Gazi University Journal of Science
GU J Sci
29(1):109-113 (2016)

Bivariate Fibonacci and Lucas Like Polynomials

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Received:28/10/2015 *Accepted:01/12/2015*

ABSTRACT

In this article, we study the generalized bivariate Fibonacci (GBF) and generalized bivariate Lucas (GBL) polynomials from specifying $p(x,y)$ and $q(x,y)$, classical bivariate Fibonacci and Lucas polynomials $((p(x,y) = x \text{ and } q(x,y) = y)$. Afterwards, we obtain the some properties of the GBF and GBL polynomials.

Keywords: Bivariate Fibonacci Polynomial, Binet Formula.

1. INTRODUCTION

Large classes of polynomial can be defined by Fibonacci-like recurrence relations and yield Fibonacci numbers. Such polynomials, called the Fibonacci polynomials, were studied in 1883 by E. Charles Catalan and E. Jacobsthal. Also, Lucas polynomials orginally studied in 1970 by Bicknell [7].

MacHenry [8-10] defined the generalized Fibonacci polynomial (GFP) and generalized Lucas polynomial (GLP). The GFPs are polynomials defined recursively by

$$F_{k,n}(t_1,t_2,\cdots,t_k) = t_1F_{k,n-1} + \cdots + t_{k-1}F_{k,n-k-1} + t_kF_{k,n-k}$$

and The GLPs are the polynomials defined recursively by

$$G_{k,n}(t_1,t_2,\cdots,t_k) = t_1G_{k,n-1} + \cdots + t_{k-1}G_{k,n-k-1} + t_kG_{k,n-k}$$

where $F_{k,0}(t_1,t_2,\cdots,t_k) = 1$, $G_{k,0}(t_1,t_2,\cdots,t_k) = k$ and $t = (t_1,t_2,\cdots,t_k)$ is set of coefficients of the monic polynomial $X^k - t_1X^{k-1} - \cdots - t_k = [t_1,t_2,\cdots,t_k]$ the companion or core polynomial, the coefficients t_j being regarded as interminate parameters.

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In [1,2,6], the authors introduced bivariate Fibonacci and Lucas polynomials and give the some properties. Also, Catalini [3] considered the generalized bivariate Fibonacci and Lucas polynomials. In [12], Swamy show that there exists an intimate relationship between the network functions of certain ladder one-port and two-port networks and a set of generalized bivariate Fibonacci and Lucas polynomials. In [13], authors defined the bivariate and trivariate Fibonacci polynomials. Some properties of these polynomials are derived and these polynomials in special cases are studied. In [4,5], Djordjevic considered the generating functions, explicit formulas and partial derivative sequences of the generalized Fibonacci and Lucas polynomials.

In [11], Nalli and Haukkanen defined the $h(x)$ −Fibonacci and Lucas polynomials which $h(x)$ is a polynomial with real coefficients. Also, given the some properties of $h(x)$ −Fibonacci and Lucas polynomials. The $h(x)$ −Fibonacci and Lucas polynomials are defined recursively by

$$F_{h,n+1}(x) = h(x)F_{h,n}(x) + F_{h,n-1}(x);\; F_{h,0}(x) = 0, F_{h,1}(x) = 1$$

and

$$L_{h,n+1}(x) = h(x)L_{h,n}(x) + L_{h,n-1}(x);\; L_{h,0}(x) = 2, L_{h,1}(x) = h(x)$$

In [14], the authors defined the bivariate Fibonacci and Lucas p −polynomials ($p \geq 0$is integer) and obtained the some properties of bivariate Fibonacci and Lucas p −polynomials.

In this study, based on the definitions Nalli and Haukkanen [11] and Tuglu, et al. [14], we make a new generalization of bivariate Fibonacci and Lucas polynomials.

2. GENERALIZED BIVARIATE FIBONACCI AND LUCAS POLYNOMIALS

Let $\boldsymbol{p(x,y)}$ and $\boldsymbol{q(x,y)}$ be polynomials with real coefficients. For $\boldsymbol{n} \geq 2$, the generalized bivariate Fibonacci polynomials (GBF) are defined by the recurrence relation

$$H_n(x,y) = p(x,y)H_{n-1}(x,y) + q(x,y)H_{n-2}(x,y) \quad (2.1)$$

where $\boldsymbol{H_0(x,y) = 0, H_1(x,y) = 1}$.

