On Left Primary and Weakly Left Primary Ideals in Γ-LA- Rings

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ABSTRACT

In this paper, we study left ideals, left primary and weakly left primary ideals in Γ-LA-rings. Some characterizations of left primary and weakly left primary ideals are obtained. Moreover, we investigate relationships left primary and weakly left primary ideals in Γ-LA-rings. Finally, we obtain necessary and sufficient conditions of a weakly left primary ideal to be a left primary ideals in Γ-LA-rings.

Keywords: Γ-LA-ring, left primary ideal, weakly left primary ideal, left ideal.

1. INTRODUCTION

Abel-Grassmann’s groupoid (AG-groupoid) is the generalization of semigroup theory with the wide range of usages in theory of flocks [6]. The fundamentals of this non-associative algebraic structure were the first discovered by Kazim and Naseeruddin [1]. AG-groupoid is a non-associative algebraic structure mid way between a groupoid and a commutative semigroup. It is interesting to note that an AG-groupoid with right identity becomes a commutative monoid [4]. This structure is closely related with a commutative semigroup because if an AG-groupoid contains a right identity, then it becomes a commutative monoid [4]. A left identity in an AG-groupoid is unique. Ideals in AG-groupoids have been discussed by Mushtaq and Yousuf [4, 5].

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In 1981, the notion of \( \Gamma \)-semigroups was introduced by Sen. Let \( S \) and \( \Gamma \) be any nonempty sets. If there exists a mapping \( S \times \Gamma \times S \rightarrow S \) written \((a,\alpha, c)\) by \( a\alpha c \). \( S \) is called a \( \Gamma \)-semigroup if \( S \) satisfies the identity:

\[
(ab)c = a(bc)
\]

for all \( a,b,c \in S \) and \( \alpha, \beta \in \Gamma \). A \( \Gamma \)-AG-groupoid analogous to \( \Gamma \)-semigroups. A groupoid \( S \) is called a \( \Gamma \)-AG-groupoid if it satisfies the left inverse law:

\[
(\alpha b)c = (\alpha b)\delta u
\]

for all \( a,b,c,d \in S \) and \( \gamma, \delta \in \Gamma \) [2]. This structure is also known as left almost semigroup (LA-semigroup).

S.M. Yusuf in [18] introduces the concept of a left almost ring (LA-ring). That is, a non-empty set \( R \) with two binary operations \( + \) and \( \cdot \) is called a left almost ring, if \( (R,+) \) is a \( \Gamma \)-AG-groupoid and distributive laws of \( \cdot \) over \( + \) holds. Further in [12] T. Shah and I. Rehman generalize the notions of commutative semigroup rings into \( \Gamma \)-semigroup LA-rings. However T. Shah and Fazal ur Rehman in [12] generalize the notion of a LA-ring into an nLA-ring. A near left almost ring (nLA-ring) \( N \) is a LA-group under \( + \) and distributive property of \( \cdot \) over \( + \) holds.

T. Shah, Fazal ur Rehman and M. Raees asserted that a commutative ring \((R,+,\cdot)\), we can always obtain a LA-ring \((R,\oplus,\otimes)\) by defining, for \( a,b,c \in R, a\otimes b = b-a \) and \( a\cdot b \) is same as in the ring. Furthermore, in this paper we characterize the left primary and weakly left primary ideals in \( \Gamma \)-LA-rings. Moreover, we investigate relationships left primary and weakly left primary ideals in \( \Gamma \)-LA-rings. Finally, we obtain necessary and sufficient conditions of a weakly left primary ideal to be a left primary ideals in \( \Gamma \)-LA-rings.

2. IDEALS IN \( \Gamma \)-LA-RINGS

The results of the following lemmas seem play an important role to study \( \Gamma \)-LA-ring; these facts will be used so frequently that normally we shall make no reference to this lemma.

