Fixed Point Theorem Through $\Omega$-distance of Suzuki Type Contraction Condition

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ABSTRACT
In this article, we utilize the notion of $\Omega$-distance in the sense of Saadati et al [ R. Saadati, S.M. Vaezpour, P. Vetro and B.E. Rhoades, Fixed point theorems in generalized partially ordered G-metric spaces, Mathematical and Computer Modeling, 52, 797-801, 2010 ] to introduce and prove some fixed point results of self-mapping under contraction conditions of the form $\Omega$-Suzuki-contractions.  
Key Words: $\Omega$-Distance, Fixed Point Theory, G-Metric Space.

1. INTRODUCTION
G-metric space was introduced by Mustafa and Sims [1] in 2006, which is a generalization of metric space. Since 2006, many researchers have worked on G-metric spaces; see for example [2]-[10].

Samet et al in [11] and [12] proved that many results in G-metric spaces can be derived from known results of the corresponding usual metric space. Moreover, the notion of $\Omega$-distance related to a complete G-metric space was considered by Saadati et.al. [13] in 2010.

Recently, many researchers studied several fixed point results using $\Omega$-distance mappings; see for example, [14]-[17]. It is worth mentioning that the interesting method of Samet et. al. [11] and [12] doesn’t work in the fixed point results involving $\Omega$-distance.

In this paper, we prove new results of fixed point theorem using the map $\Omega$ in a complete G-metric space under contractive conditions of the form $\Omega$-Suzuki-contraction.

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Definition 1.1. [1]. Let $X$ be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}^+$ be a function that satisfies the following conditions:

(G1) $G(x, y, z) = 0$ if $x = y = z$;
(G2) $G(x, y, z) > 0$ for all $x, y \in X$ with $x \neq y$;
(G3) $G(x, y, z) \leq G(x, z, y)$ for all $x, y, z \in X$ with $x \neq z$;
(G4) $G(x, y, z) = G(p(x, y, z))$, for any permutation of $x, y, z$;
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then the function $G$ is called a generalized metric space, or more specifically $G$-metric on $X$, and the pair $(X, G)$ is called a $G$-metric space.

The notion of convergence and Cauchy sequences in the setting of a $G$-metric space are given as follows:

Definition 1.2. [1]. Let $(X, G)$ be a $G$-metric space, and let $(x_n)$ be a sequence of points of $X$. We say that $(x_n)$ is $G$-Cauchy if for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x_n, x_m, x_k) < \varepsilon$ for all $n, m \geq k$.

Definition 1.3. [1]. Let $(X, G)$ be a $G$-metric space. A sequence $(x_n)$ in $X$ is said to be $G$-Cauchy if for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x_n, x_m, x_k) < \varepsilon$ for all $n, m, k \geq k$.

Definition 1.4. [5]. A $G$-metric space $(X, G)$ is said to be $G$-complete or a complete $G$-metric space if every $G$-Cauchy sequence in $(X, G)$ is $G$-convergent in $(X, G)$.

In 2010, Saadati et al. [13] introduced the notion of $\Omega$-distance related to a complete $G$-metric space and proved many results.

Definition 1.5. [13]. Let $(X, G)$ be a $G$-metric space. Then a function $\Omega : X \times X \times X \to [0, \infty)$ is called an $\Omega$-distance on $X$ if the following conditions are satisfied:

(a) $\Omega(x, y, z) \leq \Omega(x, a, a) + \Omega(a, y, z)$ for all $x, y, z, a \in X$;
(b) for any $x, y \in X$, the functions $\Omega(x, \cdot, y) : X \to [0, \infty)$ are lower semi continuous;
(c) for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $\Omega(x, a, a) \leq \delta$ and $\Omega(a, y, z) \leq \delta$, then $\Omega(x, y, z) \leq \varepsilon$.

Definition 1.6. [13]. Let $(X, G)$ be a $G$-metric space and $\Omega$ be an $\Omega$-distance on $X$. Then we say that $X$ is $\Omega$-bounded if there exists $M > 0$ such that $\Omega(x, y, z) \leq M$ for all $x, y, z \in X$.

