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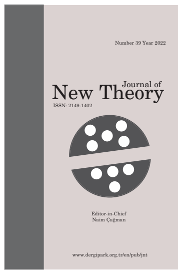

A Generalization of p-Adic Factorial

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Abstract — In this paper, we establish a new approach of the p-adic analogue of Roman factorial, called p-adic Roman factorial. We define this new concept and demonstrate its properties and some properties of p-adic factorial.

Keywords — *Roman factorial, p-adic number, p-adic factorial, p-adic gamma function*

Mathematics Subject Classification (2020) — 05A10, 11D88

1. Introduction

In the literature, the Roman factorial in the real case is one of the generalizations of the classical factorial for negative integers. This concept has been used by Steve Roman [1] to study the formal series and the harmonic logarithm. It has also been studied by Loeb and Rota in [2], and [3]. The above authors have used the notation $\lfloor m\rceil!$ to define the Roman factorial of an integer $m \in \mathbb{Z}$.
The p-adic domain has an important applications in a cryptography, number theory, algebraic geometry, and arithmetic dynamics. However, the definition of the p-adic factorial of a positive integer was considered by Alain Robert in [4] as restricted factorial, and denoted by

$$n!^* = \prod_{1\leq j\leq n, p\nmid j} j$$

Another notation for the p-adic factorial $(n!)_p$ was adopted by Menken and Çolakoğlu [5]. Both of Robert and Menken have used the p-adic factorial only to define the p-adic gamma function, without giving its properties. Furthermore, Aidagulov and Alekseyev in [6] have also used the so-called modified (p-adic) factorial, with the notation $n!_p$, to study the modified (p-adic) binomial coefficients. It can be remarked that the previous authors have given the definition of p-adic factorial without giving the properties.
Taken into previous considerations, in the present paper, we firstly demonstrate some properties of p-adic factorial (see Lemma 2.3, Theorem 2.4, Proposition 2.7, Proposition 2.8, Corollary 2.9, and Corollary 2.10). Secondly, we propose a definition of p-adic analogue of Roman factorial named *p-adic Roman factorial* (see Definition 3.1). Next, we demonstrate some combinatorial properties of this factorial, using the concept of p-adic gamma function (see Lemma 3.2, Theorems 3.4-3.6, Corollaries 3.7-3.8, Theorems 3.9-3.11). Finally, some numerical examples are given (Examples 2.5 and 3.12).

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2. Preliminary

Throughout this paper, p is a prime number, $\mathbb{Z}$ is the set of all the real integers, $\mathbb{Z}_-$ (resp. $\mathbb{Z}_+$) is the set of all the negative real integers (resp. all the positive real integers), $\mathbb{N}$ is the set of all the non-negative integers, $\mathbb{Q}$ is the field of rational numbers, and $\mathbb{R}$ is the field of real numbers. We use $|.|$ to denote the ordinary absolute value, $[.]$ the real integer part, ν_p the p-adic valuation, and $|.|_p$ the p-adic absolute value. The field of p-adic numbers $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the p-adic absolute value. The ring of p-adic integers $\mathbb{Z}_p$ is such that $|x|_p \leq 1$.

2.1. Roman Factorial in Real Domain

Roman in [1] proposed the factorial of negative integer $n \in \mathbb{Z}_-$ as $\lfloor n \rceil ! = \dfrac{(-1)^{-n-1}}{(-n-1)!}$. So, for $n \in \mathbb{Z}_+$ we have $\lfloor n \rceil ! = n!$. Also, the Roman factorial satisfies a characteristic functional equation $\lfloor n \rceil ! = \lfloor n \rceil \cdot \lfloor (n-1) \rceil !$, where $\lfloor n \rceil = n$ if $n \neq 0$, and $\lfloor 0 \rceil = 1$ is called Roman n.

For example, we give the Roman factorial of some integers in table1:

Table 1. oman factorial of some integers

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
$\lfloor n \rceil !$	1	1	2	6	24	120	720	5040
$-n$	0	-1	-2	-3	-4	-5	-6	-7
$\lfloor -n \rceil !$	1	1	-1	$\frac{1}{2}$	$-\frac{1}{6}$	$\frac{1}{24}$	$-\frac{1}{120}$	$\frac{1}{720}$

The complement formula of the factorial function, known as Knuth's theorem [7], is as follows:

$$\lfloor n \rceil ! \, \lfloor -n \rceil ! = (-1)^n \, |n|$$

and the Roman factorial can be rewritten using the gamma function Γ as follows:

$$\lfloor n \rceil ! = \begin{cases} \Gamma(n+1), & \text{for} \quad n \geq 0 \\ \dfrac{(-1)^{-n-1}}{\Gamma(-n)}, & \text{for} \quad n < 0 \end{cases} \tag{1}$$

2.2. p-adic Factorial and p-adic Gamma Function

In this subsection, we provide definitions of p-adic analogue of factorial function and gamma function and some of their basic properties, to be needed in the next section.