**Definition 2.1.** Let \((R,+)\) and \((\Gamma,\cdot)\) be two LA-groups, \( R \) is called a \( \Gamma \)-left almost ring (LA-ring) if there exists a mapping \( R\times\Gamma\times R \rightarrow R \) by \((a,\alpha, b)\rightarrow\alpha ab\), for all \( a,b \in R \) and \( \alpha \in \Gamma \) satisfying the following conditions

1. \( aa(b+c) = aab + aac \)
2. \( (a+b)xc = aac + bac \)
3. \( a(\alpha + \beta) b = aab + \alpha b \)
4. \( (a\beta)b c = (cab)\beta a, \) for all \( a,b,c \in R \) and \( \alpha,\beta \in \Gamma \).

**Lemma 2.2.** If \( R \) is a \( \Gamma \)-LA-ring with left identity, then \( a\gamma b = a\beta b, \) for all \( a,b \in R \) and \( \gamma, \beta \in \Gamma \).

**Proof.** Let \( R \) be a \( \Gamma \)-LA-ring and \( e \) be the left identity of \( a,b \in R \) and let \( \gamma, \beta \in \Gamma \) therefore we have

\[
\alpha e b = \alpha (e\beta b) = e(\alpha \beta b) = a\beta b.
\]

Hence \( a\gamma b = a\beta b \).

**Lemma 2.3.** Let \( R \) be a \( \Gamma \)-LA-ring with left identity \( e \). Then \( R\Gamma R = R \) and \( R = e\Gamma R = R\Gamma e \).

**Proof.** Let \( R \) be a \( \Gamma \)-LA-ring with left identity \( e \) and let \( r \in R \) then \( e\alpha r \in R\Gamma R \), for all \( \alpha \in \Gamma \), so that \( R \subseteq R\Gamma R \).

Since \( R \) is a \( \Gamma \)-LA-ring, we have \( R\Gamma R \subseteq R\). Thus \( R\Gamma R = R \) and \( R = e\Gamma R \). Now as \( e \) is a left identity in \( R, e\alpha a = a \), for all \( a \in R \) and \( \alpha \in \Gamma \). Then \( R = e\Gamma R \). Since \((a\alpha)b \beta c = (c\beta)\alpha b a, \) for all \( a,b,c \in R \) and \( \alpha,\beta \in \Gamma \), we have \( (R\Gamma R)e = (e\Gamma R)\Gamma R \).

\[
R\Gamma e = (R\Gamma R)e = (e\Gamma R)\Gamma R = R\Gamma R = R.
\]

Hence \( R = e\Gamma R = R\Gamma e \).

**Definition 2.4.** A nonempty subset \( I \) of a \( \Gamma \)-LA-ring \( R \) is a subring of \( R \) if under the binary operations in \( R \), form a \( \Gamma \)-LA-ring.

**Definition 2.5.** A subring \( I \) of \( R \) is called a left (right) ideal of \( R \) if \( R\Gamma I \subseteq I \) (\( I\Gamma R \subseteq I \) ) and is called ideal if it is left as well as right ideal.

**Lemma 2.6.** If \( R \) is a \( \Gamma \)-LA-ring with left identity, then every right ideal is a left ideal.

**Proof.** Let \( R \) be a \( \Gamma \)-LA-ring with left identity and \( A \) be a right ideal of \( R \). Then for \( a \in A, r \in R \) and \( \alpha \in \Gamma \), consider

\[
r\alpha a = (e\beta)\alpha a = (a\beta)e\alpha = (\Lambda\Gamma R)\alpha R \subseteq \Lambda \Gamma R \subseteq A,
\]

where \( e \) is a left identity and \( \beta \in \Gamma \), that is \( r\alpha a \in A \). Therefore \( A \) is left ideal of \( R \).

**Lemma 2.7.** If \( I \) is a left ideal of a \( \Gamma \)-LA-ring \( R \) with left identity, and if for any \( a \in R, \gamma \in \Gamma \), then \( \alpha I \) is a left ideal of \( R \).

