Fixed Point, Coincidence Point and Common Fixed Point Theorems under Various Expansive Conditions in Parametric Metric Spaces and Parametric b-Metric Spaces

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ABSTRACT

In this article, we establish some fixed point, common fixed point and coincidence point theorems for expansive type mappings in parametric metric spaces and parametric b-metric spaces. The presented theorems extend, generalize and improve many existing results in the literature. Also, we introduce some examples the support the validity of our results.

Keywords: Coincidence point, fixed point, common fixed point, parametric metric space, parametric b-metric space, weakly compatible maps.

1. INTRODUCTION

Fixed point theory has fascinated many researchers since 1922 with the celebrated Banach's fixed point theorem. There exists a vast literature on the topic and is a very active field of research at present. Theorems concerning

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the existence and properties of fixed points are known as fixed point theorems. Such theorems are very important tool for proving the existence and eventually the uniqueness of the solutions to various mathematical models (integral and partial differential equations, variational inequalities).

The concept of metric spaces has been generalized in many directions. The notion of a b-metric space was studied by Czerwik in [25-26] and a lot of fixed point results for single and multivalued mappings by many authors have been obtained in (ordered) b-metric spaces (see, e.g., [27]-[28]). The concept of fuzzy set was introduced by Zadeh [1] in 1965. In 1975, Kramosil and Michalek [2] introduced the notion of fuzzy metric space, which can be regarded as a generalization of the statistical (probabilistic) metric space. This work has provided an important basis for the construction of fixed point theory in fuzzy metric spaces.

In 2004, Park introduced the notion of intuitionistic fuzzy metric space [12]. He showed that, for each intuitionistic fuzzy metric space \((X, M, N, \ast, *)\), the topology generated by the intuitionistic fuzzy metric \((M, N)\) coincides with the topology generated by the fuzzy metric \(M\). For more details on intuitionistic fuzzy metric space and related results we refer the reader to [12-20]. The study of expansive mappings is a very interesting research area in fixed point theory. In 1984, Wang et al. [37] introduced the concept of expanding mappings and proved some fixed point theorems in complete metric spaces. In 1992, Daffer and Kaneko [36] defined an expanding condition for a pair of mappings and proved some common fixed point theorems for two mappings in complete metric spaces. Chintaman and Jagannath [38] introduced several meaningful fixed point theorems for one expanding mapping.

In this paper, we present some new fixed point, coincidence point and common fixed point theorems under various expansive conditions in parametric metric spaces and parametric b-metric spaces. These results improve and generalize some important known results in [29-38]. Some related results and illustrative some examples to highlight the realized improvements are also furnished.

2. DEFINITIONS AND PRELIMINARIES

Throughout this paper \(\mathbb{R}\) and \(\mathbb{R}^+\) will represents the set of real numbers and nonnegative real numbers, respectively.

In 2014, Hussain et al. [16] defined and studied the concept of parametric metric space as follows.

**Definition 2.1** Let \(X\) be a nonempty set and \(\mathcal{P} : X \times X \times (0, +\infty) \to [0, +\infty)\) be a function. We say \(\mathcal{P}\) is a parametric metric on \(X\) if,

1. \(\mathcal{P}(x, y, t) = 0\) for all \(t > 0\) if and only if \(x = y\);  
2. \(\mathcal{P}(x, y, t) = \mathcal{P}(y, x, t)\) for all \(t > 0\);  
3. \(\mathcal{P}(x, y, t) \leq \mathcal{P}(x, z, t) + \mathcal{P}(z, y, t)\) for all \(x, y, z \in X\) and all \(t > 0\).

and one says the pair \((X, \mathcal{P})\) is a parametric metric space.

The following definitions are required in the sequel which can be found in [16].

**Definition 2.2** Let \(\{x_n\}_{n=1}^\infty\) be a sequence in a parametric metric space \((X, \mathcal{P})\).

1. \(\{x_n\}_{n=1}^\infty\) is said to be convergent to \(x \in X\), written as \(\lim_{n \to \infty} x_n = x\), for all \(t > 0\), if \(\lim_{n \to \infty} \mathcal{P}(x_n, x, t) = 0\).
2. \(\{x_n\}_{n=1}^\infty\) is said to be a Cauchy sequence in \(X\) if for all \(t > 0\), if \(\lim_{m,n \to \infty} \mathcal{P}(x_n, x_m, t) = 0\).
3. \((X, \mathcal{P})\) is said to be complete if every Cauchy sequence is a convergent sequence.

**Definition 2.3** Let \((X, \mathcal{P})\) be a parametric metric space and \(T : X \to X\) be a mapping. We say \(T\) is a continuous mapping at \(x \in X\), if for any sequence \(\{x_n\}_{n=1}^{\infty}\) in \(X\) such that \(\lim_{n \to \infty} x_n = x\), then \(\lim_{n \to \infty} T x_n = T x\).

**Example 2.4** Let \(X\) denote the set of all functions \(f : (0, +\infty) \to \mathbb{R}\). Define \(\mathcal{P} : X \times X \times (0, +\infty) \to [0, +\infty)\) by

\[
\mathcal{P}(f, g, t) = |f(t) - g(t)|
\]

\(\forall f, g \in X\) and all \(t > 0\). Then \(\mathcal{P}\) is a parametric metric on \(X\) and the pair \((X, \mathcal{P})\) is a parametric metric space.

Very recently, Hussain et al. [21] introduced the concept of parametric b-metric space as follows.

**Definition 2.5** Let \(X\) be a nonempty set, \(s \geq 1\) be a real number and \(\mathcal{P} : X \times X \times (0, +\infty) \to [0, +\infty)\) be a function. We say \(\mathcal{P}\) is a parametric b-metric on \(X\) if,

1. \(\mathcal{P}(x, y, t) = 0\) for all \(t > 0\) if and only if \(x = y\);  
2. \(\mathcal{P}(x, y, t) = \mathcal{P}(y, x, t)\) for all \(t > 0\);  
3. \(\mathcal{P}(x, y, t) \leq s[\mathcal{P}(x, z, t) + \mathcal{P}(z, y, t)]\) for all \(x, y, z \in X\) and all \(t > 0\), where \(s \geq 1\).

and one says the pair \((X, \mathcal{P}_s)\) is a parametric metric space with parameter \(s \geq 1\).

Obviously, for \(s = 1\), parametric b-metric reduces to parametric metric.

The following definitions will be needed in the sequel which can be found in [21].

**Definition 2.6** Let \(\{x_n\}_{n=1}^{\infty}\) be a sequence in a parametric b-metric space \((X, \mathcal{P}_s)\).
1. \( \{x_n\}_{n=1}^{\infty} \) is said to be convergent to \( x \in X \), written as \( \lim_{n \to \infty} x_n = x \), for all \( t > 0 \), if 
\[ \lim_{n \to \infty} \mathcal{P}(x_n, x, t) = 0. \]
2. \( \{x_n\}_{n=1}^{\infty} \) is said to be a Cauchy sequence in \( X \) for all \( t > 0 \), if \( \lim_{n,m \to \infty} \mathcal{P}(x_n, x_m, t) = 0. \)
3. \( (X, \mathcal{P}) \) is said to be complete if every Cauchy sequence in \( X \) is a convergent sequence.

**Example 2.6** Let \( X = [0, +\infty) \) and define \( \mathcal{P} : X \times X \times (0, +\infty) \to [0, +\infty) \) by \( \mathcal{P}(x, y, t) = t(x - y)^2 \). Then \( \mathcal{P} \) is a parametric \( b \)-metric with constant \( s = 2t \). In fact, we only need to prove (3) in Definition 2.5 as follows: let \( x, y, z \in X \). Set \( u = x - z, v = z - y \), so \( u + v = x - y \). From the inequality \( (a + b)^2 \leq (2 \max(a, b))^2 \leq 2^2(a^p + b^p) \), \( \forall a, b \geq 0 \), we have
\[ \mathcal{P}(x, y, t) = t(x - y)^2 \]
\[ = t(u + v)^2 \]
\[ \leq 2^2 t(a^p + b^p) \]
\[ = 2^2 t(x - z)^2 + t(z - y)^2 \]
\[ = s^2 (\mathcal{P}(x, z, t) + \mathcal{P}(z, y, t)) \]
with \( s = 2t > 1 \).