Definition 2.1. [4] The p-adic factorial of $n \in \mathbb{N}$ is defined by $0!_p = 1$ and for $n > 0$

$$n!_p = \prod_{\substack{j=1 \\ (p,j)=1}}^{n} j \tag{2}$$

Remark 2.2. If $1 \leq n \leq p-1$, then $(p,j) = 1$, for all $1 \leq j \leq n$. Then, $n!_p = n!$.

Lemma 2.3. For $p = 2$, then we have $(2k)!_2 = (2k-1)!_2$.

Proof. The result comes from the fact that if $p = 2$, we have $n!_2 = \prod\limits_{\substack{j=1 \\ j \text{ is odd}}}^{n} j$. □

As in the real case, we define the p-adic Roman of a positive integer n as

$$\lfloor n \rceil_p = \begin{cases} n, & \text{if } \ |n|_p = 1 \\ 1, & \text{if } \ |n|_p < 1 \end{cases} \tag{3}$$

Therefore, the first property similar to that of the real factorial is given by the following

Theorem 2.4. Let $n \in \mathbb{N}$, with $n \geq 1$. Then $n!_p = \lfloor n \rceil_p (n-1)!_p$.

PROOF. Two cases are considered.

1) We suppose $|n|_p = 1$, so $(p, n) = 1$. Thus

$$n!_p = \prod_{\substack{j=1 \\ (p,j)=1}}^{n} j = n \prod_{\substack{j=1 \\ (p,j)=1}}^{n-1} j = \lfloor n \rceil_p (n-1)!_p$$

2) We suppose $|n|_p < 1$, so $(p, n) \neq 1$. Thus

$$n!_p = \prod_{\substack{j=1 \\ (p,j)=1}}^{n} j = 1 \cdot \prod_{\substack{j=1 \\ (p,j)=1}}^{n-1} j = \lfloor n \rceil_p (n-1)!_p$$

□

Example 2.5. In Tables 2-5, we calculate some p-adic factorials of some positive integers. For $p = 2, 3, 5, 7$.

Table 2. The 2-adic factorial

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
$n!_2$	1	1	1	3	3	15	15	105	105	945	945	10395

Table 3. The 3-adic factorial

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
$n!_3$	1	1	2	2	8	40	40	280	2240	2240	22400	246400

Table 4. The 5-adic factorial

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
$n!_5$	1	1	2	6	24	24	144	1008	8064	72576	72576	798336

Table 5. The 7-adic factorial

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
$n!_7$	1	1	2	6	24	120	720	720	5760	51840	518400	5702400

The next theorem represents a generalization of the Wilson congruence; it's the key of some results in this section.

Theorem 2.6. [4] Let $a \in \mathbb{Z}$ and $s \in \mathbb{Z}_+$. Then

1) For $p \geq 3$ and $s \geq 1$, we have $\prod\limits_{\substack{j=a \\ (p,j)=1}}^{a+p^s-1} j \equiv -1 \, (mod \ p^s)$.

2) For $p = 2$ and $s \geq 3$, we have $\prod_{\substack{j=a \\ j odd}}^{a+2^s-1} j \equiv 1 \, (mod \; 2^s)$.

From this generalization of the classical Wilson theorem, we obtain the following congruences:

Proposition 2.7. Let $n \in \mathbb{N}$ and $s \in \mathbb{Z}_+$.

1) If $p \geq 3$ and $s \geq 1$, then $\dfrac{(n+p^s)!_p}{n!_p} \equiv -1 \, (mod \; p^s)$.

2) If $p = 2$ and $s \geq 3$, then $\dfrac{(n+2^s)!_2}{n!_2} \equiv 1 \, (mod \; 2^s)$.

Proof. We have

$$\frac{(n+p^s)!_p}{n!_p} = \prod_{\substack{j=n+1 \\ (p,j)=1}}^{n+p^s} j$$

From the case 1 of Theorem 2.6 with $a = n+1$, we obtain the congruence for $p \geq 3$ and $s \geq 1$. From the case 2 of the same Theorem with $a = n+1$, we obtain the congruence for $p = 2$ and $s \geq 3$. □

More generally, we have the following theorem:

Proposition 2.8. Let $n \in \mathbb{N}$, and $m, s \in \mathbb{Z}_+$.