The following lemma plays an important role in the development of the results in this article.

Lemma 1.1. [13]. Let $X$ be a metric space with metric $G$ and $\Omega$ be an $\Omega$-distance on $X$. Let $(x_n), (y_n)$ be sequences in $X$ and $(a_n), (b_n)$ be sequences in $[0, \infty)$ converging to zero. Then for all $x, y, z, a \in X$, we have the following:

(1) If $\Omega(y_n, x_n, x_0) \leq a_0$ and $\Omega(x_n, y, z) \leq b_0$ for $n \in \mathbb{N}$, then $\Omega(x_n, y, z) \leq s$ and hence $y_n \to z$;
(2) If $\Omega(y_n, x_n, x_0) \leq a_0$ and $\Omega(x_n, y_m, z) \leq b_0$ for all $m > n \in \mathbb{N}$, then $\Omega(y_n, y_m, z) \to 0$ and hence $y_n \to z$;
(3) If $\Omega(x_n, x_m, x_0) \leq a_0$ then the sequence $(x_n)$ is a $G$-Cauchy sequence, for all $m, n, l \in \mathbb{N}$ with $n \leq m \leq l$;
(4) If $\Omega(x_n, a, a) \leq a_0$ for any $n \in \mathbb{N}$, then $(x_n)$ is a $G$-Cauchy sequence.

2. MAIN RESULT

Definition 2.7. [19] A nondecreasing continuous function $\varphi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following condition holds: $\varphi(t) = 0$ if and only if $t = 0$.

Definition 2.8. A mapping $T : X \to X$ of a $G$-metric space $(X, G)$ is called an $\Omega$-Suzuki contraction if there exists $k \in (0, 1)$ and an altering distance function $\varphi$ such that for all $x, y, z \in X$ and $p, q \in \mathbb{N}$ with $q \geq p$, the following condition holds

if $(1 - k) \Omega(x, T^p(x), T^q(x)) \leq \Omega(x, y, z)$, then $\varphi(\Omega(x, y, z)) \leq k \varphi(\Omega(x, y, z))$.

Theorem 2.2. Let $(X, G)$ be a complete $G$-metric space and $\Omega$ be an $\Omega$-distance on $X$ such that $X$ is $\Omega$-bounded. Let $T : X \to X$ be an $\Omega$-Suzuki-contraction mapping that satisfies the following condition:

for all $u \in X$ if $Tu = u$, then $\inf\{\Omega(u, x, u) : x \in X\} > 0$. (2.1)

Then $T$ has a fixed point in $X$. Moreover, for any fixed point $x \in X$ of $T$, we have $\Omega(x, x, x) = 0$.

Proof. Let $x_0 \in X$ and define a sequence $(x_n)$ in $X$ inductively by setting $x_{n+1} = Tx_n$, $n \in \mathbb{N}$.

For $p = q = 1$, since $(1 - k) \Omega(x, Tx, Tx) \leq \Omega(x, Tx, Tx)$, we have $\varphi(\Omega(Tx, T^2x, T^2x)) \leq k \varphi(\Omega(x, Tx, Tx))$.

Substituting $x = x_{n-1}$ in the inequality (2.2), gives us

$\varphi(\Omega(x_n, x_{n+1}, x_{n+1})) = \varphi(\Omega(Tx_{n-1}, Tx_n, x_n)) \leq k \varphi(\Omega(x_{n-1}, x_{n-1}, x_{n-1}))$. (2.3)

Since $k < 1$ and $\varphi$ is an altering distance function, the sequence $(\varphi(\Omega(x_n, x_{n+1}, x_{n+1}))) : n \in \mathbb{N}$ is a non-increasing sequence of nonnegative real numbers. Therefore, there is $r \geq 0$ such that

$\lim_{n \to \infty} \Omega(x_n, x_{n+1}, x_{n+1}) = r$.