The recursive relation for he generalized bivariate Lucas polynomials (GBL) is

$$K_n(x,y) = p(x,y)K_{n-1}(x,y) + q(x,y)K_{n-2}(x,y) \quad (2.2)$$

with the initial conditions $K_0(x,y) = 2, K_1(x,y) = p(x,y)$.

The characteristic equation of the generalized bivariate Fibonacci and Lucas polynomials is

$$t^2 - p(x,y)t - q(x,y) = 0$$

(2.3) The Binet's formulas for the GBF polynomial $H_n(x,y)$ and GBL polynomias $K_n(x,y)$ are

$$H_n(x,y) = \frac{\alpha^n(x,y) - \left(-q(x,y)\right)^n \alpha^{-n}(x,y)}{\alpha(x,y) + q(x,y)\alpha^{-1}(x,y)}$$

and

$$K_n(x,y) = \alpha^n(x,y) + \left(-q(x,y)\right)^n \alpha^{-n}(x,y)$$

Where $\alpha(x,y)$ and $\alpha^{-1}(x,y)$ are roots of characteristic equation (2.3).

The generating functions of GBF and GBL polynomials are

$$h(z) = \frac{z}{1 - p(x,y)z - q(x,y)z^2}$$

and

$$k(z) = \frac{2 - p(x,y)z}{1 - p(x,y)z - q(x,y)z^2}$$

GBF and GBL polynomials for the negative values of n are

$$H_{-n}(x,y) = (-1)^{n+1}q^{-n}(x,y)H_n(x,y)$$

and

$$K_{-n}(x,y) = (-1)^n q^{-n}(x,y)K_n(x,y).$$

For the different $p(x,y)$ and $q(x,y)$, the recursive relation generates different polynomial sequences. These polynomial sequences are given as follows.

$p(x,y)$	$q(x,y)$	$H_n(x,y)$	$K_n(x,y)$
x	y	Bivariate.Fibonacci, $F_n(x,y)$	Bivariate.Lucas, $L_n(x,y)$
x	1	Fibonacci,$F_n(x)$	Lucas, $L_n(x)$
$2x$	1	Pell, $P_n(x)$	Pell-Lucas, $Q_n(x)$
1	$2x$	Jacobsthal, $J_n(x)$	Jaco-Lucas, $j_n(x)$
$2x$	-1	Chebyshev of 2nd kind , $U_{n-1}(x)$	Chebyshev of 1st kind, $2T_n(x)$
$3x$	-2	Fermat, $F_n(x)$	Fermat-Lucas, $f_n(x)$

Theorem 1. The explicit formulas of the GBF and GBL polynomials are given as

$$H_n(x,y) = \sum_{j=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \binom{n-j-1}{j} p^{n-2j-1}(x,y)q^j(x,y) \quad (2.4)$$

and

$$K_n(x,y) = \sum_{j=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{n}{n-j}\binom{n-j}{j} p^{n-2j}(x,y)q^j(x,y) \quad (2.5)$$

Proof. Using the mathematical induction principle, the formula (2.4) trivially true for $n = 2$. Assume it is true for $n = k$. Shortly, let $p(x,y) = p$ and $q(x,y) = q$ in recurrence relation (2.1) for $n = k + 1$. Then, we have

$$H_{k+1}(x,y) = pH_k(x,y) + qH_{k-1}(x,y)$$

$$H_{k+1}(x,y) = \mathrm{p}\sum_{j=0}^{\left\lfloor\frac{k-1}{2}\right\rfloor}\binom{k-j-1}{j}p^{k-2j-1}q^j + q\sum_{j=0}^{\left\lfloor\frac{k-2}{2}\right\rfloor}\binom{k-j-2}{j}p^{k-2j-2}q^j$$

$$= p\left[\binom{k-1}{0}p^{k-1} + \binom{k-2}{1}p^{k-3}q + \cdots + \binom{\frac{k-1}{2}}{\frac{k-1}{2}}q^{\frac{k-1}{2}}\right]$$

$$+q\left[\binom{k-2}{0}p^{k-2} + \binom{k-3}{1}p^{k-4}q + \cdots + \binom{\frac{k-2}{2}}{\frac{k-2}{2}}q^{\frac{k-2}{2}}\right]$$

$$= \binom{k-1}{0}p^k + \left[\binom{k-2}{0} + \binom{k-2}{1}\right]p^{k-2}q + \cdots + \binom{\frac{k-2}{2}}{\frac{k-2}{2}}q^{\frac{k}{2}}$$

From the relation $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ of binomial coefficients, we have

$$H_{k+1}(x,y) = \binom{k-1}{0}p^k + \binom{k-1}{1}p^{k-2}q + \cdots + \binom{\frac{k-2}{2}}{\frac{k-2}{2}}q^{\frac{k}{2}}$$

Therefore

$$H_{k+1}(x,y) = \sum_{j=0}^{\left\lfloor\frac{k}{2}\right\rfloor}\binom{k-j}{j}p^{k-2j}(x,y)q^j(x,y)$$

The formula holds, and the proof is completed. The proof for the GBL polynomials is similar.