1) If $p \geq 3$ and $s \geq 1$, then $\dfrac{(n+mp^s)!_p}{n!_p} \equiv (-1)^m \, (mod \; p^s)$.

2) If $p = 2$ and $s \geq 3$, then $\dfrac{(n+m2^s)!_2}{n!_2} \equiv 1 \, (mod \; 2^s)$.

Proof. The proof is done by induction on m. □

Corollary 2.9. For $p \geq 3$, $n \in \mathbb{N}$ and $s \in \mathbb{Z}_+$, we have $|n!_p|_p = 1$ and

$$|(n+p^s)!_p + n!_p|_p \leq \frac{1}{p^s}$$

Corollary 2.10. For $p = 2$, $n \in \mathbb{N}$ and $s \in \mathbb{Z}_+$ with $s \geq 3$, we have $|n!_2|_2 = 1$ and

$$|(n+2^s)!_2 - n!_2|_2 \leq \frac{1}{2^s}$$

In dynamic system and string theory, the p-adic gamma function has been well used. This function studied by [8], [9] and [10], to give some properties of polynomials.
The function $n!$ cannot be extended by continuity on $\mathbb{Z}_p$, because $\lim_{n \to +\infty} n! = 0$ in $\mathbb{Z}_p$. So, we have the definition of p-adic gamma function as follows:

Definition 2.11. [11] The p-adic gamma function is defined by Morita as the continuous function

$$\Gamma_p : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$$

as an extension of the following sequence, with $n \in \mathbb{Z}_+$

$$\Gamma_p(n) = (-1)^n \prod_{j=1,(p,j)=1}^{n-1} j \tag{4}$$

Furthermore,

$$\Gamma_p(z) = \lim_{\substack{n \to z \\ \text{in } \mathbb{Z}_p}} \Gamma_p(n) = \lim_{\substack{n \to z \\ \text{in } \mathbb{Z}_p}} (-1)^n \prod_{\substack{j=1 \\ (p,j)=1}}^{n-1} j$$

Here, we cite some properties of Γ_p that we need to prove the theorems in the next section.

Proposition 2.12. [4] The function Γ_p satisfies the following properties:

1) $\Gamma_p(0) = 1$, $\Gamma_p(1) = -1$, $\Gamma_p(2) = 1$

2) $\Gamma_p(n+1) = (-1)^{n+1} n!_p$, $\forall n \in \mathbb{N}$

Other some important arithmetic formulas are given in the following proposition:

Proposition 2.13. [4] Let $n \geq 1$, its p-adic expansion be $\sum_{i=0}^{\ell} n_i p^i$, and the sum of digits be $S_n = \sum_{i=0}^{\ell} n_i$. Then,

1) $\Gamma_p(n+1) = \dfrac{(-1)^{n+1}\, n!}{\left[\frac{n}{p}\right]! \times p^{\left[\frac{n}{p}\right]}}$. In particular, $\Gamma_p(p^n) = \dfrac{(-1)^p\, p^n!}{p^{n-1}! \times p^{p^{n-1}}}$.

2) $\Gamma_p(np+k+1) = \dfrac{(-1)^{np+k+1}\,(np+k)!}{n! \times p^n}$, for $0 \leq k < p$.

3) $n! = (-1)^{n+1-\ell}\,(-p)^{\frac{n-S_n}{p-1}} \prod_{i=0}^{\ell} \Gamma_p\left(\left[\frac{n}{p^i}\right]+1\right)$.

3. Main Results and Proofs

Inspired by the works of Roman [1], Loeb and Rota [2], we will establish a p-adic analogue of the Roman factorial, so-called *the p-adic generalized factorial*, or *the p-adic Roman factorial*. We define of this new concept and demonstrate some of its properties.

Definition 3.1. For $n \in \mathbb{Z}$, we define the p-adic Roman factorial of n as

$$\lfloor n \rceil !_p = \begin{cases} n!_p, & \text{for } n \geq 0 \\ \dfrac{(-1)^{-n-1}}{(-n-1)!_p}, & \text{for } n < 0 \end{cases} \tag{5}$$

Remark 3.2. It can be remarked that

1) If $0 \leq n \leq p-1$, the we have $n!_p = n!$. Then, $\lfloor n \rceil !_p = \lfloor n \rceil ! = n!$.

