Results in Nonlinear Analysis 5 (2022) No. 1, 72–77.
https://doi.org/10.53006/rna.1086597
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

Remarks on saddle points of vector-valued functions

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Abstract

Using a coincidence theorem for multimaps, we prove the existence of a saddle point for vector-valued functions in topological vector spaces by means of scalarized maps. Moreover, we discuss minimax theorems as a consequence of the saddle point theorem for real-valued functions.

Keywords: saddle point, minimax theorem, coincidence theorem, upper (lower) semicontinuity, multimap.
2010 MSC: 49J35, 54H25, 54C60.

1. Introduction

Minimax theorem which goes back to von Neumann [10] plays an important role in optimization theory and game theory. The minimax theorem assures the existence of a saddle point under certain conditions, where the proof of the theorem is mainly based on a fixed point theorem for multimaps; see [4, 18] for instance. In game theory, the optimal strategies for two persons are described by saddle points.

Nieuwenhuis [11] introduced the notion of a saddle point for vector-valued functions in finite dimensional spaces. Various existence results on cone saddle points of vector-valued functions were established in [1, 14, 16, 17]. Loose saddle point theorems for multimaps via scalarization were considered in [6, 9]. Some ε-saddle point and saddle point theorems based on the coincidence theorem were investigated in [8, 13, 15]. For minimax problems on vector-valued functions, see [2, 3, 5].

In the present paper, we study a saddle point theorem for vector-valued functions in topological vector spaces by means of scalarized maps, motivated by the work [6]. The main tool is a coincidence theorem for multimaps due to Park and Kim [12]. Actually, the idea is to first find a saddle point for scalarized maps which can be transformed into a saddle point for given vector-valued functions. It is remarkable that the condition "compact convex set" in the underlying space is replaced by weaker condition "compact set" or "convex set". Moreover, we discuss some minimax theorems as a consequence of the saddle point theorem for real-valued functions.

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Received February 04, 2022, Accepted March 9, 2022, Online March 11, 2022

2. Preliminaries

A *multimap* $F : X \multimap Y$ is a function from a set X into the set of all nonempty subsets of a set Y. For topological spaces X and Y, a multimap $F : X \multimap Y$ is said to be *upper semicontinuous* if the set $\{x \in X : Fx \subset A\}$ is open in X for each open set A in Y.

A function $f : X \to \mathbb{R}$ on a topological space X is said to be *lower semicontinuous* if the set $\{x \in X : f(x) > \alpha\}$ is open in X for every real number α; and *upper semicontinuous* if the set $\{x \in X : f(x) < \alpha\}$ is open in X for every real number α.

Let E be a real topological vector space with a partial order $\leq$; that is, a reflexive transitive binary relation. Let A be a nonempty set in E. A point $a_0 \in A$ is said to be a *minimal point* of A if for any $a \in A$, $a \leq a_0$ implies $a = a_0$; and a *maximal point* of A if for any $a \in A$, $a_0 \leq a$ implies $a = a_0$. The set of minimal [resp. maximal] points of A is denoted by $\min A$ [resp. $\max A$].

Let f be a vector-valued function from a product $X \times Y$ to E. For $x \in X$ and $y \in Y$ we write $f(X, y) := \{f(x, y) : x \in X\}$ and $f(x, Y) := \{f(x, y) : y \in Y\}$. A point $(x_0, y_0) \in X \times Y$ is said to be a *saddle point* of f on $X \times Y$ if $f(x_0, y_0) \in \max f(X, y_0) \cap \min f(x_0, Y)$.

Let f be a real-valued function defined on the product $X \times Y$. A point (x_0, y_0) is said to be a *saddle point* of f on $X \times Y$ if $f(x, y_0) \leq f(x_0, y_0) \leq f(x_0, y)$ for all $x \in X$ and all $y \in Y$.

Let E be a real topological vector space with a partial order $\leq$. A real-valued function $g : E \to \mathbb{R}$ is said to be *strictly monotone* if $g(a) < g(b)$ for $a < b$, where $a < b$ means $a \leq b$ and $a \neq b$.

A nonempty topological space is said to be *acyclic* if all of its reduced Čech homology groups over rationals vanish. Note that every nonempty convex or star-shaped subset of a topological vector space is acyclic.

As a key tool of our main result, we need the following coincidence theorem which is a particular case of [12, Theorem 1]; see [7, Theorem 1].

