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Non-PT Symmetric Potentialsand (1 + 1) Dirac Equation

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ABSTRACT

The Dirac equation in (1 + 1) dimension with the complex vector potential coupling that leads to an effective Hulthen potential model is solved. Polynomial solutions are obtained using the method of Nikiforov-Uvarov. Energy spectrum and corresponding wave-functions are obtained.

Key Words: Dirac equation, Hulthen potential, PT symmetry

1. INTRODUCTION

The parity and time reversal PT symmetry has been an active research topic over the last decade [1, 2] and there have been considerable works on non-Hermitian Hamiltonians [3, 4, 5, 6, 7, 8]. Non-Hermitian Hamiltonians having PT symmetry (P parity, T time reversal) which admit real and discrete spectrum for exact PT symmetry form a special class and energy is complex conjugate pairs when this space-time symmetry is spontaneously broken. The parity operator is linear and its effect is $x \to -x$, $p \to -p$, the time reversal operator is anti-linear and has the effect $p \to -p$, $x \to x$, $i \to -i$. A potential V (x) is known to be PT symmetric if $V(-x) = V^{*}(x)$ or $[V(x), PT] = 0$. Recent years have witnessed a growing interest in there search fields for the PT–symmetric quantum systems with a constant mass to the relativistic PT –symmetric position-dependent effective mass quantum systems [9, 10, 11, 12, 13, 14, 15, 16]. Moreover, Dirac equation is studied with reflectionless PT symmetric potentials [17], one can find interesting works on PT symmetry in relativistic quantum mechanics [18, 19, 20, 21, 22, 23]. Just as non-relativistic quantum mechanics problems include a number of solvable potentials in which all the energy eigenvalues and wave-functions are explicitly known, so does relativistic quantum mechanics. Some elegant methods can also be applied to solve relativistic problems such as operator methods [24], supersymmetric quantum mechanics [25], analytical methods [26], the asymptotic iteration method (AIM) [27] etc. To our knowledge, exact solutions of PT symmetric complexification of the singular Hulthen potential and complex Morse potential were studied first by Znojil [28]. This kind of potentials can be extended to the relativistic scheme. Thus, this study is based on the idea which is a general approach to transforming one dimensional Dirac equation into a Klein-Gordon like equation leads to complex effective potentials. The relativistic scheme related interesting works can be found in [29], [30], [31], [32], [33], [34] and a matrix polynomial approach can be found in [35].

In our work, Dirac equation is studied in the presence of complex vector and scalar potentials which are given as general complex Hulthen potential is derived as an effective potential. The applications of complex potentials can be found in applied physics literature such as the meson-nucleus interaction can be described by an optical potential which has both real and

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imaginary parts as U=V+iW where the imaginary part of the meson-nucleus potential corresponds to half of the in-medium width [36]. The polynomial solutions are used to get relativistic energy levels and wave-functions.

2. THE NİKİFOROV-UVAROV METHOD

The Nikiforov Uvarov method received much interest and usually applied to both relativistic and non-relativistic quantum mechanics [26]. After a coordinate transformation in a Sturm-Liouville type equations as x = x(s), for example a Klein-Gordon-like equation becomes

$$\frac{d^2\psi}{ds^2}+\frac{\widetilde{\tau(s)}}{\sigma(s)}\frac{d\psi}{ds}+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s)=0 \tag{1}$$

Where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most of second degree, and $\tilde{\tau}(s)$is a polynomial, at most of first degree in this generalized hypergeometric differential equation. If the mapping$\psi(s)=\chi(s)y(s)$ is used in (1), we get

$$\sigma(s)y''(s)+\tau(s)y'(s)+\lambda(s)y(s)=0 \tag{2}$$

Where

$$\frac{\chi'(s)}{\chi(s)}=\frac{\pi(s)}{\sigma(s)} \tag{3}$$

And

$$\tau(s)=\tilde{\tau}(s)+2\pi(s) \tag{4}$$

$$\lambda=\frac{\bar{\sigma}(s)}{\sigma(s)} \tag{5}$$

$$\bar{\sigma}(s)=\tilde{\sigma}(s)+\pi^2(s)+\pi(s)\big(\tilde{\tau}(s)-\sigma'(s)\big)+\pi'(s)\sigma(s) \tag{6}$$