Theorem 2. The sum of the GBF and GBL polynomials are

$$\sum_{k=0}^{n}H_k(x,y) = \frac{H_{n+1}(x,y) + q(x,y)H_n(x,y) - 1}{p(x,y) + q(x,y) - 1}$$

and

$$\sum_{m=0}^{n}K_m(x,y) = \frac{K_{n+1}(x,y) + q(x,y)K_n(x,y) + p(x,y) - 2}{p(x,y) + q(x,y) - 1}$$

where $p(x,y) + q(x,y) \neq 1$

Proof. Taking the Binet's formulas for $H_n(x,y)$ and $K_n(x,y)$ the proof is clear.

Teorem 3. (Catalan Identity) Let $H_n(x,y)$ be nth GBF polynomial. Then

$$H_{n+k}(x,y)H_{n-k}(x,y) - H_n^2(x,y) = -\left(-q(x,y)\right)^{n-k}H_k^2(x,y)$$

where $n \geq 0, n \geq k$.

Proof. Let $\alpha(x,y) = \alpha$, $\alpha^{-1}(x,y) = \alpha^{-1}$ and $q(x,y) = q$ Using the Binet's formula fort he left hend side (LHS), we have

$$(LHS) = \left(\frac{\alpha^{n+k} - (-q)^{n+k}\alpha^{-n-k}}{\alpha + q\alpha^{-1}}\right)\left(\frac{\alpha^{n-k} - (-q)^{n-k}\alpha^{-n+k}}{\alpha + q\alpha^{-1}}\right) - \left(\frac{\alpha^n - (-q)^n\alpha^{-n}}{\alpha + q\alpha^{-1}}\right)^2$$

$$= -(-q)^{n-k}\left(\frac{\alpha^{2k} - 2(-q)^k + (-q)^{2k}\alpha^{-2k}}{(\alpha + q\alpha^{-1})^2}\right)$$

$$= -(-q)^{n-k}\left(\frac{\alpha^k - (-q)^k\alpha^{-k}}{\alpha + q\alpha^{-1}}\right)^2$$

$$= -(-q)^{n-k}H_k^2(x,y).$$

Theorem 4. (d'Ocagne's Identity) Let $H_n(x,y)$ be nth GBF polynomial. The d'Ocagne's identity is

$$H_n(x,y)H_{m+1}(x,y) - H_m(x,y)H_{n+1}(x,y) = \left(-q(x,y)\right)^m H_{n-m}(x,y)$$

where $n \geq 0, m \geq 0$.

Proof. Using recurrence relation (2.1) to left hand side (LHS), we have

$$(LHS) = H_n(x,y)[p(x,y)H_m(x,y) + q(x,y)H_{m-1}(x,y)] - H_m(x,y)[p(x,y)H_n(x,y) + q(x,y)H_{n-1}(x,y)]$$

$$= -q(x,y)[H_m(x,y)H_{n-1}(x,y) - H_n(x,y)H_{m-1}(x,y)]$$

Similarly, using recurrences for $H_m(x,y)$ and $H_n(x,y)$, we obtain

$$(LHS) = \left(-q(x,y)\right)^2[H_{m-1}(x,y)H_{n-2}(x,y) - H_{n-1}(x,y)H_{m-2}(x,y)]$$

Repeated process for m times, we have

$$(LHS) = \left(-q(x,y)\right)^m[H_{n-m}(x,y)H_1(x,y) - H_{n-m+1}(x,y)H_0(x,y)]$$

Therefore, d'Ocagne's identity of GBF polynomial is

$H_n(x,y)H_{m+1}(x,y) - H_m(x,y)H_{n+1}(x,y) = \left(-q(x,y)\right)^m H_{n-m}(x,y).$

Theorem 5. Let $H_n(x,y)$ and $K_n(x,y)$ be nth GBF and GBL polynomials. Then

$H_{n+k}(x,y) - H_k(x,y)K_n(x,y) = \left(-q(x,y)\right)^k H_{n-k}(x,y)$ (2.6)

where $n \geq 0, n \geq k$.