**Definition 2.7** Let \( (X, \mathcal{P}, s) \) be a parametric \( b \)-metric space and \( T : X \to X \) be a mapping. We say \( T \) is a continuous mapping at \( x \in X \), if for any sequence \( \{x_n\}_{n=1}^{\infty} \) in \( X \) such that \( \lim_{n \to \infty} x_n = x \), then \( \lim_{n \to \infty} T x_n = T x \).

In general, a parametric \( b \)-metric function for \( s > 1 \) is not jointly continuous in all its variables.

3. FIXED POINT RESULTS IN PARAMETRIC METRIC SPACE

In this section, we first prove some unique fixed point results satisfying expansive condition by considering surjective self-mapping in the context of parametric metric space.

We begin with a simple but a useful lemma.

**Lemma 3.1** Let \( \{x_n\}_{n=1}^{\infty} \) be a sequence in a parametric metric space \( (X, \mathcal{P}) \) such that

(1) \( \mathcal{P}(x_n, x_{n+1}, t) \leq \lambda \mathcal{P}(x_{n-1}, x_n, t) \)

where \( \lambda \in [0, 1) \) and \( n = 1, 2, \ldots \). Then \( \{x_n\}_{n=1}^{\infty} \) is a Cauchy sequence in \( (X, \mathcal{P}) \).

**Proof** Let \( m > n \geq 1 \). It follows that

(2) \( \mathcal{P}(x_n, x_m, t) \leq \mathcal{P}(x_n, x_{n+1}, t) + \mathcal{P}(x_{n+1}, x_{n+2}, t) + \cdots + \mathcal{P}(x_{m-1}, x_m, t) \)
\[ \leq (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}) \mathcal{P}(x_0, x_1, t) \]
\[ = \frac{\lambda^n}{1-\lambda} \mathcal{P}(x_0, x_1, t) \]
for all \( t > 0 \). Since \( \lambda < 1 \). Assume that \( \mathcal{P}(x_0, x_1, t) > 0 \). By taking limit as \( m, n \to +\infty \) in above inequality we get

(3) \( \lim_{n,m \to \infty} \mathcal{P}(x_n, x_m, t) = 0. \)

Therefore, \( \{x_n\}_{n=1}^{\infty} \) is a Cauchy sequence in \( X \). Also, if \( \mathcal{P}(x_n, x_t, t) = 0 \), then \( \mathcal{P}(x_n, x_m, t) = 0 \) for all \( m > n \) and hence \( \{x_n\}_{n=1}^{\infty} \) is a Cauchy sequence in \( X \).

Now, our first main results as follows.

**Theorem 3.2** Let \( (X, \mathcal{P}) \) be a complete parametric metric space and \( T : X \to X \) be a surjection. Suppose that there exist \( a, b, c \geq 0 \) with \( a + b + c > 1 \) such that

(4) \( \mathcal{P}(T x, T y, t) \geq a \mathcal{P}(x, y, t) + b \mathcal{P}(x, T x, t) + c \mathcal{P}(y, T y, t) \)
\[ \forall x, y \in X \text{ with } x \neq y \text{ and all } t > 0. \]
Then \( T \) has a fixed point in \( X \).

**Proof** Under the assumption. It is clear that \( T \) is injective. Let \( G \) be the inverse mapping of \( T \). Choose \( x_0 \in X \) set \( x_1 = G(x_0), x_2 = G(x_1), \ldots, x_{n+1} = G(x_n) = G^{n+1}(x_0) \ldots \).

Without loss of generality, we assume that \( x_{n+1} \neq x_n \) for all \( n = 1, 2, \ldots \) (otherwise, if there exists some \( n_0 \) such that \( x_{n+1} = x_n \) then \( x_{n_0} \) is a fixed point of \( T \)).

It follows that from condition (4)

(5) \( \mathcal{P}(x_{n-1}, x_n, t) = \mathcal{P}(T T^{-1}x_{n-1}, T T^{-1}x_n, t) \)
\[ \geq a \mathcal{P}(T T^{-1}x_{n-1}, T T^{-1}x_n, t) \]
\[ + b \mathcal{P}(T T^{-1}x_{n-1}, T T^{-1}x_{n-1}, t) \]
\[ + c \mathcal{P}(T T^{-1}x_n, T T^{-1}x_n, t) \]
\[ = a \mathcal{P}(G x_{n-1}, G x_n, t) \]
\[ + b \mathcal{P}(G x_{n-1}, G x_{n-1}, t) \]
\[ + c \mathcal{P}(G x_n, x_n, t) \]
\[ = \mathcal{P}(x_{n-1}, x_n, t) + b \mathcal{P}(x_n, x_{n-1}, t) + c \mathcal{P}(x_n, x_n, t) \]
\[ \forall x, y \in X \text{ with } x \neq y \text{ and all } t > 0. \]

Hence

(6) \( (1 - b) \mathcal{P}(x_{n-1}, x_n, t) \geq a \mathcal{P}(x_{n-1}, x_n, t) \)

If \( a + c = 0 \), then \( b > 0 \). The above inequality implies that a negative number is greater then or equal to zero. This is impossible. So, \( a + c \neq 0 \) and \( (1 - b) > 0 \).

Therefore,

(7) \( \mathcal{P}(x_{n+1}, x_n, t) \leq k \mathcal{P}(x_n, x_{n-1}, t) \)

where \( k = \frac{1-b}{a+c} < 1 \) for all \( n \in \mathbb{N} \cup \{0\} \) and \( t > 0 \). Repeating (7) \( n \)-times, we get

(8) \( \mathcal{P}(x_{n+1}, x_n, t) \leq k^n \mathcal{P}(x_0, x, t) \)

for all \( t > 0 \). By Lemma 3.1, \( \{x_n\}_{n=1}^{\infty} \) is a Cauchy sequence. Since \( (X, \mathcal{P}) \) is a complete parametric metric space, there exists \( x^* \in X \) such that \( x_n \to x^* \) as \( n \to \infty \).
Now since \( T \) is surjective map. So there exists a point \( y \) in \( X \) such that \( x^* = Ty \). Consider
\[
(9) \quad \mathcal{P}(x_n, x^*, t) = \mathcal{P}(Tx_n, Ty, t)
\]
\[
\geq a \mathcal{P}(x_{n+1}, y, t) + b \mathcal{P}(x_n, Tx_{n+1}, t)
\]
\[
+ c \mathcal{P}(y, Ty, t)
\]
\[
= a \mathcal{P}(x_{n+1}, y, t) + b \mathcal{P}(x_{n+1}, x_n, t)
\]
\[
+ c \mathcal{P}(y, x^*, t)
\]
which implies that as \( n \to +\infty \)
\[
(10) \quad 0 \geq (a + c) \mathcal{P}(y, x^*, t)
\]
Hence \( y = x^* \). This gives that \( x^* \) is a fixed point of \( T \). This completes the proof.

Now we give an example illustrating Theorem 3.2.

**Example 3.3** Let \( X = [0, +\infty) \) be endowed with parametric metric,
\[
\mathcal{P}(x, y, t) = \begin{cases} 
|t - \max(x, y)|, & x \neq y \\
0, & x = y
\end{cases}
\]
for all \( x, y \in X \) and \( t > 0 \). Define \( T: X \to X \) by \( Tx = \frac{x + y}{2} \).

Obviously, \( T \) is continuous surjective map on \( X \). So for \( a = 4, b = -2 \) and \( c = \frac{3}{4} \) all the conditions of Theorem 3.2 are satisfied. Therefore \( x^* = 0 \) is the unique fixed point of \( T \).

Setting \( b = c \) and \( a = k \) in Theorem 3.2, we can obtain the following result.

**Corollary 3.4** Let \((X, \mathcal{P})\) be a complete parametric metric space and \( T: X \to X \) be a surjection. Suppose that there exist a constant \( k > 1 \) such that
\[
(11) \quad \mathcal{P}(Tx, Ty, t) \geq k \mathcal{P}(x, y, t)
\]
\( \forall x, y \in X \) and \( \forall t > 0 \). Then \( T \) has a unique fixed point in \( X \).

**Proof** From Theorem 3.2, it follows that \( T \) has a fixed point \( x^* \) in \( X \) by setting \( b = c = 0 \) and \( a = k \) in condition (4).

Uniqueness. Suppose that \( x^* \neq y^* \) is also another fixed point of \( T \), then from condition (11), we obtain
\[
(12) \quad \mathcal{P}(x^*, y^*, t) = \mathcal{P}(Tx^*, Ty^*, t)
\]
\[
\geq k \mathcal{P}(x^*, y^*, t)
\]
which implies \( \mathcal{P}(x^*, y^*, t) = 0 \), that is \( x^* = y^* \). This completes the proof.