To find $\pi(s)$ and $\lambda(s)$, (5) is written as

$$\pi^2(s)+\pi(s)\big(\tilde{\tau}(s)-\sigma'(s)\big)-k\sigma(s)=0 \tag{7}$$

Where

$$k=\lambda-\pi'(s). \tag{8}$$

From this quadratic equation, $\pi(s)$ is given by

$$\pi(s)=\frac{\sigma'-\tilde{\tau}}{2}\pm\sqrt{\left(\frac{\sigma'(s)-\tilde{\tau}(s)}{2}\right)^2-\tilde{\sigma}(s)+k\sigma(s)}\ . \tag{9}$$

If the discriminant of the expression under the square root is zero, $\pi(s)$ becomes a polynomial of degree at most one. On the other hand, boundary conditions require$\tau'(s)<0$ that means , $\pi(s)$ and $k(s)$ has to be chosen according to these conditions. Using the family of particular solutions of (2), $y(z)=y_n(z)$ [26], the form of the λ is

$$\lambda=\lambda_n=-n\tau'(s)-\frac{n(n-1)}{2}\sigma''(s) \tag{10}$$

which determines the eigenvalues and n=0,1,2,…The polynomial solutions of (2) are given by the Rodrigues relation

$$y_n(s)=\frac{B_n}{\rho(s)}\frac{d^n}{ds^n}(\sigma^n(s)\rho(s)) \tag{11}$$

Where $\rho(s)$ satisfies the relation

$$\frac{d(\sigma(s)\rho(s))}{ds}=\tau(s)\rho(s). \tag{12}$$

3. DIRAC EQUATION

The one dimensional time independent Dirac equation is given as

$$\big(\vec{\alpha}.\vec{p}+\beta\big(m_0+S(x)\big)+V\big)\Psi(x)=E\Psi(x) \tag{13}$$

where Ψ is the two component spinor wave-function, E is the energy, $\vec{p}$ is the momentum operator, m_0 is the mass of the particle, V(x) and S(x) denote the vector and scalar potentials correspondingly and α,β are 2X2 Dirac matrices in Standard representation and $\hbar=c=1$ atomic units are chosen. Let us Show the upper and lower components by $\phi(x),\ \theta(x)$. Using $\alpha=\sigma_3,\ \beta=\sigma_1$ where σ_1 and σ_3 are Pauli matrices, and multiplying (13) by σ_1, we obtain

$$-i\frac{d\theta}{dx}+\big(E-V(x)\big)\theta-\big(m+S(x)\big)\phi=0 \tag{14}$$

$$i\frac{d\phi}{dx}+\big(E-V(x)\big)\phi-\big(m+S(x)\big)\theta=0\ . \tag{15}$$

If we terminate the lower component in above coupled equations, one gets

$$-\frac{d^2\phi}{dx^2}+\frac{1}{m_0+S(x)}\frac{dS}{dx}\frac{d\phi}{dx}+\left(2EV(x)-V(x)^2-i\frac{dV}{dx}-i\frac{1}{m_0+S(x)}\frac{dS}{dx}(E-V(x))\right)\phi=(E^2-(m_0+S(x))^2)\phi\ . \tag{16}$$

Let us transform (16) into a Klein-Gordon-like equation using$\phi(x)=\sqrt{m_0+S(x)}\varphi(x)$ in (16):

$$-\frac{d^2\varphi}{dx^2}+V_{eff}(x)\varphi=E^2\varphi \tag{17}$$

where

$$V_{eff}(x)=-V^2-i\frac{dV}{dx}+\big(m_0+S(x)\big)^2+i\frac{V}{m_0+S(x)}\frac{dS}{dx}+E\left(2V-\frac{i}{m_0+S(x)}\frac{dS}{dx}\right)-\frac{1}{2(m_0+S(x))}\frac{d^2S}{dx^2}+\frac{3}{4}\left(\frac{1}{(m_0+S(x))}\frac{dS}{dx}\right)^2. \tag{18}$$