2) If $-p \leq n \leq -1$, the we have $(-n-1)!_p = (-n-1)!$. Then, $\lfloor n \rceil !_p = \lfloor n \rceil !$.

Lemma 3.3. For $p = 2$, then

$$\lfloor n \rceil !_p = \begin{cases} \lfloor n-1 \rceil !_p, & \text{for } n = 2k \geq 0 \\ -\lfloor n-1 \rceil !_p, & \text{for } n = -2k < 0 \end{cases}$$

Proof. From Lemma 2.3, we have $(2k)!_2 = (2k-1)!_2$, thus $\lfloor 2k \rceil !_p = \lfloor 2k-1 \rceil !_p$. For the second case, we have $(-2k-1)!_2 = (-2k)!_2$, thus $\lfloor -2k-1 \rceil !_p = \lfloor -2k \rceil !_p$ □

We keep the notation of the p-adic Roman for a negative integer $n \in \mathbb{Z}_-$ and define it as

$$\lfloor n \rceil_p = \begin{cases} n, & \text{if} \quad |n|_p = 1 \\ -1, & \text{if} \quad |n|_p < 1 \end{cases} \tag{6}$$

So, it can easily verified that $\lfloor -n \rceil_p = - \lfloor n \rceil_p$.

Therefore, the first property similar to that of the real Roman factorial is as follows:

Theorem 3.4. For all $n \in \mathbb{Z}$, we have $\lfloor n+1 \rceil !_p = \lfloor n+1 \rceil_p \lfloor n \rceil !_p$.

PROOF. We consider the following three cases:
1) If $n \geq 0$, then $n+1 \geq 1$. Then, from Proposition 2.4 we have

$$\lfloor n+1 \rceil !_p = (n+1)!_p = \lfloor n+1 \rceil_p n!_p = \lfloor n+1 \rceil_p \lfloor n \rceil !_p$$

2) If $n < -1$, then $n+1 < 0$. Then, from Proposition 2.4 we have

$$\lfloor n+1 \rceil !_p = \frac{(-1)^{-n} \lfloor -n-1 \rceil_p}{\lfloor -n-1 \rceil_p (-n-2)!_p} = \frac{(-1)^{-n-1} \lfloor n+1 \rceil_p}{(-n-1)!_p} = \lfloor n+1 \rceil_p \lfloor n \rceil !_p$$

3) If $n = -1$, then, we have in the left side $\lfloor n+1 \rceil !_p = 0!_p = 1$, and in the right side $\lfloor n+1 \rceil_p \lfloor n \rceil !_p = 1 \cdot \lfloor -1 \rceil !_p = 1$. □

The following congruences hold from the properties of p-adic factorial.

Theorem 3.5. Let $n \in \mathbb{Z}$ and $s \in \mathbb{Z}_+$. Then

1) If $p \geq 3$ and $s \geq 1$, then we have

$$\begin{cases} \dfrac{\lfloor n+p^s \rceil !_p}{\lfloor n \rceil !_p} \equiv -1 \,(mod\ p^s)\,, & \text{if} \quad n \geq 0 \\ \dfrac{\lfloor n \rceil !_p}{\lfloor n-p^s \rceil !_p} \equiv -1 \,(mod\ p^s)\,, & \text{if} \quad n < 0 \end{cases}$$

2) If $p = 2$ and $s \geq 3$, then we have

$$\begin{cases} \dfrac{\lfloor n+2^s \rceil !_2}{\lfloor n \rceil !_2} \equiv 1 \,(mod\ 2^s)\,, & \text{if} \quad n \geq 0 \\ \dfrac{\lfloor n \rceil !_2}{\lfloor n-2^s \rceil !_2} \equiv 1 \,(mod\ 2^s)\,, & \text{if} \quad n < 0 \end{cases}$$

PROOF. The case $n \geq 0$ comes from the Proposition 2.8. It only remains to explain the case $n < 0$. Indeed, we have

$$\frac{\lfloor n \rceil !_p}{\lfloor n-p^s \rceil !_p} = (-1)^{p^s} \frac{(-n-1+p^s)!_p}{(-n-1)!_p}$$