**Corollary 3.5** Let \((X, \mathcal{P})\) be a complete parametric metric space and \( T: X \to X \) be a surjection. Suppose that there exist a positive integer \( n \) and a real number \( k > 1 \) such that
\[
(13) \quad \mathcal{P}(T^n x, T^n y, t) \geq k \mathcal{P}(x, y, t)
\]
\( \forall x, y \in X \) and all \( t > 0 \). Then \( T \) has a unique fixed point in \( X \).

**Proof** From Corollary 3.4, \( T^n \) has a fixed point \( x^* \). But \( T^n(Tx^*) = T^n(x^*) \). So \( Tx^* \) is also a fixed point of \( T^n \). Hence \( Tx^* = x^* \). \( x^* \) is a fixed point of \( T \). Since the fixed point of \( T \) is also fixed point of \( T^n \), the fixed point of \( T \) is unique.

**Theorem 3.6** Let \((X, \mathcal{P})\) be a complete parametric metric space and \( T: X \to X \) be a continuous surjection. If there exist a constant \( k > 1 \) such that for any \( x, y \in X \), there is
\[
(14) \quad \mathcal{M}_t(x, y) \in [\mathcal{P}(x, y, t), \mathcal{P}(x, Ty, t), \mathcal{P}(y, Ty, t)]
\]
satisfying
\[
(15) \quad \mathcal{P}(Tx, Ty, t) \geq k \mathcal{M}_t(x, y)
\]
\( \forall x, y \in X \) and all \( t > 0 \). Then \( T \) has a unique fixed point in \( X \).

**Proof** Similar to the proof of the Theorem 3.2, we can obtain a sequence \( \{x_n\}_{n=1}^\infty \) such that \( x_{n+1} = Tx_n \). Without loss of generality, we assume that \( x_{n+1} \neq x_n \) for all \( n = 1, 2, \ldots \) (otherwise, if there exists some \( n_0 \) such that \( x_{n_0+1} = x_{n_0} \), then \( x_{n_0} \) is a fixed point of \( T \)). It follows that from condition (15)
\[
(16) \quad \mathcal{P}(x_{n-1}, x_n, t) \geq k \mathcal{M}_t(x_n, x_{n+1})
\]
where
\[
\mathcal{M}_t(x_n, x_{n+1}) = \{\mathcal{P}(x_n, x_{n+1}, t), \mathcal{P}(x_n, x_{n-1}, t)\}
\]
Now we have to consider the following two cases:

- If \( \mathcal{M}_t(x_n, x_{n+1}) = \mathcal{P}(x_n, x_{n-1}, t) \) then from (14), we have
  \[
  \mathcal{P}(x_{n-1}, x_{n+1}, t) \geq k \mathcal{P}(x_n, x_{n-1}, t)
  \]
which implies \( \mathcal{P}(x_{n-1}, x_{n+1}, t) = 0 \) that is \( x_{n-1} = x_n \). This is a contradiction.

- If \( \mathcal{M}_t(x_n, x_{n+1}) = \mathcal{P}(x_n, x_{n+1}, t) \) then from (14), we have
  \[
  \mathcal{P}(x_{n-1}, x_{n+1}, t) \geq k \mathcal{P}(x_n, x_{n+1}, t)
  \]
Proceeding like Theorem 2.2, we obtain that \( \{x_n\}_{n=1}^\infty \) is a Cauchy sequence in complete parametric metric space \((X, \mathcal{P})\). So the sequence \( \{x_n\}_{n=1}^\infty \) converges to a point \( x^* \in X \). Since \( T \) is continuous, it is clear that \( x^* \) is a fixed point of \( T \). This is completes of the proof.

4. COMMON FIXED POINT RESULTS IN PARAMETRIC METRIC SPACE

In this section, now we establish that common fixed points for a pair of two weakly compatible self-mappings satisfying expansive condition are proved in the frame of parametric metric spaces.

**Definition 4.1** Let $S$ and $T$ be two self-mappings on a nonempty set $X$. Then $S$ and $T$ are said to be weakly compatible if they commute at all of their coincidence points; that is, $Sx = Tx$ for some $x \in X$ and then $STx = TSx$.

Now, let us prove our result.

**Theorem 4.2** Let $(X, \mathcal{P})$ be a complete parametric metric space. Let $S$ and $T$ be a weakly compatible mappings of $X$ and $T(X) \subseteq S(X)$. Suppose that there exist $k > 1$ such that

$$\mathcal{P}(Sx, Sy, t) \geq k \mathcal{P}(Tx, Ty, t), \quad \forall x, y \in X.$$  

If one of the subspace $T(X)$ or $S(X)$ is complete, then $S$ and $T$ have a unique common fixed point in $X$.

**Proof** Let $x_0 \in X$. Since $T(X) \subseteq S(X)$, choose $x_1$ such that $y_1 = Sx_1 = Tx_0$. In general choose $x_{n+1} = Sx_n = Tx_n$, then form condition (17),

$$\mathcal{P}(y_{n+1}, y_{n+2}, t) = \mathcal{P}(Tx_n, Tx_{n+1}, t) \leq \frac{1}{k} \mathcal{P}(Sx_n, Sx_{n+1}, t) \leq \frac{1}{k} \mathcal{P}(Tx_{n-1}, Tx_n, t) \leq \frac{1}{k} \mathcal{P}(y_n, y_{n+1}, t) \leq \lambda \mathcal{P}(y_n, y_{n+1}, t)$$

where $\lambda = \frac{1}{k} < 1$. Repeating (18) $(n + 1)$ -times, we obtain

$$\mathcal{P}(y_{n+1}, y_{n+2}, t) \leq \lambda^{n+1} \mathcal{P}(y_0, y_1, t)$$

Hence for $n > m$, we have

$$\mathcal{P}(y_n, y_m, t) \leq \mathcal{P}(y_n, y_{n+1}, t) + \mathcal{P}(y_{n+1}, y_{n+2}, t) + \ldots + \mathcal{P}(y_{m+1}, y_m, t) \leq (\lambda^{n+1} + \lambda^{n+2} + \ldots + \lambda^{m+1}) \mathcal{P}(y_{n+1}, y_{n+1}, t) \leq \frac{\lambda^{n+1}}{1-\lambda} \mathcal{P}(y_0, y_1, t)$$

for all $t > 0$. By taking $n, m \to \infty$ in the above inequality, we get $\lim_{m,n \to \infty} \mathcal{P}(y_n, y_m, t) = 0$. Therefore, \{\seq{x_n}\}$_{n=1}^\infty$ is a Cauchy Sequence. Since $(X, \mathcal{P})$ is a complete parametric metric space, there exists $x^* \in X$ such that $y_n \to x^*$ as $n \to \infty$. Hence

$$\lim_{n \to \infty} y_n = \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} Sx_n = x^*.$$  

Since $T(X)$ or $S(X)$ is complete and $T(X) \subseteq S(X)$, there exists a point $p \in X$ such that $Sp = x^*$. Now from (17), for all $t > 0$,

$$\mathcal{P}(Tp, Tx_n, t) \leq \frac{1}{k} \mathcal{P}(Sp, Sx_n, t)$$

Proceeding to the limit as $n \to +\infty$ in (22), we have

$$\mathcal{P}(Tp, x^*, t) \leq \frac{1}{k} \mathcal{P}(Sp, x^*, t)$$

for all $t > 0$, which implies that $Tp = x^*$. Therefore $Sp = x^*$. Since $T$ and $S$ are weakly compatible, then $STp = TSp$, that is $Sx^* = Tx^*$.

Now we show that $x^*$ is a fixed point of $S$ and $T$. From (17), we have

$$\mathcal{P}(Sx^*, Sx^*, t) \geq k \mathcal{P}(Tx^*, Tx^*, t)$$

Proceeding to the limit as $n \to \infty$ in (24), we have

$$\mathcal{P}(Sx^*, x^*, t) \geq k \mathcal{P}(Tx^*, x^*, t)$$

for all $t > 0$, which implies that $Sx^* = x^*$. Hence $Sx^* = Tx^* = x^*$.

Uniqueness. Suppose that $x^* \neq y^*$ is also another common fixed point of $S$ and $T$. Then

$$\mathcal{P}(Sx^*, Sy^*, t) \geq k \mathcal{P}(Tx^*, Ty^*, t)$$

for all $t > 0$, which implies that $x^* = y^*$. This completes the proof.