Taking the limit as $n \to \infty$ in 2.3, implies that $r \leq k \varphi r$ and thus $r = 0$, since $k < 1$. Hence

$\lim_{n \to \infty} \Omega(x_n, x_{n+1}, x_{n+1}) = 0$. (2.4)

Moreover, for $p = 1$, and $q \geq 1$, since $(1 - k)\Omega(x, Tx, T^qTx) \leq \Omega(x, Tx, T^qTx)$ holds for every $x \in X$, then

$\varphi(\Omega(Tx, T^2x, T^{q+1}x)) \leq k \varphi(\Omega(x, Tx, T^qTx))$. (2.5)

For $n, s \in \mathbb{N}$ with $s \geq 1$, substituting $x = x_{n-1}$ in (2.5), implies that
φΩ((x_n, x_{n+1}, x_{n+2}) = φΩ(T_{n-1} T_n T_{n+1}) ≤ k φΩ(x_{n-1}, x_n, x_{n+1}).

Since k < 1 and φ is an altering distance function, the sequence (Ω(x_n, x_{n+1}, x_{n+2}); n ∈ N) is a non-increasing sequence of nonnegative real numbers. Therefore, there is \( r \geq 0 \) such that

\[
\lim_{n \to \infty} \Omega(x_n, x_{n+1}, x_{n+2}) = r.
\]

Applying the limit as \( n \to \infty \) to the inequality 2.6, gives us \( φr \leq k φr \). Since \( k < 1 \), we have \( r = 0 \) and hence

\[
\lim_{n \to \infty} \Omega(x_n, x_{n+1}, x_{n+2}) = 0, \quad \forall s \geq 1. \tag{2.7}
\]

Considering the Definition 1.5, implies that

\[
\Omega(x_n, x_m, x_l) ≤ \Omega(x_n, x_{n+1}, x_{n+1}) + \Omega(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + \Omega(x_{m-1}, x_m, x_l),
\]

for all \( l, m, n \in \mathbb{N} \) with \( l ≥ m ≥ n, m = n + s \) and \( l = m + t \).

By taking the limit of the above inequality as \( n \to \infty \), we get

\[
\lim_{n, m, l \to \infty} \Omega(x_n, x_m, x_l) = 0.
\]

Lemma 1.1 implies that \( (x_n) \) is a G-Cauchy sequence and hence \( (x_n) \) converges to an element \( u \in X \). For all \( ε > 0 \), since \( (x_n) \) is a G-Cauchy sequence, there exists \( N \in \mathbb{N} \) such that \( \Omega(x_n, x_m) < ε \), for all \( n, m, l ≥ N \).

\[
\lim_{l \to \infty} \inf \Omega(x_n, x_m, x_l) ≤ ε, \quad \forall \, n, \, m, \, n ≥ N.
\]

The lower semi-continuity of \( \Omega \) implies that

\[
\Omega(x_n, x_m, u) \leq \lim_{l \to \infty} \inf \Omega(x_n, x_m, x_l) ≤ ε, \quad \forall \, n, \, m, \, n ≥ N.
\]

Considering \( m = n + 1 \) in (2.8), gives us \( \Omega(x_n, x_{n+1}, u) ≤ ε, \quad \forall \, n, \, m, \, l ≥ N \).

Assume that \( Tu ≠ u \). Then 2.1 implies that

\[
0 < \inf \{ \Omega(x, T x, u) : x \in X \} ≤ \inf \{ \Omega(x, x_{n+1}, u) : n ≥ N \} ≤ ε, \quad \forall \, ε > 0 \text{ which is a contradiction.}
\]

Then \( Tu = u \). Let \( z = T z \). Then by (2.2), we have

\[
Ω(z, z, z) = Ω(T z, T^2 z, T^3 z) ≤ k φΩ(z, T z, T z) = k φΩ(z, z, z).
\]

Since \( k < 1 \) and \( φ \) is an altering distance function, we have \( Ω(z, z, z) = 0 \).