Theorem 2.1. *Let X be a nonempty convex set in a topological vector space and Y a Hausdorff compact topological space. Suppose that $A : X \multimap Y$ is an upper semicontinuous multimap with closed acyclic values and $B : Y \multimap X$ has convex values and open fibers, that is, $B^- x = \{y \in Y : x \in By\}$ is open in Y for each $x \in X$. Then there exist points $x_0 \in X$ and $y_0 \in Y$ such that $y_0 \in Ax_0$ and $x_0 \in By_0$.*

The following elementary result on semicontinuity will be often used, taken from [7, Lemma]; see [6, Lemma 3.1].

Lemma 2.2. *Let X and Y be topological spaces and $f : X \times Y \to \mathbb{R}$ a real-valued function on the product space $X \times Y$. Then the following statements hold:*

(a) If $f(x, \cdot)$ is bounded below on Y for each $x \in X$ and $f(\cdot, y)$ is upper semicontinuous on X for each $y \in Y$, then the real-valued function $g : X \to \mathbb{R}$, $g(x) := \inf f(x, Y)$, is upper semicontinuous on X.
(b) If $f(\cdot, y)$ is bounded above on X for each $y \in Y$ and $f(x, \cdot)$ is lower semicontinuous on Y for each $x \in X$, then the function $h : Y \to \mathbb{R}$, $h(y) := \sup f(X, y)$, is lower semicontinuous on Y.
(c) If f is lower semicontinuous on $X \times Y$ and $f(\cdot, y)$ is upper semicontinuous on X for each $y \in Y$ and if Y is a Hausdorff compact space, then the multimap $F : X \multimap Y$, $Fx := \{y \in Y : f(x, y) = \min f(x, Y)\}$, is upper semicontinuous on X.

3. Main Result

In this section, we study a saddle point theorem for vector-valued functions, where the main method is to use the coincidence theorem stated in the previous section.

Motivated by the work [6], we prove the existence of a saddle point for vector-valued functions via scalarization.

Theorem 3.1. *Let X be a nonempty convex set in a Hausdorff topological vector space, Y a nonempty compact set in a Hausdorff topological vector space, and E a topological vector space with a partial order $\leq$. Let $f : X \times Y \to E$ be a vector-valued function on the product space $X \times Y$. Suppose that there is a strictly monotone function $g : E \to \mathbb{R}$ such that*

(1) $g \circ f$ is lower semicontinuous on $X \times Y$;
(2) $g \circ f(\cdot, y)$ is bounded above and upper semicontinuous on X for each $y \in Y$;
(3) $g \circ f(x, \cdot) - \sup g \circ f(X, \cdot)$ is lower semicontinuous on Y for each $x \in X$;
(4) $g \circ f(\cdot, y)$ is quasiconcave for each $y \in Y$;
(5) $\{y \in Y : g \circ f(x, y) = \min g \circ f(x, Y)\}$ is acyclic for each $x \in X$; and
(6) for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, there exist a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ and a point $\hat{x} \in X$ such that

$$g \circ f(\hat{x}, y) \geq \limsup_{k\to\infty} g \circ f(x_{n_k}, y) \qquad \text{for all } y \in Y.$$

Then f has a saddle point (x_0, y_0) in $X \times Y$.

Proof. Let $A : X \multimap Y$ be a multimap defined by

$$Ax := \{y \in Y : g \circ f(x, y) = \min g \circ f(x, Y)\} \qquad \text{for } x \in X.$$

By Lemma 2.2(c), the multimap A is upper semicontinuous on X and has closed acyclic values by assumption (5).

Let $h : Y \to \mathbb{R}$ be a real-valued function defined by

$$h(y) := \sup g \circ f(X, y) \qquad \text{for } y \in Y.$$

Since $g \circ f(x, \cdot)$ is lower semicontinuous on Y for each $x \in X$, Lemma 2.2(b) implies that the function h is lower semicontinuous on Y.