**Example 4.3** Let $X = [0, +\infty)$ be endowed with parametric metric,

$$\mathcal{P}(x, y, t) = \begin{cases} t \max(x, y), & x \neq y \\ 0, & x = y \end{cases}$$

for all $x, y \in X$ and all $t > 0$. Define $S, T : X \to X$ by $S(x) = \frac{x}{2}, T(x) = \frac{x}{6}$ for all $x \in X$. Then $T(X) \subseteq S(X)$ and $S(X)$ is complete. Further

$$\mathcal{P}(Sx, Sy, t) = \max(Sx, Sy)$$

$$= \max\left(\frac{x}{2}, \frac{y}{6}\right)$$

$$= \frac{1}{2} \max(x, y)$$

$$\geq k \max\left(\frac{x}{2}, \frac{y}{6}\right)$$

$$= k \mathcal{P}(Tx, Ty, t)$$

for $1 < k < 3$ and (17) is satisfied. Moreover mappings are weakly compatible at $x = 0$. Thus all conditions of Theorem 4.2 are satisfied and $x^* = 0$ is the unique common fixed point of $S$ and $T$.

Now, we prove the following common fixed point theorem, which is generalization of Theorem 2.2 of W. Shatanwi and F. Awawdeh [42] in the setting of parametric metric space.

**Theorem 4.4** Let $T, S : X \to X$ be two surjective mappings of a complete parametric metric space $(X, \mathcal{P})$. Suppose that $T$ and $S$ satisfying the following inequalities

$$\mathcal{P}(Sx, Sy, t) \geq k \mathcal{P}(Tx, Ty, t)$$

for all $t > 0$, which implies that $Sp = x^*$. Now from (17), for all $t > 0$,
(27) \( \mathcal{P}(T(Sx), Sx, t) + k\mathcal{P}(T(Sx), x, t) \geq \mathcal{P}(Sx, x, t) \)
and
(28) \( \mathcal{P}(S(Tx), Tx, t) + k\mathcal{P}(S(Tx), x, t) \geq b\mathcal{P}(Tx, x, t) \)
for all \( x \in X \), all \( t > 0 \) and some nonnegative real numbers \( a, b \) and \( k \) with \( a > 1 + 2k \) and \( b > 1 + 2k \). If \( T \) or \( S \) is continuous, then \( T \) and \( S \) have a common fixed point.

**Proof** Let \( x_0 \) be an arbitrary point in \( X \). Since \( T \) is surjective, there exists \( x_1 \in X \) such that \( x_0 = Tx_1 \). Also since \( S \) is surjective, there exists \( x_2 \in X \) such that \( x_2 = Sx_1 \). Continuing this process, we construct a sequence \( \{x_n\}_{n=1}^{\infty} \) in \( X \) such that

(29) \( x_{2n} = Tx_{2n+1} \) and \( x_{2n+1} = Sx_{2n+2} \)
for all \( n \in \mathbb{N} \cup \{0\} \). Now for \( n \in \mathbb{N} \cup \{0\} \), we have

(30) \( \mathcal{P}(T(Sx_{2n+2}), Sx_{2n+2}, t) + k\mathcal{P}(T(Sx_{2n+2}), x_{2n+2}, t) \geq a\mathcal{P}(Sx_{2n+2}, x_{2n+2}, t) \)
Thus we have

(31) \( \mathcal{P}(x_{2n}, x_{2n+1}, t) + k\mathcal{P}(x_{2n-1}, x_{2n+1}, t) \geq a\mathcal{P}(x_{2n-1}, x_{2n+1}, t) \)

Since

(32) \( \mathcal{P}(x_{2n-1}, x_{2n+1}, t) \leq \frac{1 + k}{a - k}\mathcal{P}(x_{2n}, x_{2n+2}, t) \)
On the other hand, we have

(33) \( \mathcal{P}(S(Tx_{2n+1}), Tx_{2n+1}, t) + k\mathcal{P}(S(Tx_{2n+1}), x_{2n+1}, t) \geq b\mathcal{P}(Tx_{2n+1}, x_{2n+1}, t) \)
Thus, we have

(34) \( \mathcal{P}(x_{2n-1}, x_{2n}, t) + k\mathcal{P}(x_{2n-2}, x_{2n+1}, t) \geq b\mathcal{P}(x_{2n-2}, x_{2n+1}, t) \)

Since

(35) \( \mathcal{P}(x_{2n-2}, x_{2n+1}, t) \leq \frac{1 + k}{b - k}\mathcal{P}(x_{2n-1}, x_{2n}, t) \)
Let

(36) \( \lambda = \max\left\{\frac{1 + k}{a - k}, \frac{1 + k}{b - k}\right\} \)
Then by combining (32) and (35), we have

(37) \( \mathcal{P}(x_n, x_{n+1}, t) \leq \lambda\mathcal{P}(x_{n-1}, x_n, t) \)
for all \( n \in \mathbb{N} \cup \{0\} \) and for all \( t > 0 \). Repeating (34) \( n \)-times, we get

(38) \( \mathcal{P}(x_n, x_{n+1}, t) \leq \lambda^n\mathcal{P}(x_0, x_1, t) \)
for all \( n \in \mathbb{N} \cup \{0\} \) and for all \( t > 0 \). By Lemma 3.1, \( \{x_n\}_{n=1}^{\infty} \) is a Cauchy sequence in the complete parametric metric space \((X, \mathcal{P})\). Then there exists \( x^* \in X \) such that \( x_n \to x^* \) as \( n \to +\infty \). Therefore, \( x_{2n+1} \to x^* \) and \( x_{2n+2} \to x^* \) as \( n \to +\infty \). Without loss of generality, we may assume that \( T \) is continuous, then \( Tx_{2n+1} \to Tx^* \) as \( n \to +\infty \). Thus, we have \( Tx^* = x^* \). Since \( S \) is surjective, there exists \( u \in X \) such that \( Su = x^* \). Now

(39) \( \mathcal{P}(x^*, u, t) \leq \frac{1}{a}\mathcal{P}(x^*, u, t) \)
Since \( a > k \), we conclude that \( \mathcal{P}(x^*, u, t) = 0 \). So \( x^* = u \).
Hence \( Tx^* = Sx^* = x^* \). Therefore \( x^* \) is a common fixed point of \( T \) and \( S \).

By taking \( b = a \) in above Theorem 4.4, we have the following result.

**Corollary 4.5** Let \( T, S : X \to X \) be two surjective mappings of a complete parametric metric space \((X, \mathcal{P})\). Suppose that \( T \) and \( S \) satisfying the following inequalities

(40) \( \mathcal{P}(T(Sx), Sx, t) + k\mathcal{P}(T(Sx), x, t) \geq a\mathcal{P}(Sx, x, t) \)
and

(41) \( \mathcal{P}(S(Tx), Tx, t) + k\mathcal{P}(S(Tx), x, t) \geq a\mathcal{P}(Tx, x, t) \)
for all \( x \in X \), all \( t > 0 \) and some nonnegative real numbers \( a \) and \( k \) with \( a > 1 + 2k \). If \( T \) or \( S \) is continuous. Then \( T \) and \( S \) have a common fixed point.

By taking \( S = T \) in above Corollary 4.5, we have the following result.

**Corollary 4.7** Let \( T : X \to X \) be a surjective mapping of a complete parametric metric space \((X, \mathcal{P})\). Suppose that \( T \) satisfying the following inequality

(42) \( \mathcal{P}(T(Tx), Tx, t) + k\mathcal{P}(T(Tx), x, t) \geq a\mathcal{P}(Tx, x, t) \)
for all \( x \in X \), all \( t > 0 \) and some nonnegative real numbers \( a \) and \( k \) with \( a > 1 + 2k \). If \( T \) is continuous. Then \( T \) has a fixed point.

Now, we give an example.

**Example 4.8** Let \( X = \mathbb{R}^+ \) be endowed with parametric metric \( \mathcal{P}(x, y, t) = t|x - y| \) for all \( x, y \in X \) and all \( t > 0 \). Then \((X, \mathcal{P})\) is a complete parametric metric space. Define \( T : X \to X \) by \( T(x) = 2x \) for all \( x \in X \). Then \( T \) is a surjection on \( X \). Note that

(43) \( \mathcal{P}(T(Tx), Tx, t) + \mathcal{P}(T(Tx), x, t) \)
where \( k = 1 \) and \( \alpha = 4 \). Clearly \( 4 = \alpha > 1 + 2k = 3 \). Then (42) is satisfied. Thus all conditions of Corollary 2.7 are satisfied and \( x^* = 0 \in X \) is a fixed point of \( T \).