If we take $k = 1$ in (2.6), we obtain the relation between GBF and GBL polynomials as follows.

$$K_n(x,y) = H_{n+1}(x,y) + q(x,y)H_{n-1}(x,y)$$

Corollary 1. Let $K_n(x,y)$ be nth GBL polynomials. Then

$K_n^2(x,y) = K_{2n}(x,y) + 2\left(-q(x,y)\right)^n.$

Corollary 2. Let $K_n(x,y)$ be nth GBL polynomials. Then

$K_n^2(x,y) = \left(p^2(x,y) + 4q(x,y)\right)H_n^2(x,y) + 4\left(-q(x,y)\right)^n.$

In [7], the $Q-$matrix associated with Fibonacci numbers is defined by $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. In [2,3], the author defined the $A-$ matrix associated with bivariate Fibonacci polynomials. The Matrix A is $A = \begin{pmatrix} x & 1 \\ y & 0 \end{pmatrix}$. Now, we define the matrix $Q_{p,q}(x,y)$**.** The matrix $Q_{p,q}(x,y)$ that plays the role of $Q-$matrix and $A-$matrix. The matrix $Q_{p,q}(x,y)$ is

$$Q_{p,q}(x,y) = \begin{pmatrix} p(x,y) & 1 \\ q(x,y) & 0 \end{pmatrix}$$

It's note that, the determinant of matrix $Q_{p,q}(x,y)$ is $-q(x,y)$. By easy induction

$$\left(Q_{p,q}(x,y)\right)^n = \begin{pmatrix} H_{n+1}(x,y) & H_n(x,y) \\ q(x,y)H_n(x,y) & q(x,y)H_{n-1}(x,y) \end{pmatrix}$$

Now, we give Cassini identity which is special case of Catalan identity.

Teorem 8. (Cassini Identity) Let $H_n(x,y)$ be nth GBF polynomial. The Cassini identity is

$$H_{n+1}(x,y)H_{n-1}(x,y) - H_n^2(x,y) = -\left(-q(x,y)\right)^{n-1}$$

where $n \geq 1$.

Proof. Since $det\left(\left(Q_{p,q}(x,y)\right)^n\right) = \left(-q(x,y)\right)^n$, we have

$$q(x,y)\left(H_{n+1}(x,y)H_{n-1}(x,y) - H_n^2(x,y)\right) = \left(-q(x,y)\right)^n$$

Hence

$$H_{n+1}(x,y)H_{n-1}(x,y) - H_n^2(x,y) = -\left(-q(x,y)\right)^{n-1}$$

Theorem 9. (Honsberger Identity) Let $H_n(x,y)$ be nth GBF polynomial. Then

$H_{n+m}(x,y) = q(x,y)H_n(x,y)H_{m-1}(x,y) + H_m(x,y)H_{n+1}(x,y)$

where $n \geq 0, m \geq 0.$

Proof. From the identity

$$\left(Q_{p,q}(x,y)\right)^{n+m} = \left(Q_{p,q}(x,y)\right)^n \left(Q_{p,q}(x,y)\right)^m$$

and matrix equality, the result is clear.

Taking $m = n$ in Honsberger identity, we have

$H_{2n}(x,y) = H_n(x,y)K_n(x,y)$

If we take $n + 1$ instead of m in Honsberger identity,we obtain

$H_{2n+1}(x,y) = H_{n+1}^2(x,y) + q(x,y)H_n^2(x,y).$

Theorem 10. The eigenvalues of $\left(Q_{p,q}(x,y)\right)^n$ are $\alpha^n(x,y)$ and $\alpha^{-n}(x,y)$.

Proof. The characteristic equation of $\left(Q_{p,q}(x,y)\right)^n$ is

$$\det\left(\left(Q_{p,q}(x,y)\right)^n - \mu I\right) = \mu^2 - \mu K_n(x,y) + \left(-q(x,y)\right)^n$$

The roots of this equation are

$$\mu = \frac{K_n(x,y) \pm \sqrt{K_n^2(x,y) - 4\left(-q(x,y)\right)^n}}{2}$$

From Corallary 2, we have

$$\mu = \frac{K_n(x,y) \pm \sqrt{p^2(x,y) + 4q(x,y)}H_n(x,y)}{2}$$

Therefore, we obtain the eigenvalues as $\alpha^n(x,y)$ and $\alpha^{-n}(x,y)$.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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