The result comes from Proposition 2.8, for two cases $p \geq 3$ and $p = 2$. □

More generally, we have the following Theorem

Theorem 3.6. Let $n \in \mathbb{Z}$ and $s, m \in \mathbb{Z}_+$. Then,

1) If $p \geq 3$ and $s \geq 1$, then we have

$$\begin{cases} \dfrac{\lfloor n+mp^s\rceil!_p}{\lfloor n\rceil!_p} \equiv (-1)^m \,(mod\ p^s)\,, & \text{if} \quad n \geq 0 \\ \\ \dfrac{\lfloor n\rceil!_p}{\lfloor n-mp^s\rceil!_p} \equiv (-1)^m \,(mod\ p^s)\,, & \text{if} \quad n < 0 \end{cases}$$

2) If $p = 2$ and $s \geq 3$, then we have

$$\begin{cases} \dfrac{\lfloor n+m2^s\rceil!_2}{\lfloor n\rceil!_2} \equiv 1 \,(mod\ 2^s)\,, & \text{if} \quad n \geq 0 \\ \\ \dfrac{\lfloor n\rceil!_2}{\lfloor n-m2^s\rceil!_2} \equiv 1 \,(mod\ 2^s)\,, & \text{if} \quad n < 0 \end{cases}$$

PROOF. Easy recursion on m. □

The following corollaries follow from the two previous theorems.

Corollary 3.7. Let $p \geq 3$, $n \in \mathbb{Z}$ and $s, m \in \mathbb{Z}_+$. Then, $|\lfloor n\rceil!_p|_p = 1$ and

$$\begin{cases} |\lfloor n+mp^s\rceil!_p + \lfloor n\rceil!_p|_p \leq \dfrac{1}{p^s}, & \text{if} \quad n \geq 0 \\ \\ |\lfloor n-mp^s\rceil!_p + \lfloor n\rceil!_p|_p \leq \dfrac{1}{p^s}, & \text{if} \quad n < 0 \end{cases}$$

Corollary 3.8. Let $p = 2$, $n \in \mathbb{Z}$, and $s, m \in \mathbb{Z}_+$ with $s \geq 3$. Then, $|\lfloor n\rceil!_2|_2 = 1$ and

$$\begin{cases} |\lfloor n+m2^s\rceil!_2 - \lfloor n\rceil!_2|_2 \leq \dfrac{1}{2^s}, & \text{if} \quad n \geq 0 \\ \\ |\lfloor n-m2^s\rceil!_2 - \lfloor n\rceil!_2|_2 \leq \dfrac{1}{2^s}, & \text{if} \quad n < 0 \end{cases}$$

Next, we give the p-adic complement formula for p-adic Roman factorial function, in other words, the p-adic version of Knuth's theorem

Theorem 3.9. (p-adic Knuth's theorem)
For all $n \in \mathbb{Z}$, we have

$$\lfloor n\rceil!_p \, \lfloor -n-1\rceil!_p = \begin{cases} (-1)^n\,, & \text{for} \quad n \geq 0 \\ \\ (-1)^{n+1}\,, & \text{for} \quad n < 0 \end{cases}$$

PROOF. If $n \geq 0$, then $-n-1 < 0$. From Definition 3.1, we have $\lfloor n\rceil!_p = n!_p$ and the result comes from

$$\lfloor -n-1\rceil!_p = \frac{(-1)^n}{n!_p}$$

For the case $n < 0$, we use the same reasoning. □

As we have seen before for p-adic factorial, we can rewrite the p-adic Roman factorial using the p-adic gamma function, as follows:

Theorem 3.10. Let $n \in \mathbb{Z}$. Then, the relationship between p-adic Roman factorial and p-adic gamma function is given by

$$\lfloor n \rceil !_p = (-1)^{\delta(n)} \Gamma_p(n+1)$$

where

$$\delta(n) = \begin{cases} n+1, & \text{for} \quad n \geq 0 \\ n+1+\left[-\frac{n+1}{p}\right], & \text{for} \quad n < 0 \end{cases}$$

Proof. For the case of $n \geq 0$, the result comes from Proposition 2.12 (2). We show the theorem only for negative integers. Indeed, we proof $n < 0$, so $-n-1 > 1$. From Proposition 2.12, we have $(-n-1)!_p = (-1)^{-n}\Gamma_p(-n)$. On the other hand, from the complement formula of the p-adic gamma function (see [4]), we have

$$\Gamma_p(n+1)\Gamma_p(-n) = (-1)^{-n-[-\frac{n+1}{p}]}$$

Hence, we obtain

$$\begin{aligned} \lfloor n \rceil !_p &= \frac{-1}{\Gamma_p(-n)} \\ &= \frac{-\Gamma_p(n+1)}{(-1)^{-n-[-\frac{n+1}{p}]}} \\ &= (-1)^{n+1+\left[-\frac{n+1}{p}\right]}\Gamma_p(n+1) \end{aligned}$$

□

In the following theorem, we give some properties related to the p-adic gamma function.

