fixed-point theorem. The only mathematical prerequisite is a version of the Bolzano-Weierstrass theorem: a sequence in a compact subset of n-dimensional Euclidean space has a convergent subsequence with a limit in that set. Our main tool is a 'no-bullying' lemma for agents with preferences over indivisible goods. What does this lemma claim? Consider a finite number of children, each with a single indivisible good (a toy) and preferences over those toys. Let us say that a group of children, possibly after exchanging toys, could bully some poor kid if all group members find their own current toy better than the toy of this victim. The no-bullying lemma asserts that some group S of children can redistribute their toys among themselves in such a way that all members of S get their favorite toy from S, but they cannot bully anyone.

(XXVI) Shimalo [35] in 2018 gave a new combinatorial labeling lemma, generalizing Sperner's lemma, and is used it to derive a simple proof for Kakutani's fixed point theorem. The proof is constructive and can be easily applied to numerically approximate the location of fixed points.

(XXVII) Le et al. [16] in 2020 showed that Sperner lemma implies the Brouwer and Kakutani fixed point theorems and the Gale-Nikaido-Debreu lemma.. Additionally, Kakutani theorem is shown as a corollary of Gale-Nikaido-Debreu lemma. Moreover, they provide another proof on the existence of a general equilibrium using only Sperner lemma and without a need to call on the fixed point theorems or the lemma.

(XXVIII) In 2020, Park [30] generalizes his 1991 theorem [23] to the class of half-continuous multifunctions due to Bich in 2006 and shows that this new result implies and improves fixed point results of Termwuttipong and Kaewtem in 2010 and Park in 1992. Consequently, a large number of generalizations or equivalents of the Brouwer theorem are unified.

(XXIX) Bueno and Cortina [3] in 2021: We study the existence of projected solutions for generalized Nash equilibrium problems defined in Banach spaces, under mild convexity assumptions for each loss function and without lower semicontinuity assumptions on the constraint maps. Our approach is based on Himmelberg's fixed point theorem. As a consequence, we also obtain existence of projected solutions for quasi-equilibrium problems and quasi-variational inequalities. Finally, we show the existence of projected solutions for Single-Leader–Multi-Follower games.

(XXX) Fierro [9] in 2021: We extend, to the framework of topological vector spaces, two results by Horvath and Kuratowski related to conditions for a family of closed sets to have compact and nonempty intersection. This extension enables us to introduce a number of applications such as the existence of maximal elements in preordered spaces, issues related to KKM functions, fixed point theorems, a variant of a matching theorem by Fan, and mainly the improvement of some minimax and variational inequalities.

(XXXI) Idzik, Kulpa, and Mackowiak [14] in 2021: Equivalents of the Brouwer fixed point theorem are proved. They involve formulations either for the standard simplex or for the cube. Characterizations of continuous functions defined on the standard simplex are also presented. The famous Steinhaus chessboard theorem is generalized.

In Section 3 we provide some equivalent versions of the labeled Sperner lemma and related results due to Bapat and Gale. In Section 4 we prove that an indexed closed (open) cover theorem, the Eilenberg-Otto theorem, the Poincaré theorem and the Bohl-Brouwer theorem are equivalent. Next, we present properties of continuous functions defined on the standard closed simplex. We also prove a lemma on the collapse. Finally, we generalize the Steinhaus chessboard theorem and the Gale theorem on hexagonal tiling, for an arbitrary finite tiling of the square.

(XXXII) Muu and Le [22] in 2021: The equilibrium problem defined by the Nikaido-Isoda-Fan inequality contains a number of problems such as optimization, variational inequality, Kakutani fixed point, Nash equilibria, and others as special cases. This paper presents a picture for the relationship between the fixed points of the Moreau proximal mapping and the solutions of the equilibrium problem that satisfies some kinds of monotonicity and Lipschitz-type condition.

(XXXIII) Park [31] in 2021: This is an enlarged version of our previous "Ninety years of the Brouwer fixed point theorem" [1999].

(XXXIV) Cortina and Fierro [6] in 2022: We deal with direct and inverse maximum theorems. Alternative versions to the Berge theorem are provided, by relaxing the compactness condition of the constraint correspondence. Also, inverse maximum theorems are introduced. These are of two types, according to their generality. First, we consider the framework consisting of topological spaces without linear structure and, on the other hand, the convex case, i.e., when the range space is a vector space is separately considered. By means of one of our inverse maximum theorem, we generalize a corresponding result by Komiya. In the field of applications, we prove the equivalences of some remarkable results existing in the literature.

5. New equivalents due to Jinlu Li

Recently Jinlu Li [18, 19] obtained two new fixed point theorems extending the Tychonoff fixed point theorem based on the 1961 KKM lemma of Ky Fan [8]. In this section, we show that these new theorems also are equivalents of the Brouwer theorem. Similarly, any paper listed in Section 4 can be used to obtain such equivalents, but it takes time.