5. COINCIDENCE POINT RESULTS IN PARAMETRIC METRIC SPACE

In this section, we establish a new theorem of coincidence point for expansive mappings in parametric metric spaces under a set of conditions, which is generalization of Theorem 2.1 of W. Shatanawi and F. Awadheh [42] in the setting of parametric metric space.

Theorem 5.1 Let \((X, \mathcal{P})\) be a parametric metric space. Let \(T, f : X \to X\) be mappings satisfying

\[
\mathcal{P}(Tx, Ty, t) \geq a \mathcal{P}(fx, fy, t) + b \mathcal{P}(fx, Tx, t) + c \mathcal{P}(fy, Ty, t)
\]

for all \(x, y \in X\) and all \(t > 0\), where \(a, b, c \geq 0\) with \(a + b + c > 1\). Suppose the following hypotheses:

1) \( b > 1 \) or \( c < 1 \);
2) \( f(X) \subseteq T(X) \);
3) \( T(X) \) is a complete subspace of \( X \).

Then \( T \) and \( f \) have a coincidence point.

Proof: Let \( x_0 \in X \). Since \( f(X) \subseteq T(X) \), we choose \( x_1 \in X \) such that \( Tx_1 = fx_0 \). Again we can choose \( x_2 \in X \) such that \( Tx_2 = fx_1 \). Continuing in the same way, we construct a sequence \( \{x_n\}_{n=1}^{\infty} \) in \( X \) such that \( Tx_{n+1} = fx_n \forall n \in \mathbb{N} \cup \{0\} \). If \( fx_{m-1} = fx_m \) for \( m \in \mathbb{N} \), then \( Tx_m = fx_m \). Thus \( x_m \) is a coincidence point of \( T \) and \( f \).

Now assume that \( x_{n-1} \neq x_n \) for all \( n \in \mathbb{N} \). We have the following two cases:

Case (1) Suppose \( b < 1 \). By (43), we have

\[
\mathcal{P}(fx_{n-1}, fx_n, t) = \mathcal{P}(Tx_n, Tx_{n+1}, t)
\]

\[
\geq a \mathcal{P}(fx_n, fx_{n+1}, t) + b \mathcal{P}(fx_n, Tx_n, t) + c \mathcal{P}(fx_{n+1}, Tx_{n+1}, t)
\]

\[
= a \mathcal{P}(fx_n, fx_{n+1}, t) + b \mathcal{P}(fx_n, fx_{n+1}, t) + c \mathcal{P}(fx_{n+1}, fx_n, t)
\]

Thus, we have

\[
(1 - b)\mathcal{P}(fx_{n-1}, fx_n, t) \geq (a + c)\mathcal{P}(fx_{n+1}, fx_n, t)
\]

Hence

\[
\mathcal{P}(fx_{n+1}, fx_n, t) \leq \frac{1 - b}{a + c} \mathcal{P}(fx_{n-1}, fx_n, t)
\]

Case (2) Suppose \( c < 1 \). Also from (43), we have

\[
\mathcal{P}(fx_n, fx_{n-1}, t) = \mathcal{P}(Tx_n, Tx_{n+1}, t)
\]

\[
\geq a \mathcal{P}(fx_n, fx_{n+1}, t) + b \mathcal{P}(fx_n, Tx_n, t) + c \mathcal{P}(fx_{n+1}, Tx_{n+1}, t)
\]

\[
= a \mathcal{P}(fx_n, fx_{n+1}, t) + b \mathcal{P}(fx_n, fx_{n+1}, t) + c \mathcal{P}(fx_{n+1}, fx_n, t)
\]

Thus, we have

\[
(1 - b)\mathcal{P}(fx_n, fx_{n-1}, t) \geq (a + c)\mathcal{P}(fx_{n+1}, fx_n, t)
\]

Hence

\[
\mathcal{P}(fx_{n+1}, fx_n, t) \leq \frac{1 - c}{a + b} \mathcal{P}(fx_{n-1}, fx_n, t)
\]

In case (1), we let \( \lambda_1 = \frac{1 - b}{a + c} \) and in case (2), we let \( \lambda_2 = \frac{1 - c}{a + b} \). Define \( \lambda = \max\{\lambda_1, \lambda_2\} \). Thus in both cases, we have \( \lambda < 1 \). Hence

\[
\mathcal{P}(fx_{n+1}, fx_n, t) \leq \lambda \mathcal{P}(fx_{n-1}, fx_n, t)
\]

Repeating (48), \( n \)-times, we obtain

\[
\mathcal{P}(fx_{n+1}, fx_n, t) \leq \lambda^n \mathcal{P}(fx_0, fx_1, t)
\]

So for \( m > n \), we have

\[
\mathcal{P}(fx_m, fx_n, t) \leq \mathcal{P}(fx_m, fx_{m+1}, t) + \mathcal{P}(fx_{m+1}, fx_{m+2}, t) + \cdots + \mathcal{P}(fx_{m-n}, fx_{m-t}, t) + \lambda^{m-n} \mathcal{P}(fx_0, x_{1-t})
\]

\[
\leq \lambda^m \mathcal{P}(fx_0, fx_1, t)
\]

for all \( t > 0 \). By taking limit as \( m, n \to +\infty \) in above inequality (50), we get \( \lim_{m,n \to +\infty} \mathcal{P}(fx_m, fx_n, t) = 0 \) for all \( t > 0 \). Therefore \( \{Tx_n\}_{n=1}^{\infty} \) is a Cauchy sequence in \( TX \). Since \( TX \) is a complete subspace of \( X \), there is \( x^* \in X \) such that \( \{Tx_n\}_{n=1}^{\infty} \) converges to \( x^* \) as \( n \to +\infty \). Hence \( fx_n \) converges to \( Tx^* \) as \( n \to +\infty \). Since \( a + b + c > 1 \), we have \( a, b \) and \( c \) are not all 0. So we have the following cases:

Case (1) If \( a \neq 0 \), then

\[
\mathcal{P}(Tx_n, Tx^*, t) \geq a \mathcal{P}(fx_n, fx^*, t)
\]

\[
+ b \mathcal{P}(fx_n, Tx_n, t)
\]

\[
+ c \mathcal{P}(fx^*, Tx^*, t)
\]

\[
\geq a \mathcal{P}(fx_n, fx)
\]
Hence

\[ \mathcal{P}(fx_n, fx^*, t) \leq \frac{1}{a} \mathcal{P}(Tx_n, Tx^*, t) \]  

Since \( \frac{1}{a} \mathcal{P}(Tx_n, Tx^*, t) \to 0 \) as \( n \to +\infty \). Thus \( fx_n \to fx^* \) as \( n \to +\infty \). By uniqueness of limit, we have \( Tx^* = fx^* \). Therefore \( T \) and \( f \) have a coincidence point.

Case (2) If \( b \neq 0 \), then

\[
\mathcal{P}(Tx^*, Tx_n, t) \geq a \mathcal{P}(fx_n, fx^*, t) + b \mathcal{P}(fx^*, Tx^*, t) + c \mathcal{P}(Tx_n, Tx^*, t) \geq b \mathcal{P}(fx^*, Tx^*, t)
\]

Hence

\[ \mathcal{P}(fx^*, Tx^*, t) \leq \frac{1}{b} \mathcal{P}(Tx^*, Tx_n, t) \]

As similar proof of case (1), we can show that \( fx^* = Tx^* \). thus \( f \) and \( T \) have a coincidence point.

Case (3) If \( c \neq 0 \), then

\[
\mathcal{P}(Tx_n, Tx^*, t) \geq a \mathcal{P}(fx_n, fx^*, t) + b \mathcal{P}(fx^*, Tx^*, t) + c \mathcal{P}(Tx^*, Tx_n, t) \geq c \mathcal{P}(fx^*, Tx^*, t)
\]

Hence

\[ \mathcal{P}(fx^*, Tx^*, t) \leq \frac{1}{c} \mathcal{P}(Tx_n, Tx^*, t) \]

for all \( t > 0 \). As similar proof of case (1), we can show that \( fx^* = Tx^* \). thus \( f \) and \( T \) have a coincidence point.

Setting \( b = c = 0 \) in Corollary 5.1, we can obtain the following result.