**Proof.** Let \( I \) be a left ideal of \( R \), consider

\[
\alpha \gamma (\alpha I) = (e\gamma)\gamma (\alpha I) = (e\gamma)\alpha \gamma I = (\alpha \gamma)\alpha \gamma I \subseteq \alpha I.
\]

and \( (\alpha \gamma) \gamma + (\alpha \gamma) j \gamma = \alpha \gamma (i + j) \gamma \alpha I \). Hence \( \alpha \gamma I \) is a left ideal of \( R \).

**Lemma 2.8.** Let \( R \) be a \( \Gamma \)-LA-ring with left identity, and \( a \in R, \gamma \in \Gamma \). Then \( R\Gamma R \) is a left ideal of \( R \).

**Proof.** Let \( R \) be a \( \Gamma \)-LA-ring with left identity, and \( a \in R, \gamma \in \Gamma \). Then
Lemma 2.9. If $I$ is an ideal of a $\Gamma$ -LA-ring $R$ with left identity, and if for any $a \in R, \gamma \in \Gamma$, then $a^2 \gamma I$ is an ideal of $R$.

Proof. By Lemma 2.7, we have $a^2 \gamma I$ is a left ideal of $R$. Now consider 

\[
(a \gamma r) \gamma s = ((a \gamma)(a \gamma) r) \gamma s = (s \gamma)(r \gamma) \gamma s = (r \gamma)(s \gamma) \gamma e = (r \gamma)(s \gamma) \gamma \gamma e = (r \gamma)(s \gamma) \gamma \gamma e = \gamma \gamma (r \gamma s) = a^2 \gamma (r \gamma s) = a^2 \gamma I.
\]

Hence $a^2 \gamma I$ is an ideal of $R$.

Lemma 2.10. Let $R$ be a $\Gamma$ -LA-ring with left identity, and $a \in R, \gamma \in \Gamma$. Then $R \gamma a^2$ is an ideal of $R$.

Proof. Let $R$ be a $\Gamma$ -LA-ring with left identity, and $a \in R, \gamma \in \Gamma$. Now consider 

\[
R \gamma a^2 = (R \gamma R) \gamma a^2 = a^2 \gamma (R \gamma R) = a^2 \gamma R
\]

By Lemma 2.9, we have $R \gamma a^2$ is an ideal of $R$.

Lemma 2.11. Let $R$ be a $\Gamma$ -LA-ring with left identity, and let $A, B$ be left ideals of $R$. Then $(A: \Gamma B) = A \subseteq B$ is a left ideal in $R$, where $(A: \Gamma B) = \{r \in R : \gamma r \in A \}$.

Proof. Suppose that $R$ is a $\Gamma$ -LA-ring. Let $s \in R$ and let $a, b \in (\Gamma A : B)$. Then $B \gamma a \subseteq A$ and $B \gamma b \subseteq A$ so that 

\[
B \gamma (a + b) = (B \gamma a) + (B \gamma b) \subseteq A + A = A
\]

and 

\[
B \gamma (\gamma a) = s \gamma (B \gamma a) = s \gamma A = A.
\]

Therefore $a + b \in (A: \Gamma B)$ and $s \gamma a \in (A: \Gamma B)$ so that $R \gamma (A: \Gamma B) \subseteq (A: \Gamma B)$. Hence $(A: \Gamma B)$ is a left ideal in $R$.

Corollary 2.12. Let $R$ be a $\Gamma$ -LA-ring with left identity, and let $A$ be left ideals of $R$ Then $(A : \Gamma : B)$ is a left ideal in $R$, where $(A : \Gamma : B) = \{r \in R : \gamma b \in A \}$.

Proof. This follows from Lemma 2.11.

Remark 1. Let $R$ be a $\Gamma$ -LA-ring and let $A$ be a left ideal of $R$. It is easy to verify that $A \subseteq (A: \Gamma : r)$.

1. Let $R$ be a $\Gamma$ -LA-ring with left identity $e$, and let $A$ be a proper left (right) ideal of $R$. By Corollary 2.12 , we have $e \in (A : \Gamma : r)$, where $r \in R - A$.