For each $n \in \mathbb{N}$, let $B_n : Y \multimap X$ be a multimap defined by

$$B_n y := \left\{x \in X : g \circ f(x, y) > h(y) - \frac{1}{n}\right\} \qquad \text{for } y \in Y.$$

Then it follows from assumptions (3) and (4) that the multimap B_n has convex values and open fibers. By Theorem 2.1, there exist points $x_n \in X$ and $y_n \in Y$ such that $y_n \in Ax_n$ and $x_n \in B_n y_n$. This implies

$$g \circ f(x_n, y) \geq g \circ f(x_n, y_n) > h(y_n) - \frac{1}{n} \qquad \text{for all } y \in Y. \tag{3.1}$$

In view of assumption (6), we can choose a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ and a point $x_0 \in X$ such that

$$g \circ f(x_0, y) \geq \limsup_{k\to\infty} g \circ f(x_{n_k}, y) \qquad \text{for all } y \in Y. \tag{3.2}$$

By the compactness of Y, there are a subnet $\{y_\alpha\}$ of $\{y_{n_k}\}_{k\in\mathbb{N}}$ and a point $y_0 \in Y$ which converges to y_0. Note that $\min g \circ f(x_0, Y) = g \circ f(x_0, \hat{y})$ for some $\hat{y} \in Y$. Since h is lower semicontinuous on Y, we obtain from (3.1) and (3.2) that

$$\begin{aligned} g \circ f(x_0, y_0) &\geq \min g \circ f(x_0, Y) = g \circ f(x_0, \hat{y}) \\ &\geq \limsup_{k\to\infty} g \circ f(x_{n_k}, \hat{y}) \geq \limsup_{\alpha} g \circ f(x_\alpha, \hat{y}) \\ &\geq \liminf_{\alpha} \left(h(y_\alpha) - \alpha^{-1}\right) \geq h(y_0) \\ &\geq g \circ f(x_0, y_0), \end{aligned}$$

which implies

$$g \circ f(x_0, y_0) = \min g \circ f(x_0, Y) = h(y_0) = \max g \circ f(X, y_0). \tag{3.3}$$

Thus, (x_0, y_0) is a saddle point of the real-valued function $g \circ f : X \times Y \to \mathbb{R}$. Since g is strictly monotone, we obtain from (3.3) that

$$f(x_0, y_0) \in \min f(x_0, Y) \quad \text{and} \quad f(x_0, y_0) \in \max f(X, y_0).$$

In fact, if $f(x_0, y_0)$ is not a minimal point of $f(x_0, Y)$, then $w < f(x_0, y_0)$ for some $w \in f(x_0, Y)$ and hence by the strict monotonicity $g(w) < g \circ f(x_0, y_0)$, which contradicts the fact that $g \circ f(x_0, y_0) = \min g \circ f(x_0, Y)$. A similar argument shows that $f(x_0, y_0) \in \max f(X, y_0)$. Therefore, $(x_0, y_0) \in X \times Y$ is a saddle point of the vector-valued function f. This completes the proof. □

Remark 3.2. In Theorem 3.1, it is emphasized that the hypothesis "compact convex set" in [6] was replaced by "compact set" or "convex set".

The following result is the real-valued function case of Theorem 3.1. When X is a generalized convex space, it was studied in [7, Theorem 2].

Corollary 3.3. *Let X be a nonempty convex set in a Hausdorff topological vector space and Y a nonempty compact set in a Hausdorff topological vector space. Suppose that $f : X \times Y \to \mathbb{R}$ is a lower semicontinuous function on $X \times Y$ such that*

(1) $f(\cdot, y)$ is bounded above and upper semicontinuous on X for each $y \in Y$;
(2) $f(x, \cdot) - \sup f(X, \cdot)$ is lower semicontinuous on Y for each $x \in X$;
(3) $f(\cdot, y)$ is quasiconcave for each $y \in Y$;
(4) $\{y \in Y : f(x, y) = \min f(x, Y)\}$ is acyclic for each $x \in X$; and
(5) for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, there are a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ and a point $\hat{x} \in X$ such that

$$f(\hat{x}, y) \geq \limsup_{k\to\infty} f(x_{n_k}, y) \qquad \text{for all } y \in Y.$$

Then f has a saddle point $(x_0, y_0) \in X \times Y$.

Proof. Apply Theorem 3.1 with $E = \mathbb{R}$ and $g = id$, the identity map. □

Remark 3.4. Corollary 3.3 holds true when assumption (5) is replaced by

(5′) $\{f(x, \cdot) : x \in X\}$ is sequentially compact in which convergence is uniform on Y.

This means that for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, there exist a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ and a point $\hat{x} \in X$ such that $\{f(x_{n_k}, \cdot)\}$ converges to $f(\hat{x}, \cdot)$ uniformly on Y.

Next we give a minimax theorem as a consequence of the saddle point theorem for real-valued functions; see [7, Theorem 4].