**Corollary 5.2** Let \((X, \mathcal{P})\) be a parametric metric space. Let \( T, f : X \to X \) be mappings satisfying

\[ \mathcal{P}(Tx, Ty, t) \geq a \mathcal{P}(fx, fy, t) + b \mathcal{P}(fx, Tx, t) \]

for all \( x, y \in X \) and all \( t > 0 \), where \( a, b \geq 0 \) with \( a + b > 1 \) and \( b > 1 \). Suppose the following hypotheses:

1) \( f(X) \subseteq T(X) \);
2) \( T(X) \) is a complete subspace of \( X \).

Then \( T \) and \( f \) have a coincidence point.

Setting \( b = c = 0 \) in Theorem 5.1, we can obtain the following corollary.

**Corollary 5.3** Let \((X, \mathcal{P})\) be a parametric metric space. Let \( T, f : X \to X \) be mappings satisfying

\[ \mathcal{P}(Tx, Ty, t) \geq a \mathcal{P}(fx, fy, t) \]

for all \( x, y \in X \) and all \( t > 0 \), where \( a > 1 \). Suppose the following hypotheses:

1) \( f(X) \subseteq T(X) \);
2) \( T(X) \) is a complete subspace of \( X \).

Then \( T \) and \( f \) have a coincidence point.

Setting \( T = S \) in Theorem 5.1, we have the following corollary.

**Corollary 5.4** Let \((X, \mathcal{P})\) be a parametric metric space. Let \( T : X \to X \) be a surjective mapping satisfying

\[ \mathcal{P}(Tx, Ty, t) \geq a \mathcal{P}(x, y, t) + b \mathcal{P}(x, Tx, t) + c \mathcal{P}(y, Ty, t) \]

for all \( x, y \in X \) and all \( t > 0 \), where \( a, b, c \geq 0 \) with \( a + b + c > 1 \). Suppose \( b > 1 \) or \( c < 1 \). Then \( T \) has a fixed point.

Setting \( b = c = 0 \) in Corollary 5.4, we can obtain the following corollary.

**Corollary 5.5** Let \((X, \mathcal{P})\) be a parametric metric space. Let \( T : X \to X \) be a surjective mapping satisfying

\[(56) \quad \mathcal{P}(Tx, Ty, t) \geq a \mathcal{P}(x, y, t) \]

for all \( x, y \in X \) and all \( t > 0 \), where \( a > 1 \). Then \( T \) has a fixed point.

**Corollary 5.6** Let \((X, \mathcal{P})\) be a parametric metric space. Let \( T : X \to X \) be a surjective mapping satisfying

\[ \mathcal{P}(Tx, Ty, t) \geq a \mathcal{P}(x, y, t) + b \mathcal{P}(x, Tx, t) \]

for all \( x, y \in X \) and all \( t > 0 \), where \( a, b \geq 0 \) with \( a + b > 1 \). Suppose \( b > 1 \). Then \( T \) has a fixed point.

**Corollary 5.7** Let \((X, \mathcal{P})\) be a parametric metric space. Let \( T : X \to X \) be a surjective mapping satisfying

\[ \mathcal{P}(Tx, Ty, t) \geq a \mathcal{P}(x, y, t) + b \mathcal{P}(y, Ty, t) \]

for all \( x, y \in X \) and all \( t > 0 \), where \( a, b \geq 0 \) with \( a + b > 1 \). Suppose \( b > 1 \). Then \( T \) has a fixed point.

Now, we introduce an example to support the validity of Corollary 5.3.

**Example 5.8** Let \( X = [0,1] \) be endowed with parametric metric \( \mathcal{P}(x, y, t) = t|x - y| \) for all \( x, y \in X \) and all \( t > 0 \). Then \((X, \mathcal{P})\) is a complete parametric metric space. Define \( T, f : X \to X \) by \( T(x) = \frac{x}{4} \) and \( f(x) = \frac{x}{16} \) for all \( x \in X \). Then \( T \) and \( f \) are surjection on \( X \). Then \( T(X) \subseteq S(X) \) and \( S(X) \) is complete. Further

\[
\mathcal{P}(Tx, Ty, t) = t \left| \frac{x}{4} - \frac{y}{16} \right|
\]

\[
= 4t \left| \frac{x}{16} - \frac{y}{16} \right|
\]

\[
\geq 3\mathcal{P}(fx, fy, t)
\]

\[
= a \mathcal{P}(fx, fx, t)
\]
for all \(x, y \in X\) and all \(t > 0\), where \(a = 3 > 1\). Then (54) is satisfied. Thus all conditions of Corollary 5.3 are satisfied and \(x^* = 0\) is a coincidence point of \(T\) and \(f\).

6. FIXED POINT RESULTS IN PARAMETRIC B-METRIC SPACE

In this section, we establish some unique fixed point results satisfying expansive condition by considering surjective self-mapping in the context of parametric b-metric space. Our results generalize and extend some old and recent fixed point result in the literature.

We begin with following some lemmas.

**Lemma 6.1** Let \((X, \mathcal{P}, s)\) be a b-metric space with the coefficient \(s \geq 1\) and let \(\{x_n\}_{n=1}^{\infty}\) be a sequence in \(X\). If \(\{x_n\}_{n=1}^{\infty}\) converges to \(x\) and also \(\{x_n\}_{n=1}^{\infty}\) converges to \(y\), then \(x = y\). That is the limit of \(\{x_n\}_{n=1}^{\infty}\) is unique.

**Proof** Since \(x_n \to x\) and \(x_n \to y\) as \(n \to +\infty\), that is \(\lim_{n \to +\infty} \mathcal{P}(x_n, x, t) = 0\) and \(\lim_{n \to +\infty} \mathcal{P}(x_n, y, t) = 0\). By using triangular inequality, we have

\[
\mathcal{P}(x, y, t) = s[\mathcal{P}(x_n, x, t) + \mathcal{P}(x_n, y, t)]
\]

for all \(t > 0\). By taking limit as \(n \to +\infty\), we get \(d(x, y) = 0\) and so \(x = y\).

**Lemma 6.2** Let \((X, \mathcal{P}, s)\) be a b-metric space with the coefficient \(s \geq 1\) and let \(\{x_n\}_{n=1}^{\infty}\) be a sequence in \(X\). If \(\{x_n\}_{n=1}^{\infty}\) converges to \(x\). Then

\[
\frac{1}{s} \mathcal{P}(x, y, t) \leq \lim_{n \to +\infty} \mathcal{P}(x_n, y, t) \leq s \mathcal{P}(x, y, t)
\]

\(\forall y \in X\) and all \(t > 0\).

**Proof** From triangular inequality, we have

\[
\frac{1}{s} \mathcal{P}(x, y, t) \leq \lim_{n \to +\infty} \mathcal{P}(x_n, x, t)
\]

\[
\leq \lim_{n \to +\infty} \mathcal{P}(x_n, y, t)
\]

\[
\leq s (\mathcal{P}(x, y, t) + \lim_{n \to +\infty} \mathcal{P}(x_n, x, t))
\]

and so

\[
\frac{1}{s} \mathcal{P}(x, y, t) \leq \lim_{n \to +\infty} \mathcal{P}(x_n, y, t) \leq s \mathcal{P}(x, y, t)
\]

\(\forall y \in X\) and all \(t > 0\).

**Lemma 6.3** Let \((X, \mathcal{P}, s)\) be a b-metric space with the coefficient \(s \geq 1\) and let \(\{x_k\}_{k=0}^{\infty} \subseteq X\). Then

\[
\mathcal{P}(x_n, x_m, t) \leq s \mathcal{P}(x_0, x_1, t) + s^2 \mathcal{P}(x_1, x_2, t)
\]

\[
+ \cdots + s \mathcal{P}(x_{n-1}, x_n, t) + s^{n+1} \mathcal{P}(x_{n-1}, x_{n+1}, t)
\]

From Lemma 6.3, we deduce the following result.

**Lemma 6.4** Let \((X, \mathcal{P}, s)\) be a parametric b-metric space with the coefficient \(s \geq 1\). Let \(\{x_n\}_{n=1}^{\infty}\) be a sequence of points of \(X\) such that

\[
\mathcal{P}(x_n, x_{n+1}, t) \leq \lambda \mathcal{P}(x_{n-1}, x_n, t)
\]

where \(\lambda \in [0, \frac{1}{s}]\) and \(n = 1, 2, \ldots\). Then \(\{x_n\}_{n=1}^{\infty}\) is a Cauchy sequence in \((X, \mathcal{P}, s)\).