2. Let $R$ be a $\Gamma$ -LA-ring with left identity $e$, and let $A$ be a proper left (right) ideal of $R$. By Corollary 2.12 , we have $e \in (A : \Gamma : r)$, where $r \in R - A$.

3. Let $R$ be a $\Gamma$ -LA-ring and let $A, B, C$ be left ideals of $R$. It is easy to verify that $(A : \Gamma : C) \subseteq (A : \Gamma : B)$, where $B \subseteq C$.

3. Left Primary and Weakly Left Primary Ideal in $\Gamma$ -LA-Rings

We start with the following theorem that gives a relation between left primary and weakly left primary ideal in $\Gamma$ -LA-ring. Our starting points is the following definition:

Definition 3.1. A left ideal $P$ is called left primary if $A \Gamma B \subseteq P$ implies that $((A \Gamma A) \Gamma A) \Gamma A = A^n \subseteq P$ or $B \subseteq P$ for some positive integer $n$, where $A, B$ is a left ideals of $R$.

Definition 3.2. A left ideal $P$ is called weakly left primary ideal if $A \Gamma B \subseteq P$ implies that $((A \Gamma A) \Gamma A) \Gamma A = A^n \subseteq P$ or $B \subseteq P$ for some positive integer $n$, where $A, B$ is a left ideals of $R$.

Remark. It is easy to see that every left primary ideal is weakly left primary.

Lemma 3.3. If $R$ is a $\Gamma$ -LA-ring with left identity, then a left ideal $P$ of $R$ is left primary if and only if $a \gamma b \in P$ implies that $a^n \in P$ or $b \in P$ for some positive integer $n$, where $\gamma \in \Gamma$ and $a, b \in R$.

Proof. Let $P$ be a left ideal of $\Gamma$ -LA-ring $R$ with left identity. Now suppose that $a \gamma b \in P$. Then by Definition of left ideal, we get 

\[
(R \gamma a) (R \gamma b) = (R \gamma R) (a \gamma b) = R \gamma (a \gamma b) \subseteq R \gamma P \subseteq P.
\]
Then \( a = (\gamma a) \in (R \gamma a)^{\gamma} \subseteq P \) or \( b = eabaRab \subseteq P \), for some positive integer \( \gamma \). Consequently, the proof is easy.

**Corollary 3.4.** If \( R \) is a \( \gamma \) -LA-ring with left identity, then a left ideal \( P \) of \( R \) is weakly left primary if and only if \( 0 \neq \gamma b \in P \) implies that \( \gamma a \in P \) or \( b \in P \) for some positive integer \( \gamma \), where \( \gamma \in \Gamma \) and \( a, b \in R \).

**Proof.** This follows from Lemma 3.3. Let \( R \) be a \( \gamma \) -LA-ring and \( A \) be a subset of \( R \). We write
\[
\sqrt{A} = \{ a \in R : a^i \in A, \text{ for some positive integer } k \}.
\]

**Theorem 3.5.** Let \( R \) be a \( \gamma \) -LA-ring with left identity, and let \( P \) be an ideal of \( R \). If \( P \) is a weakly left primary ideal that is not left primary. Then \( \sqrt{P} = \sqrt{0} \).

**Proof.** Let \( R \) be a \( \gamma \) -LA-ring with left identity. First, we prove that \( \sqrt{P} = 0 \). Suppose that \( \sqrt{P} \neq 0 \). Then there exists an element \( p' \in P \) such that \( p' \gamma b \neq 0 \), then either \( \sqrt{P} = 0 \) or \( \sqrt{P} = \sqrt{0} \).