Theorem 3.5. *Under the assumptions of Corollary 3.3, we have*

$$\min_{y\in Y} \sup_{x\in X} f(x, y) = \sup_{x\in X} \min_{y\in Y} f(x, y).$$

Proof. Corollary 3.3 implies that there is a point (x_0, y_0) of $X \times Y$ such that

$$\sup_{x\in X} f(x, y_0) = f(x_0, y_0) = \min_{y\in Y} f(x_0, y). \tag{3.4}$$

From Lemma 2.2 we know that $\sup f(X, \cdot)$ is lower semicontinuous on the compact set Y. Obviously, $\min_{y\in Y} \sup_{x\in X} f(x, y)$ exists and $\min f(x, Y)$ exists for each $x \in X$. We have

$$\min_{y\in Y} \sup_{x\in X} f(x, y) \geq \min f(x, Y) \qquad \text{for every } x \in X.$$

This implies

$$\min_{y\in Y}\sup_{x\in X} f(x,y) \geq \sup_{x\in X}\min_{y\in Y} f(x,y).$$

On the other hand, it follows from (3.4) that

$$\min_{y\in Y}\sup_{x\in X} f(x,y) \leq \sup_{x\in X} f(x,y_0) = \min_{y\in Y} f(x_0,y) \leq \sup_{x\in X}\min_{y\in Y} f(x,y).$$

The proof is complete. □

Corollary 3.6. *Let X be a nonempty compact convex set in a Hausdorff topological vector space and Y a nonempty compact set in a Hausdorff topological vector space. Suppose that $f : X \times Y \to \mathbb{R}$ is a lower semicontinuous function on $X \times Y$ such that*

(1) $f(\cdot, y)$ is upper semicontinuous on X for each $y \in Y$;
(2) $f(x, \cdot) - \max f(X, \cdot)$ is lower semicontinuous on Y for each $x \in X$;
(3) $f(\cdot, y)$ is quasiconcave for each $y \in Y$;
(4) $\{y \in Y : f(x,y) = \min f(x,Y)\}$ is acyclic for each $x \in X$; and
(5′) $\{f(x,\cdot) : x \in X\}$ is sequentially compact in which convergence is uniform on Y.

Then we have the minimax theorem

$$\min_{y\in Y}\max_{x\in X} f(x,y) = \max_{x\in X}\min_{y\in Y} f(x,y).$$

Proof. In view of Lemma 2.2, $\max f(X,\cdot)$ is lower semicontinuous on the compact set Y and $\min f(\cdot, Y)$ is upper semicontinuous on the compact set X. It is clear that $\min_{y\in Y}\max_{x\in X} f(x,y)$ and $\max_{x\in X}\min_{y\in Y} f(x,y)$ exist. We conclude by Theorem 3.5 that

$$\min_{y\in Y}\max_{x\in X} f(x,y) = \max_{x\in X}\min_{y\in Y} f(x,y).$$

□

Corollary 3.7. *Let X and Y be nonempty compact convex sets in Hausdorff topological vector spaces, respectively. Suppose that $f : X \times Y \to \mathbb{R}$ is a continuous function on $X \times Y$ such that*

(1) $f(\cdot, y)$ is quasiconcave for each $y \in Y$;
(2) $f(x, \cdot)$ is quasiconvex for each $x \in X$; and
(3) $\{f(x,\cdot) : x \in X\}$ is sequentially compact in $C(Y)$, where $C(Y)$ denotes the Banach space of all continuous real-valued functions defined on Y equipped with the supremum norm.

Then we have

$$\min_{y\in Y}\max_{x\in X} f(x,y) = \max_{x\in X}\min_{y\in Y} f(x,y).$$

Proof. Notice that if f is continuous on $X \times Y$ and X is compact, then the function $h : Y \to \mathbb{R}$, $h(y) := \max f(X,y)$, is continuous on Y; see [6, Lemma 2.2]. Moreover, it follows from the quasiconvexity of $f(x,\cdot)$ that the set $\{y \in Y : f(x,y) = \min f(x,Y)\}$ is convex for each $x \in X$. Since all the assumptions of Corollary 3.6 are satisfied, this implies

$$\min_{y\in Y}\max_{x\in X} f(x,y) = \max_{x\in X}\min_{y\in Y} f(x,y).$$

□

We have shown a saddle point theorem for vector-valued functions in topological vector spaces which implies minimax theorems for real-valued functions. For minimax theorems on convex sets in topological vector spaces, see [4, Theorem 4].

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