**Proof** Let \(m > n \geq 1\). It follows that

\[
\mathcal{P}(x_n, x_m, t) \leq s \mathcal{P}(x_n, x_{n+1}, t) + s^2 \mathcal{P}(x_{n+1}, x_{n+2}, t)
\]

\[
+ \cdots + s^{m-n} \mathcal{P}(x_{m-1}, x_m, t)
\]

\[
\leq s \lambda^n + s^2 \lambda^{n+1} + \cdots + s^{m-n} \lambda^{m-1} \mathcal{P}(x_0, x_1, t)
\]

\[
\leq s \lambda^n(1 + s \lambda + \cdots + (s \lambda)^{m-n-1}) \mathcal{P}(x_0, x_1, t)
\]

\[
\leq \frac{s \lambda^n}{1-s \lambda} \mathcal{P}(x_0, x_1, t)
\]

for all \(t > 0\). Since \(s \lambda < 1\). Assume that \(\mathcal{P}(x_0, x_1, t) > 0\). By taking limit as \(m, n \to +\infty\) above inequality we get

\[
\lim_{n, m \to \infty} \mathcal{P}(x_0, x_m, t) = 0.
\]

Therefore, \(\{x_n\}_{n=1}^{\infty}\) is a Cauchy sequence in \(X\). Also, if \(\mathcal{P}(x_0, x_1, t) = 0\), then \(\mathcal{P}(x_n, x_m, t) = 0\) for all \(m > n\) and hence \(\{x_n\}_{n=1}^{\infty}\) is a Cauchy sequence in \(X\).

Now, we have the following fixed point theorem in parametric b-metric space.

**Theorem 6.5** Let \((X, \mathcal{P}, s)\) be a complete parametric b-metric space with the coefficient \(s \geq 1\). Assume that the mapping \(T : X \to X\) is surjection and satisfies

\[
\mathcal{P}(T(x), y, t) \geq \lambda \mathcal{P}(x, y, t)
\]

\(\forall x, y \in X\) and all \(t > 0\) where \(\lambda > s\). Then \(T\) has a fixed point.

**Proof** Let \(x_0 \in X\), since \(T\) is surjection on \(X\), then there exists \(x_1 \in X\) such that \(x_0 = Tx_1\). By continuing this process, we get

\[
x_n = Tx_n, \quad \forall n \in \mathbb{N} \cup \{0\}.
\]

In case \(x_n = x_{n+1}\) for some \(n_0 \in \mathbb{N} \cup \{0\}\), then it is clear that \(x_{n_0}\) is a fixed point of \(T\). Now assume that \(x_n \neq x_{n-1}\) for all \(n\). Consider,

\[
\mathcal{P}(x_{n-1}, x_n, t) = \mathcal{P}(Tx_{n-1}, Tx_n, t)
\]

Now by (67) and definition of the sequence

\[
\mathcal{P}(x_{n-1}, y, t) = \mathcal{P}(Tx_{n-1}, y, t)
\]

\[
\geq \lambda \mathcal{P}(x_{n-1}, t)
\]

and so

\[
\mathcal{P}(x_{n-1}, x_n, t) \leq \frac{1}{\lambda} \mathcal{P}(x_{n-1}, x_n, t)
\]

\[
= h \mathcal{P}(x_{n-1}, x_n, t)
\]
for all \( n \in \mathbb{N} \cup \{0\} \) and all \( t > 0 \) where \( h = \frac{1}{4} < \frac{1}{t} \). Repeating (68) \( n \)-times, we get
\begin{equation}
(69) \quad \mathcal{P}(x_{n+1}, x_n, t) \leq h^n \mathcal{P}(x_0, x, t)
\end{equation}
for all \( t > 0 \). By Lemma 6.4, \( \{x_n\}_{n=1}^{\infty} \) is a Cauchy sequence in \( X \). Since \( X \) is a complete parametric b-metric space, there exists \( x^* \in X \) such that \( x_n \rightarrow x^* \) as \( n \rightarrow +\infty \). Now since \( T \) is surjective map. So there exists a point \( p \) in \( X \) such that \( x^* = Tp \). Consider from (65), we have
\begin{equation}
(70) \quad \mathcal{P}(x_n, x^*, t) = \mathcal{P}(T(x_{n+1}, Tp, t) \\
\geq \lambda \mathcal{P}(x_{n+1}, p, t)
\end{equation}
for all \( t > 0 \). Taking limit as \( n \rightarrow +\infty \) in the above inequality, we get
\[ 0 = \lim_{n \rightarrow +\infty} \mathcal{P}(x_n, x^*, t) \geq \lambda \lim_{n \rightarrow +\infty} \mathcal{P}(x_{n+1}, p, t) \]
which implies that
\begin{equation}
(71) \quad \lim_{n \rightarrow +\infty} \mathcal{P}(x_{n+1}, p, t) = 0.
\end{equation}
for all \( t > 0 \). Thus \( x_{n+1} \rightarrow p \) as \( n \rightarrow +\infty \). By lemma 6.1, we get \( x^* = p \). Hence \( x^* \) is a fixed point of \( T \).
Finally, assume \( x^* = y^* \) is also another fixed point of \( T \).
From (65), we get for all \( t > 0 \)
\begin{equation}
(72) \quad \mathcal{P}(x^*, y^*, t) = \mathcal{P}(T(x^*, y^*, t) \\
\geq \lambda \mathcal{P}(x^*, y^*, t)
\end{equation}
This is true only when \( \mathcal{P}(x^*, y^*, t) = 0 \). So \( x^* = y^* \). Hence \( T \) has a unique fixed point in \( X \).

**Example 6.6** Let \( X = [0, +\infty) \) and define \( \mathcal{P} : X \times X \times (0, +\infty) \rightarrow [0, +\infty) \) by
\[ \mathcal{P}(x, y, t) = \begin{cases} 
(\frac{t}{2}x - \frac{t}{2}y)^2 & \text{if } t \neq 0, \\
0 & \text{if } x = y,
\end{cases} \]
\[ \forall x, y \in X \] and all \( t > 0 \). It is obvious that \( \mathcal{P} \) is a parametric b-metric on \( X \) with \( s = \frac{1}{2} > 1 \) and \( (X, \mathcal{P}) \) is complete. Also, \( \mathcal{P} \) is not a parametric metric on \( X \). Define a mapping \( T : X \rightarrow X \) by
\[ Tx = \begin{cases} 
6x & \text{if } x \in [0, 1), \\
5x + 1 & \text{if } x \in [1, 2), \\
4x + 3 & \text{if } x \in [2, +\infty).
\end{cases} \]
Clearly \( T \) is a surjection on \( X \). Now we consider following cases.

- Let \( x, y \in [0, 1) \), then
\[ \mathcal{P}(Tx, Ty, t) = t(6x - 6y)^2 \\
= 36t(x - y)^2 \\
\geq 3t(x - y)^2 \\
= 3\mathcal{P}(x, y, t) \]

- Let \( x, y \in [2, +\infty) \), then
\[ \mathcal{P}(Tx, Ty, t) = t((5x + 1) - (5y + 1))^2 \\
= 25t(x - y)^2 \\
\geq 3t(x - y)^2 \\
= 3\mathcal{P}(x, y, t) \]

- Let \( x, y \in [1, 2) \), then
\[ \mathcal{P}(Tx, Ty, t) = t((4x + 3) - (4y + 3))^2 \\
= t(4x - 4y)^2 \\
= 16t(x - y)^2 \\
\geq 3t(x - y)^2 \\
= 3\mathcal{P}(x, y, t) \]

- Let \( x \in [0, 1) \) and \( y \in [1, 2) \), then
\[ \mathcal{P}(Tx, Ty, t) = t((6x - (5y + 1))^2 \\
\geq t(6x - 5y)^2 \\
\geq t(5x - 5y)^2 \\
= 25t(x - y)^2 \\
\geq 3t(x - y)^2 \\
= 3\mathcal{P}(x, y, t) \]

- Let \( x \in [0, 1) \) and \( y \in [2, +\infty) \), then
\[ \mathcal{P}(Tx, Ty, t) = t((6x - (4y + 3))^2 \\
\geq t(6x - 4y)^2 \\
\geq t(4x - 4y)^2 \\
= 16t(x - y)^2 \\
\geq 3t(x - y)^2 \\
= 3\mathcal{P}(x, y, t) \]

Hence in all above cases
\[ \mathcal{P}(Tx, Ty, t) \geq \lambda \mathcal{P}(x, y, t) \]
\[ \forall x, y \in X \text{ and all } t > 0 \text{ where } \lambda = 3 > 2 = s. \text{ The conditions of Theorem 6.5, are satisfied and } T \text{ has a unique fixed point } x^* = 0 \in X. \]

Now, motivated by the work in [40], we give the following.