For the other inclusion, suppose that \( m \in (P; \Gamma: R \Gamma a) \), so that
\[
(R \Gamma a)(R \Gamma m) = (m \Gamma R)(a \Gamma R) = (m \Gamma a) \Gamma (R \Gamma R) = (R \Gamma a) \Gamma m \subseteq P.
\]

If \( 0 \neq (R \Gamma a)(\Gamma m) \), then \( m = c \gamma m \in R \Gamma m \subseteq P \) since \( P \) is weakly left primary. If \( 0 = (R \Gamma a)(\Gamma m) \), then \( m \in (0: \Gamma: R \Gamma a) \) so we have the equality.

**Corollary 3.8.** Let \( R \) be a \( \gamma \) -LA-ring with left identity, and let \( P \) be a proper ideal of \( R \). If \( P \) is a weakly left primary ideal of \( R \), then
\[
(P; \Gamma: a) = P \cup (0: \Gamma: a),
\]
where \( a \in R \setminus \sqrt{P} \).

**Proof.** This follows from Lemma 3.7.

**Corollary 3.9.** Let \( R \) be a \( \gamma \) -LA-ring with left identity, and let \( P \) be a proper ideal of \( R \). If \( (P; \Gamma: R \Gamma a) = P \cup (0: \Gamma: R \Gamma a) \), then
\[
(P; \Gamma: R \Gamma a) = P \cup (0: \Gamma: R \Gamma a),
\]
where \( a \in R \setminus \sqrt{P} \).

**Proof.** This follows from Lemma 3.7.

**Theorem 3.10.** Let \( R \) be a \( \gamma \) -LA-ring with left identity, and let \( P \) be a proper ideal of \( R \). If \( (P; \Gamma: n) = P \cup (0: \Gamma: n) = (0: \Gamma: n) \), then \( P \) is a weakly left primary ideal of \( R \), where \( n \in R \setminus \sqrt{P} \).

**Proof.** Let \( R \) be a \( \gamma \) -LA-ring with left identity, and let \( P \) be a proper ideal of \( R \). Suppose that \( 0 \neq m \gamma n \in \sqrt{P} \), where \( m \in R \setminus \sqrt{P}. \gamma \in \Gamma \). Then
\[
m \in (P; \Gamma: n) = P \cup (0: \Gamma: n)
\]
by Corollary 3.9 hence \( m \in P \) since \( m \gamma n \neq 0 \), as required.

**Lemma 3.11.** Let \( R = R_1 \times R_2 \), where each \( R_i \) is a \( \gamma \) -LA-ring with left identity. Then the following hold:

(i) If \( A \) is a left ideal of \( R_1 \), then
\[
\sqrt{A} \times R_1 = \sqrt{A} \times R_2.
\]

(ii) If \( A \) is a left ideal of \( R_2 \), then
\[
R_1 \times \sqrt{A} = R_1 \times \sqrt{A}.
\]

**Proof.** The proof is straightforward.
Theorem 3.12. Let \( R = R_1 \times R_2 \), where each \( R_i \) is a \( -\)LA-ring with left identity. If \( P \) is a weakly left primary (left primary) ideal of \( R_1 \), then \( P \times R_2 \) is a weakly left primary (left primary) ideal of \( R \).

Proof. Suppose that \( R = R_1 \times R_2 \), where each \( R_i \) is a \( -\)LA-ring with left identity and \( P \) is a weakly left primary ideal of \( R_1 \). Let

\[
0 \neq (a, b)\gamma(c, d) = (a\gamma c, b\gamma d) \in P \times R,
\]

where \((a, b), (c, d) \in R \), \( \gamma \in \Gamma \) so either \( a \in \sqrt{P} \) or \( c \in P \) since \( P \) is weakly left primary. It follows that either

\[
(a, b) \in \sqrt{P} \times R = \sqrt{P} \times R_2 \quad \text{or} \quad (c, d) \in P \times R.
\]

By Definition of weakly left primary ideal, we have \( P \times R_2 \) is a weakly left primary ideal of \( R \).

Corollary 3.13. Let \( R = R_1 \times R_2 \), where each \( R_i \) is a \( -\)LA-ring with left identity. If \( P \) is a weakly left primary (left primary) ideal of \( R_1 \), then \( R_1 \times P \) is a weakly left primary (left primary) ideal of \( R \).