**Definition 2.9.** A mapping \( T : X → X \) of a G-metric space \((X, G)\) is called a generalized \( Ω \)-Suzuki-contraction if there exists \( k \in (0, 1) \) and an altering distance function \( φ \) such that the following condition holds:

If for all \( p, q \in \mathbb{N} \) with \( q ≥ p \),

\[
(1 - k) \, Ω(x, T^p x, T^q x) ≤ Ω(x, y, x)
\]

then we have

\[
Ω(T x, T y, T z) ≤ k \max \{ Ω(x, T x, T x), Ω(y, T y, T y), Ω(z, T z, T z) \}
\]

for all \( x, y, z \in X \).

**Lemma 2.3.** Let \( T : X → X \) be a generalized \( Ω \)-Suzuki-contraction. Then

\[
Ω(T x, T^2 x, T^3 x) ≤ k Ω(x, T x, T x) \quad \forall \, x \in X. \tag{2.9}
\]

**Proof.** Assume \( p = q = 1 \). Since \( (1 - k)\Omega(x, T x, T x) ≤ Ω(x, T x, T x) \) holds for every \( x \in X \), then we have

If \( \max \{ Ω(x, T x, T x), Ω(x, T^2 x, T x) \} = Ω(x, T^2 x, T x) \), then \( Ω(x, T^2 x, T x) ≤ k Ω(x, T^2 x, T^2 x) \) which is a contradiction, since \( k < 1 \). Therefore, \( \max \{ Ω(x, T x, T x), Ω(x, T^2 x, T^2 x) \} = Ω(x, T x, T x) \) and hence

\[
Ω(T x, T^2 x, T^3 x) ≤ k Ω(x, T x, T x) \quad \forall \, x \in X. \tag{2.10}
\]

**Lemma 2.4.** Let \( q ≥ 1 \) and \( T : X → X \) be a generalized \( Ω \)-Suzuki-contraction. Then

\[
Ω(T^q x, T^{q+1} x, T^{q+1} x) ≤ k^q \, Ω(x, T x, T x) \quad \forall \, x \in X. \tag{2.11}
\]

**Proof.** By substituting \( x \) in Lemma (2.3) by \( T^q x \), we get

\[
Ω(T^q x, T^{q+1} x, T^{q+1} x) = Ω(T (T^{q-1} x), T (T^{q-1} x), T (T^{q-1} x)) ≤ k \, Ω(T^{q-1} x, T^q x, T^q x)
\]

Thus

\[
Ω(T^q x, T^{q+1} x, T^{q+1} x) ≤ k^q \, Ω(x, T x, T x). \tag{2.12}
\]

**Theorem 2.5.** Let \((X, G)\) be a complete G-metric space and \( Ω \) be an \( Ω \)-distance on \( X \) such that \( X \) is \( Ω \)-bounded. Let \( T \) be a self-mapping on \( X \) that satisfies the following conditions:

1. \( T \) is a generalized \( Ω \)-Suzuki-contraction;
2. if for all \( u \in X \), \( Tu ≠ u \), then

\[
\inf \{ Ω(x, T x, u) : x \in X \} > 0.
\]

Then \( T \) has a fixed point in \( X \).

**Proof.** Let \( x_0 \in X \) and define a sequence \( \{ x_n \} \) in \( X \) inductively by taking \( x_n = T x_{n-1} \) for \( n \in \mathbb{N} \).