We shall give$V(x)$ in the form of a complex function as

$$V(x)=V_R(x)+iV_I(x) \tag{19}$$

Then, the effective potential becomes

$$V_{eff}(x) = -V_R{}^2 + V_I{}^2 - i\frac{dV}{dx} + \left(m_0 + S(x)\right)^2 + 2EV_R(x) - \frac{S''(x)}{2(m_0+S(x))} + \frac{3}{4}\left(\frac{M'(x)}{m_0+S(x)}\right)^2 + V_I'(x) - \frac{S'(x)}{m_0+S(x)}V_I(x) + i\left(-2V_IV_R + 2EV_I - V_R' + \frac{S'(x)}{m_0+S(x)}V_R - E\frac{S'}{m_0+S(x)}\right). \quad (20)$$

Now we may choose the imaginary component of V(x) as

$$V_I(x) = \frac{S'(x)}{2(m_0+S(x))} \quad (21)$$

and use in (20), we have

$$V_{eff}(x) = (m_0 + S(x))^2 + 2EV_R(x) - V_R^2(x) - iV_R'(x). \quad (22)$$

It is noted that $V_I(x)$ is chosen as given in (21) to get a model exactly soluble.

3.1 The Model

We can choose $V_R(x)$ and $S(x)$ as below

$$V_R(x) = V_I\frac{e^{-\alpha x}}{e^{-\alpha x}-q} \quad (23)$$

$$S(x) = S_1\frac{e^{-\alpha x}}{e^{-\alpha x}-q} \quad (24)$$

Where S_1, V_1 are constants. Then, using (19), (21) and (23), V(x) becomes

$$V(x) = V_1\frac{e^{-\alpha x}}{e^{-\alpha x}-q} + \frac{iqS_1\alpha}{2(q-e^{-\alpha x})(-S_1+m_0(e^{\alpha x}-q))} \quad (25)$$

And the expression in (22) turns into

$$V_{eff}(x) = m_0^2 + (S_1^2 - V_1^2 - iV_1\alpha)\frac{e^{-2\alpha x}}{(e^{-\alpha x}-q)^2} + (2m_0S_1 + 2V_1E + iV_1\alpha)\frac{e^{-\alpha x}}{e^{-\alpha x}-q} \quad (26)$$

Thus, one obtains the following Klein-Gordon-like second order differential equation whose solutions are related to the upper component as

$$\varphi''(x) + \left(\left(S_1{}^2 - V_1^2 - iV_1\alpha\right)\frac{e^{-2\alpha x}}{(e^{-\alpha x}-q)^2} + (2m_0S_1 + 2V_1E + iV_1\alpha)\frac{e^{-\alpha x}}{(e^{-\alpha x}-q)}\right)\varphi(x) = (E^2 - m_0^2)\varphi(x) \quad (27)$$

Here we will use $\tilde{E} = E^2 - m_0^2$ and we note that we have derived an effective potential which is a generalized complex Hulthen potential. Now we will look at the solutions of the system. Let us use the variable transformation $s = e^{-\alpha x}$ in (27), then we have

$$\varphi''(x) + \frac{1-qs}{s(1-qs)}\varphi'(x) + \frac{1}{\left(s(1-qs)\right)^2}\left((\gamma - q^2\epsilon^2 - q\beta)s^2 + (\beta + 2q\epsilon^2)s - \epsilon^2\right)\varphi(x) = 0 \quad (28)$$

Where

$$\frac{\tilde{E}}{\alpha^2} = -\epsilon^2 \quad (29)$$

$$\beta = \frac{2m_0S_1+2EV_1+iV_1\alpha}{\alpha^2} \quad (30)$$

$$\gamma = \frac{S_1^2-V_1^2-i\alpha V_1}{\alpha^2} \quad (31)$$

We use $\mu = \sqrt{q^2 - 4\gamma}$ and obtain $\pi(s)$ and $\tau(s)$ as

$$\pi(s) = \epsilon - \frac{1}{2}\left(q + (\mu + 2q\epsilon)\right)s \quad (32)$$

$$\tau(s) = 1 + 2\epsilon - (2q + \mu + 2q\epsilon)s \quad (33)$$

Then we use

$$\lambda_n = \beta - \frac{1}{2}(\mu + q) - \epsilon(\mu + q) = n^2q + 2qn\epsilon + qn + \mu n \quad (34)$$