Theorem 3.11. Let $n \in \mathbb{Z}$, and $m \in \mathbb{N}$. Then

1) We have

$$\lfloor n \rceil !_p = \begin{cases} \dfrac{\lfloor n \rceil !}{\left[\frac{n}{p}\right]! \times p^{\left[\frac{n}{p}\right]}}, & \text{for} \quad n \geq 0 \\ (-1)^n \lfloor n \rceil ! \left[-\frac{n+1}{p}\right]! \times p^{\left[-\frac{n+1}{p}\right]}, & \text{for} \quad n < 0 \end{cases}$$

2) In particular, $p^{m+1}!_p = \dfrac{p^{m+1}!}{p^m! \times p^{p^m}}$.

3) $(mp+k)!_p = \dfrac{(mp+k)!}{m!\, p^m}$, for $0 \leq k < p$.

4) $n! = (-1)^n (-p)^{\frac{n-S_n}{p-1}} \prod_{i=0}^{\ell} \left((-1)^{\left[\frac{n}{p^i}\right]} \left\lfloor \left[\frac{n}{p^i}\right] \right\rceil !_p \right)$, for $n \in \mathbb{N}$ given by its p-adic expansion $\sum_{i=0}^{\ell} n_i p^i$ and with the sum of digits $S_n = \sum_{i=0}^{\ell} n_i$.

Proof. The proof is clear from Proposition 2.13 and Theorem 3.10. □

Example 3.12. We give p-adic Roman factorial of the first ten negative integers in Tables 6-9. For positive numbers are the same that given in example 2.5.

Table 6. The 2-adic Roman factorial

n	**-1**	**-2**	**-3**	**-4**	**-5**	**-6**	**-7**	**-8**	**-9**	**-10**
$-n-1$	0	1	2	3	4	5	6	7	8	9
$(-n-1)!_2$	1	1	1	3	3	15	15	105	105	945
$\lfloor n \rceil !_2$	1	-1	1	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{15}$	$\frac{1}{15}$	$-\frac{1}{105}$	$\frac{1}{105}$	$-\frac{1}{945}$

Table 7. The 3-adic Roman factorial

n	**-1**	**-2**	**-3**	**-4**	**-5**	**-6**	**-7**	**-8**	**-9**	**-10**
$-n-1$	0	1	2	3	4	5	6	7	8	9
$(-n-1)!_3$	1	1	2	2	8	40	40	280	2240	2240
$\lfloor n \rceil !_3$	1	-1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{8}$	$-\frac{1}{40}$	$\frac{1}{40}$	$-\frac{1}{280}$	$\frac{1}{2240}$	$-\frac{1}{2240}$

Table 8. The 5-adic Roman factorial

n	**-1**	**-2**	**-3**	**-4**	**-5**	**-6**	**-7**	**-8**	**-9**	**-10**
$-n-1$	0	1	2	3	4	5	6	7	8	9
$(-n-1)!_5$	1	1	2	6	24	24	144	1008	8064	72576
$\lfloor n \rceil !_5$	1	-1	$\frac{1}{2}$	$-\frac{1}{6}$	$\frac{1}{24}$	$-\frac{1}{24}$	$\frac{1}{144}$	$-\frac{1}{1008}$	$\frac{1}{8064}$	$-\frac{1}{72576}$

Table 9. The 7-adic Roman factorial

n	**-1**	**-2**	**-3**	**-4**	**-5**	**-6**	**-7**	**-8**	**-9**	**-10**
$-n-1$	0	1	2	3	4	5	6	7	8	9
$(-n-1)!_7$	1	1	2	6	24	120	720	720	5760	51840
$\lfloor n \rceil !_7$	1	-1	$\frac{1}{2}$	$-\frac{1}{6}$	$\frac{1}{24}$	$-\frac{1}{120}$	$\frac{1}{720}$	$-\frac{1}{720}$	$\frac{1}{5760}$	$-\frac{1}{51840}$

4. Conclusion

In this article, we have given some properties of the p-adic factorial. Then, we have defined a generalization of this factorial, so-called p-adic Roman factorial, with the proof of some properties and a congruances modulo a power of a prime number. Also, a numerical examples have been given. This concept will be used to define the p-adic binomial coefficients and its generalzation, in a future paper.

Author Contributions

The author read and approved the last version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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