Substitute \( x = x_{n-1} \) in (2.10), implies that

\[
Ω(x_n, x_{n+1}, x_{n+1}) = Ω(Tx_{n-1}, Tx_n, Tx_n) ≤ k \, Ω(x_{n-1}, x_n, x_{n+1})
\]

Thus

\[
Ω(T^q x, T^{q+1} x, T^{q+1} x) ≤ k^q \, Ω(x, T x, T x). \tag{2.11}
\]

Since \( X \) is \( Ω \)-bounded, there exists \( M > 0 \) such that \( Ω(x, y, z) ≤ M \) for all \( x, y, z \in X \). Hence
\[ \Omega(x_n, x_{n+1}, x_{n+1}) \leq k^{2M}. \]

By taking the limit as \( n \to \infty \) for both sides, we get

\[ \lim_{n \to \infty} \Omega(x_n, x_{n+1}, x_{n+1}) = 0. \tag{2.13} \]

since \( k < 1 \). Also, for \( p = 1 \), and \( q \geq 1 \), since \( (1-k)\Omega(x, Tx, T^q x) \leq \Omega(x, Tx, T^q x) \) holds for every \( x \in X \), we have

\[
\Omega(Tx, T^{2}x, T^{q+1}x) \leq k \max \{ \Omega(x, Tx, Tx), \Omega(Tx, T^{2}x, T^{q}x), \Omega(T^{q}x, T^{q+1}x, T^{q+1}x) \} = k \max \{ \Omega(x, Tx, Tx), \Omega(T^{q}x, T^{q+1}x, T^{q+1}x) \}.
\]

But from 2.11, we have \( \Omega(T^{q}x, T^{q+1}x, T^{q+1}x) \leq k \Omega(Tx, Tx, Tx) \) and thus,

\[ \Omega(Tx, T^{2}x, T^{q+1}x) \leq k \Omega(x, Tx, Tx). \tag{2.14} \]

For \( n, s \in \mathbb{N} \) with \( s \geq 1 \) substitute \( x = x_{n-s-1} \) in (2.14), implies that

\[ \Omega(x_n, x_{n+s}, x_{n+s}) = \Omega(Tx_{n-s-1}, T^{2}x_{n-s-1}, T^{q}x_{n+s-1}) \leq k \Omega(x_{n-s-1}, x_{n+s}). \]

Taking the limit as \( n \to \infty \) for both sides and using 2.13, we get

\[ \lim_{n \to \infty} \Omega(x_n, x_{n+1}, x_{n+s}) = 0. \tag{2.15} \]

The Definition 1.5 implies that

\[ \Omega(x_n, x_{m}, x_{l}) \leq \Omega(x_n, x_{m}, x_{l}) + \Omega(x_{m}, x_{l}, x_{m}) \]

for all \( l, m, k \in \mathbb{N} \) with \( l \geq m \geq n, m = n+s \) and \( l = m+t \). Applying the limit as \( n \to \infty \) and using 2.13 and 2.15, we get that

\[ \lim_{n, m, k \to \infty} \Omega(x_n, x_{m}, x_{l}) = 0. \]

Lemma 1.1 implies that \( (x_n) \) is a G-Cauchy sequence and so \( (x_n) \) converges to some \( u \in X \). Since \( (x_n) \) is a G-Cauchy sequence, then for all \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that \( \Omega(x_n, x_m, x_l) \leq \varepsilon \), for all \( n, m, l \geq N \). Thus

\[ \lim_{l \to \infty} \inf \Omega(x_n, x_m, x_l) \leq \varepsilon. \]

Since \( \Omega \) is lower semi-continuous, we have

\[ \Omega(x_n, x_m, u) \leq \liminf_{l \to \infty} \Omega(x_n, x_m, x_l) \leq \varepsilon, \tag{2.16} \]

for all \( n, m \geq N \).

Considering \( m = n + 1 \) in (2.16), we get \( \Omega(x_n, x_{n+1}, u) \leq \varepsilon \), for all \( n \geq N \). Suppose that \( Tu \neq u \). Then Condition 2.12 implies that

\[ 0 < \inf \{ \Omega(x, Tx, u); x \in X \} \leq \inf \{ \Omega(x_n, x_{n+1}, u); n \geq N \} \leq \varepsilon, \]

for all \( \varepsilon > 0 \) which is a contradiction. Therefore \( Tu = u \).

**CONFLICT OF INTEREST**

No conflict of interest was declared by the authors.

**REFERENCES**