Now we arrange this equation and give

$$-(E^2 - m_0^2)\alpha^2 = \left(\frac{C+2EV_1}{D_n} - \frac{\alpha^2}{2}D_n\right)^2 \quad (35)$$

Where

$$C = 2m_0S_1 + iV_1\alpha(1 + 1/q) - \frac{1}{q}\left(S_1{}^2 - V_1^2\right) \quad (36)$$

$$D_n = 2q\left(n + \frac{1}{2}\left(1 + \frac{\mu}{q}\right)\right) \quad (37)$$

We will solve (35) to find eigenvalues En of the system but before giving the eigenvalue solutions, let us check the real spectrum conditions using (35) and (31). We will write $S_1 = S_R + iS_I$ and put into (36) and (31), we get

$$2m_0S_I + V_1\alpha(1 + 1/q) - \frac{2S_IS_R}{q} = 0 \quad (38)$$

$$2S_RS_I - V_1\alpha = 0 \quad (39)$$

$$S_1 = -m_0 - i\frac{V_1\alpha}{2m_0} \quad (40)$$

If we plug (40) into (36), we get real C and also real γ which are given by

$$C = -2m_0^2(1 + 1/q) + \frac{V_1^2 1}{q}\left(1 + \frac{\alpha^2}{4m_0^2}\right) \quad (41)$$

$$\gamma = \frac{m_0^2-V_1^2}{\alpha^2} - \frac{V_1^2}{4m_0^2}. \quad (42)$$

Moreover, $S(x), V(x)$ and $V_{eff}(x)$ turn into

$$S(x) = -\left(m_0 + i\frac{V_1\alpha}{2m_0}\right)\frac{e^{-\alpha x}}{e^{-\alpha x}-q} \quad (43)$$

$$V(x) = \frac{V_1e^{-\alpha x}}{e^{-\alpha x}-q} + \frac{e^{\alpha x}(e^{\alpha x}-q)m_0^2\alpha^2qV_1}{4m_0^4(-1+qe^{\alpha x})\left((e^{\alpha x}+1-q)^2+V_1^2\alpha^2\right)} - i\frac{e^{\alpha x}q\alpha\left(4m_0^4(1-q+e^{\alpha x})+V_1^2\alpha^2\right)}{8m_0^4(-1+qe^{\alpha x})\left((e^{\alpha x}+1-q)^2+V_1^2\alpha^2\right)} \quad (44)$$

And

$$V_{eff}(x) = 2(EV_1 - m_0^2)\frac{e^{-\alpha x}}{e^{-\alpha x}-q} + \left(m_0^2 - V_1^2 - \frac{V_1^2\alpha^2}{4m_0^2}\right)\frac{e^{-2\alpha x}}{(e^{-\alpha x}-q)^2}. \quad (45)$$

Using (35), we can get real energies of (44) as

$$E_{n,\pm} = -\frac{V_1}{D_n} \pm \frac{\sqrt{16{V_1}^2 - 4D_n(D_n(2m^2+\alpha^2)+4m^2+2\alpha^2n^2q+2\alpha^2n\mu)}}{4D_n} \quad (46)$$

We note that there exists solutions both for positive energy as well as for negative energy shown by (46). The negative energy solutions correspond to predict the existence of antiparticle, positron. If the real electrons are described by positive energy states, all negative energy states are occupied by electrons and a real electron is not allowed to fall into a negative energy state according to the Dirac theory. Now, the functions $\rho(s)$ and $f(s)$ are given by

$$\rho(s) = s^{2\epsilon}(1-qs)^{\frac{\mu}{q}} \quad (47)$$

$$f(s) = s^{\epsilon}(1-qs)^{\frac{\mu+q}{2q}} \quad (48)$$

And finally wave-function $\varphi(s)$ reads

$$\varphi_n(s) = B_n s^{\epsilon}(1-qs)^{\frac{\mu+q}{2q}} P^{\left(2\epsilon,\frac{\mu}{q}\right)}{}_n(1-2qs). \quad (49)$$