Proof. This follows from Theorem 3.12.

Corollary 3.14. Let \( R = \prod_{i=1}^{n} R_i \), where each \( R_i \) is a \( -\)LA-ring with left identity. If \( P \) is a weakly left primary (left primary) ideal of \( R_1 \), then

\[
R_1 \times R_2 \times \ldots \times R_i \times \ldots \times R_n
\]

is a weakly left primary (left primary) ideal of \( R \).

Proof. This follows from Theorem 3.12 and Corollary 3.13.

Theorem 3.15. Let \( R = R_1 \times R_2 \), where each \( R_i \) is a \( -\)LA-ring with left identity. If \( P \) is a weakly left primary ideal of \( R \), then either \( P = 0 \) or \( P \) is left primary.

Proof. Let \( R = R_1 \times R_2 \), where each \( R_i \) is a \( -\)LA-ring with identity and let \( P = P_i \times R_2 \) be a weakly left primary ideal of \( R \). We can assume that \( P \neq 0 \). So there is an element \((a, b) \) of \( P \) with \((a, b) \neq (0, 0) \). Then

\[
(0, 0) \neq (a, c)\gamma(e, b) \in P,
\]

where \( \gamma \in \Gamma \), gives either

\[
(a, e) \in \sqrt{P} = \sqrt{P} \times R = \sqrt{P} \times R_2 \quad \text{or} \quad (e, b) \in P.
\]

If \((e, b) \in P \), then \( P = P_1 \times P_2 \). We will show that \( P_1 \) is left primary hence \( P \) is weakly left primary by Corollary 3.13. Let \( cd \in P_1 \), where \( c, d \in R_2 \). Then

\[
(0, 0) \neq (c, d)\gamma(e, b) \in P,
\]

so either \((c, d) \neq (0, 0) \) or \((e, b) \neq (0, 0) \) and hence either \( c \in \sqrt{P} \) or \( d \in P_2 \). By a similar argument, \( P = R_1 \times P_2 \) is left primary.

Proposition 3.16. Let \( A \subseteq P \) be proper ideals of a \( -\)LA-ring \( R \). Then the following hold:

(i) If \( P \) is weakly left primary (left primary), then \( P/A \) is weakly left primary (left primary).

(ii) If \( A \) and \( P/A \) are weakly left primary (left primary), then \( P \) is weakly left primary (left primary).

Proof. (i) Let \( 0 \neq (a + A)\gamma(b + A) = a\gamma b + A \in P/A \), where \( a, b \in R \), \( \gamma \in \Gamma \) so \( a\gamma b \in P \). If \( a\gamma b = 0 \in A \), then

\[
(a + A)\gamma(b + A) = 0,
\]

a contradiction. So if \( P \) is weakly left primary, then either \( a \in \sqrt{P} \) or \( b \in P \), hence either \( a + A = P/A \) or \( b + A \in P/A \), as required.

(ii) Let \( 0 \neq a\gamma b \in P \), where \( a, b \in R \), so \((a + A)\gamma(b + A) \in P/A \). For \( a\gamma b \in A \), if \( A \) is weakly left primary, then either

\[
a \in A \subseteq \sqrt{P} \quad \text{or} \quad b \in A \subseteq P.
\]

So we may assume that \( a\gamma b \notin A \). Then either \( a + A \in \sqrt{P} \) or \( b + A \in P/A \). It follows that either \( a \in \sqrt{P} \) or \( b \in P \) as needed.

Theorem 3.17. Let \( P \) and \( Q \) be weakly left primary ideals of a \( -\)LA-ring \( R \) that are not left primary. Then \( P + Q \) is a weakly left primary ideal of \( R \).

Proof. Since \((P + Q)/Q \cong Q/(P \cap Q)\), we get that \((P + Q)/Q \) is weakly left primary by Proposition 3.16 (i). Now the assertion follows from Proposition 3.16 (ii).

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REFERENCES