In 2022, Li [19] defined that a topological vector space (X, τ) has the *fixed point property* if every non-empty, compact and convex subset of X has the fixed point property in the usual sense.

Theorem 5.1. (Brouwer, An alternative version) *Every n-dimension Euclidean space has the fixed point property.*

In [19], Li introduce a new concept of quasi-point-separable topological vector spaces, which is a generalization of pseudonorm adjoint topological vector spaces [18]. It immediately implies that it is a proper extension of locally convex topological vector spaces.

Definition. ([19]) Let X be a vector space with origin θ. A quasiconvex function $u : X \to \mathbb{R}^+$ is said to be *m-quasiconvex* if it satisfies $u(\theta) = 0$, and, for any elements $x_1,\ x_2$ of X, and $0 \leq \alpha \leq 1$, one has

$$u(\alpha x_1 + (1-\alpha)x_2) \leq \mathrm{Max}\{u(x_1), u(x_2)\}.$$

Definition. ([19]) Let (X, τ) be a topological vector space with dual space X^* (the space of linear and τ-continuous functions on X).

(i) If there is a subset $V \subseteq X^*$, such that, for $x \in X$,

$$v(x) = 0,\ \text{for every } v \in V,\ \text{implies } x = \theta,$$

then (X, τ) is said to be point-separable (by V) (or the set V is said to be total on X). Meanwhile, the set V is called a point-separating space for this vector space X.

(ii) Suppose that a topological vector space (X, τ) is equipped with a family of τ-continuous m-quasiconvex functions $\{u_\lambda\}_{\lambda\in\Lambda}$, where Λ is an index set. If, for $x \in X$,

$$u(x) = 0,\ \text{for every } \lambda \in \Lambda,\ \text{implies } x = \theta,$$

then (X, τ) is said to be quasi-point-separable (or the family $\{u_\lambda\}_{\lambda\in\Lambda}$ is said to be total on X). Or we say that X is quasi-point-separated by the family $\{u_\lambda\}_{\lambda\in\Lambda}$. Meanwhile, the family $\{u_\lambda\}_{\lambda\in\Lambda}$ is called a quasi-point-separating space for X.

The ideas of the proof of the main theorem of [19] is similar to the proof of the main theorem in [18]. As mentioned as in the proof of the following theorem, we see that the quasi-point-separable property of topological vector spaces plays the key factor for the considered spaces to have the fixed point property.

Theorem 5.2. ([19]) *Every quasi-point-separable Hausdorff topological vector space has the fixed point property.*

Note that Jinlu Li based the proof on the 1961 KKM Lemma of Ky Fan [8].

In [23], Park studied fixed point problems on point-separable topological vector spaces for some classes of functions, such as half-continuous functions. Here, Li used Theorem 5.2 to obtain the following fixed point theorem. Since every Hausdorff locally convex topological vector space is a point-separable topological vector space, the following theorem is also an extension of the Tychonoff fixed point theorem to quasi-point-separable topological vector spaces.

Theorem 5.3. *Every point-separable topological vector space has the fixed point property.*

Theorem 5.4. ([18]) (an alternative version) *Every Hausdorrf and total pseudonorm adjoint topological vector space has the fixed point property.*

Theorem 5.5. (Tychonoff, An alternative version) *Every Hausdorff locally convex topological vector space has the fixed point property.*

Note the following circular tour:

$$\text{Brouwer Theorem 5.1} \Longrightarrow \text{KKM Theorem} \Longrightarrow \text{Fan's 1961 Lemma}$$
$$\Longrightarrow \text{Li's Theorem 5.2} \Longrightarrow \text{Theorem 5.3 or Theorem 5.4}$$
$$\Longrightarrow \text{Tychonoff Theorem 5.5} \Longrightarrow \text{Brouwer Theorem 5.1.}$$

6. Conclusion

This article aims to let the readers know that there are hundreds of equivalent formulations of the Brouwer fixed point theorem and that there would be more and more such formulations.

Finally, in this section, we add some episodes as follows:

(1) It is quite interesting that Brouwer himself denied the truthness of his theorem later based on his intuitionism philosophy; see [24, 31].

(2) Since Hahn-Banach theorem follows from the KKM theorem, its numerous applications follow the Grand KKM Theory; see [32].

(3) Wikipedia covers several topics closely related to this article. For example, Brouwer fixed point theorem, Hex (board game), Kakutani fixed point theorem, Knaster-Kuratowski-Mazurkiewicz theorem, Nash equilibrium, Poincaré-Miranda theorem, Schauder fixed point theorem, Sperner's lemma, Tucker's lemma, etc.

(4) One of the most impressive publication is "Consequences of Axiom of Choice" by P. Howard and J. E. Rubin (*in* Mathematical Surveys and Monographs, vol. 59, Amer. Math. Soc. 1998, 432pp.). If we could edit "Consequences of the Brouwer Fixed Point Theorem", then it would need more than one thousand pages.

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