B_n is the normalization constant and the upper spinor component can be determined as

$$\phi_n(s) = N\sqrt{m_0 + S_1\frac{s}{s-q}}\, s^{\epsilon}(1-qs)^{\frac{\mu+q}{2q}} P_n^{\left(2\epsilon,\frac{\mu}{q}\right)}(1-2qs). \quad (50)$$

Now,(50) satisfies the boundary conditions $s = e^{-\alpha x}$, then, $\phi_n(x) \to 0$ becomes zero when $x \to 0, x \to \infty$ if and only if $q = 1, E^2 < m_0^2, \mu > 0, \epsilon > 0$. Equation (15) can help us to find the lower spinor wave-function as

$$\theta_n(s) = \frac{1}{\sqrt{m_0+S(x)}}\left[\left(-i\alpha\epsilon + \frac{i\alpha(\mu+1)}{2}\frac{s}{1-s} - \frac{i\alpha}{2}\frac{s(1-s)}{(1-s)(m_0+S_1)-S_1} + E - V(s)\right)\varphi_n - B_n(n+1+2\epsilon+\mu)s^{\epsilon}(1-s)^{\frac{\mu+1}{2}}P_{n-1}^{(1+2\epsilon,1+\mu)}(1-2s)\right] \quad (51)$$

4. CONCLUSION

In conclusion, we have extended one dimensional Dirac Hamiltonian to a solvable Klein-Gordon-like Hamiltonian where the vector potential is chosen as a complex function. It is seen that decomposing the vector potential and using a specific $V_I(x)$ leads to a new effective potential given by (22). We have introduced some specific forms for $V_R(x)$ and S(x) to obtain an effective potential in the form of generalized complex Hulthen potential. After then, we have applied Nikiforov-Uvarov method to (27) in order to obtain energy and wave-functions. We have seen that we can obtain a real energy spectrum under some parameter restrictions while S(x) should be a complex function but $V_{eff}(x)$ is real. In [22], the authors solved one dimensional Dirac equation for generalized Hulthen potential and found that the real spectrum was obtained by using an imaginary α, i.e. α → iα, rather than a real α. But in this study, we have seen that one can obtain real spectrum if S(x) and V (x) are complex functions when α is real. In figure 1, our results with those given in [22] agree for n = 1 states.

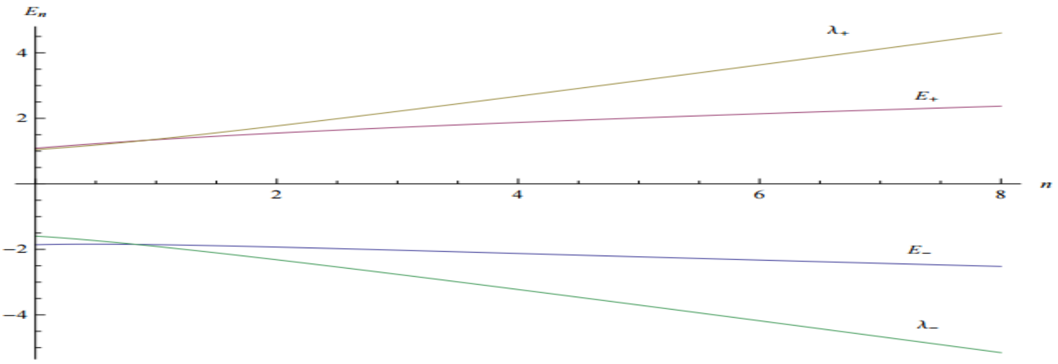

Figure 1: Graph of (46) $(E_n, -E_n)$ with respect to n shown by pink and blue lines while (λ_+, λ_-) correspond to the energy formula (33) in [22].

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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