Let \( \Psi^B \) denote the class of those function \( \mathcal{B}: (0, \infty) \to (L^2, \infty) \) which satisfy the condition \( \mathcal{B}(t_n) \to (L^2)^* \Rightarrow t_n \to 0, \text{ where } L > 0. \)

**Theorem 6.7** Let \( (X, \mathcal{P}, s) \) be a complete parametric b-metric space. Assume that the mapping \( T: X \to X \) is surjection and satisfies

\[ \mathcal{P}(Tx, Ty, t) \geq \mathcal{B}(\mathcal{P}(x, y, t)) \mathcal{P}(x, y, t) \]

\[ \forall x, y \in X \text{ and all } t > 0 \text{ where } \mathcal{B} \in \Psi^B. \text{ Then } T \text{ has a fixed point.} \]

**Proof** Let \( x_0 \in X \). Since \( T \) is surjective, choose \( x_1 \in X \) such that \( Tx_1 = x_0 \). Inductively, we can define a sequence \( \{x_n\}_{n=1}^{\infty} \in X \) such that

\[ x_n = Tx_{n-1}; \forall n \in \mathbb{N} \cup \{0\}. \]

In case \( x_n = x_{n+1} \) for some \( n \in \mathbb{N} \cup \{0\} \), then it is clear that \( x_n \) is a fixed point of \( T \). Now assume that \( x_n \neq x_{n+1} \) for all \( n \). Consider

\[ \mathcal{P}(x_{n-1}, x_n, t) = \mathcal{P}(Tx_{n-1}, Tx_n, t) \]

Now by (73) and definition of the sequence

\[ \mathcal{P}(x_{n-1}, x_n, t) = \mathcal{P}(Tx_{n-1}, Tx_n, t) \]

\[ \geq \mathcal{B}(\mathcal{P}(x_{n-1}, x_n, t)) \mathcal{P}(x_{n-1}, x_n, t) \]

\[ \geq s^2 \mathcal{P}(x_{n-1}, x_n, t) \]

\[ \geq \mathcal{P}(x_{n-1}, x_n, t) \]

Thus the sequence \( \{d(x_n, x_{n+1})\}_{n=1}^{\infty} \) is a decreasing sequence in \( \mathbb{R}^+ \) and so there exists \( r \geq 0 \) such that

\[ \lim_{n \to \infty} \mathcal{P}(x_n, x_{n+1}, t) = r \]

for all \( t > 0. \) Let us prove that \( r = 0. \) Suppose to the contrary that \( r > 0. \) By (73) we can deduce that

\[ s^2 \frac{\mathcal{P}(x_n, x_{n+1}, t)}{\mathcal{P}(x_{n-1}, x_n, t)} \geq \mathcal{B}(\mathcal{P}(x_n, x_{n+1}, t)) \geq s^2 \]

By taking limit as \( n \to +\infty \) in the above inequality, we have

\[ \lim_{n \to +\infty} \mathcal{B}(\mathcal{P}(x_n, x_{n+1}, t)) = s^2 \]

Hence by definition of \( \mathcal{B} \), we have

\[ r = \lim_{n \to +\infty} \mathcal{P}(x_n, x_{n+1}, t) = 0 \]

which is a contradiction. That is \( r = 0. \) Now, we shall show that

\[ \lim_{n,m \to +\infty} \sup \mathcal{P}(x_n, x_m, t) = 0 \]

Suppose to the contrary that \( \lim_{n,m \to +\infty} \sup \mathcal{P}(x_n, x_m, t) > 0. \)

By (73), we have

\[ \mathcal{P}(x_n, x_m, t) = \mathcal{P}(Tx_{n+1}, Tx_{m+1}, t) \]

\[ \geq \mathcal{B}(\mathcal{P}(x_{n+1}, x_{m+1}, t)) \mathcal{P}(x_{n+1}, x_{m+1}, t) \]

That is,

\[ \frac{\mathcal{P}(x_n, x_m, t)}{\mathcal{B}(\mathcal{P}(x_{n+1}, x_{m+1}, t))} \geq \mathcal{P}(x_{n+1}, x_{m+1}, t) \]

By triangular inequality, we have

\[ \mathcal{P}(x_n, x_m, t) \leq s \mathcal{P}(x_n, x_{n+1}, t) + s^2 \mathcal{P}(x_{n+1}, x_{m+1}, t) \]

\[ + s^2 \mathcal{P}(x_{m+1}, x_m, t) \]

\[ \leq s \mathcal{P}(x_n, x_{n+1}, t) + s^2 \mathcal{P}(x_{n+1}, x_{m+1}, t) \]

\[ + s^2 \mathcal{P}(x_{m+1}, x_m, t) \]

Therefore,

\[ \mathcal{P}(x_n, x_m, t) \leq \left( 1 - \frac{s^2}{\mathcal{B}(\mathcal{P}(x_{n+1}, x_{m+1}, t))} \right)^{-1} \]

\[ (s \mathcal{P}(x_n, x_{n+1}, t) + s^2 \mathcal{P}(x_{m+1}, x_m, t) \]

By taking limit as \( n, m \to +\infty \) in the above inequality, since \( \lim_{n,m \to +\infty} \sup \mathcal{P}(x_n, x_m, t) > 0 \) and \( r = 0 \), we obtain

\[ \lim_{n,m \to +\infty} \left( 1 - \frac{s^2}{\mathcal{B}(\mathcal{P}(x_{n+1}, x_{m+1}, t))} \right)^{-1} = +\infty \]

which implies that

\[ \lim_{n,m \to +\infty} \sup \mathcal{B}(\mathcal{P}(x_{n+1}, x_{m+1}, t)) = (s^2)^+ \]

and so by definition of \( \mathcal{B} \), we have

\[ \lim_{n,m \to +\infty} \sup \mathcal{P}(x_n, x_{n+1}, t) = 0 \]

which is a contradiction. Hence,

\[ \lim_{n,m \to +\infty} \sup \mathcal{P}(x_n, x_m, t) = 0 \]

Since \( \lim_{n,m \to +\infty} \sup \mathcal{P}(x_n, x_m, t) = 0 \), \( \{x_n\}_{n=1}^{\infty} \) is a Cauchy sequence. Since \( X \) is a complete parametric b-metric space, the sequence \( \{x_n\}_{n=1}^{\infty} \) in \( X \) converges to \( x^* \in X \) so that

\[ \lim_{n \to +\infty} \mathcal{P}(x_n, x^*, t) = 0 \]

As \( T \) is surjective, there exists \( p \in X \) such that \( x^* = Tp \). Let us prove that \( x^* = p \). Suppose to the contrary that \( x^* \neq p \). Then by (73), we have

\[ \mathcal{P}(x_n, x^*, t) = \mathcal{P}(T(x_n+1), Tp, t) \]

\[ \geq \mathcal{B}(\mathcal{P}(x_{n+1}, p, t)) \mathcal{P}(x_{n+1}, p, t) \]
By taking limit as \( n \to +\infty \) in the above inequality and applying Lemma 6.2, we obtain

\[
\begin{align*}
0 &= \lim_{n \to +\infty} \mathcal{P}(x_n, x^*, t) \\
&\geq \lim_{n \to +\infty} \mathcal{B}(\mathcal{P}(x_{n+1}, p, t)) \lim_{n \to +\infty} \mathcal{P}(x_{n+1}, p, t) \\
&\geq \frac{1}{2} \lim_{n \to +\infty} \mathcal{B}(\mathcal{P}(x_{n+1}, x^*, t)) \mathcal{P}(x^*, p, t)
\end{align*}
\]

and hence,

\[
\lim_{n \to +\infty} \mathcal{B}(\mathcal{P}(x_{n+1}, x^*, t)) = 0
\]

which is a contradiction. Indeed,

\[
\lim_{n \to +\infty} \mathcal{B}(\mathcal{P}(x_{n+1}, x_n, t)) \geq s^2.
\]

Since \( \mathcal{B}(t) > s^2 \) for all \( t \in [0, \infty) \), therefore \( x^* = p \). Hence \( x^* = Tp = Tx^* \).

**CONFLICT OF INTEREST**

No conflict of interest was declared by the authors.

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All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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