Journal of Universal Mathematics
Vol.5 No.2 pp.68-75 (2022)
ISSN-2618-5660
DOI:10.33773/jum.1089354

QUADRATIC MODULES OF LIE ALGEBRAS FIBRED OVER NIL(2)-MODULES OF LIE ALGEBRAS

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Abstract. In this work we illustrade that the forgetful functor mapping a quadratic module of Lie algebra to a nil(2)-module of Lie algebra is a fibration.

1. Introduction

Whitehead introduced crossed modules of groups in [1] as an algebraic models for homotopy 2-types. Using the simplicial methods given by Kan in [2], Conduché defined 2-crossed modules in [3]. Crossed modules over Lie algebras firstly given by Gertenhaber in [15]. The Lie algebraic version of a 2-crossed module is given by Ellis in [4].

Grothendieck defined the notion of fibred category in [5]. Quadratic module of groups introduced in [7] is algebraic model for homotopy 3-types. Lie algebraic variation of a quadratic module is given by Ulualan and Uslu in [10]. Baues cofibration for a quadratic module of Lie algebra defined in [13]. The relations amongsimplicial Lie algebras and 2-crossed modules are given in [8] by using simplicial properties.

Another model for homotopy 3-type is crossed squares defined in [6]. The categorical equivalency of crossed squares and quadratic modules is given in [9] for commutative algebras. Pullback and fibration for quadratic modules given in[11] and for crossed squares given in [14] [12]. In this work we give the Lie algebra adaptation of a fibration of quadratic modules. In section 3 to show that the forgetful functor $\Phi : \mathfrak{QM}_{\mathfrak{Lie}} \to \mathfrak{Nil}(2)_{\mathfrak{Lie}}$ is fibred we construct the pullback of quadratic modules of Lie algebras with a homomorphism of Nil(2)-module of Lie algebras.

2. Preliminaries

Definition 2.1. [13] The diagram of Lie algebra homomorphisms

Date: **Received:** 2022-03-17; **Accepted:** 2022-07-21.
2000 *Mathematics Subject Classification.* 18G50, 18G55, 16B50.
Key words and phrases. Crossed module, Lie algebra, Nil(2)-module, Quadratic module.

$$\begin{array}{ccccc} & & C \otimes C & & \\ & \overset{w}{\swarrow} & \downarrow \omega & & \\ C_2 & \xrightarrow[\delta]{} & C_1 & \xrightarrow[\partial]{} & C_0 \end{array}$$

which satisfies:

QML1. $\partial : C_1 \to C_0$ is a nil(2)-module of Lie algebras and the quotient map

$$C_1 \to C = C_1^{cr}/[(C_1^{cr}),(C_1^{cr})]$$

is defined as $c_1 \mapsto [c_1]$, in which $[c_1] \in C$ denotes the class represented by $c_1 \in C_1$.

QML2. For $c_2 \in C_2$, $\partial\delta(c_2) = 1$ and for $c_1, c_1' \in C_1$

$$\partial_2 w([c_1] \otimes [c_1']) = \omega([c_1] \otimes [c_1']) = \partial(c_1) \cdot c_1' - [c_1, c_1'].$$

QML3. For $c_2 \in C_2, c_0 \in C_0$

$$\partial(c_0) \cdot c_2 = w([\delta c_2] \otimes [c_0] + [c_0] \otimes [\delta c_2]).$$

QML4. For $c_2, c_2' \in C_2$

$$w([\delta c_2] \otimes [\delta c_2']) = [c_2, c_2'].$$

is called a quadratic module of Lie algebras.

If $\varphi : (f_2, f_1, f_0) : (w, \partial, \delta) \to (w', \partial', \delta')$ is a morphism of quadratic module of Lie algebras then

i.

$$\begin{array}{ccccccc} C \otimes C & \xrightarrow{w} & C_2 & \xrightarrow{\partial} & C_1 & \xrightarrow{\delta} & C_0 \\ \downarrow{\scriptstyle \varphi^* \otimes \varphi^*} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ C \otimes C & \xrightarrow[w']{} & C_2 & \xrightarrow[\partial']{} & C_1 & \xrightarrow[\delta']{} & C_0 \end{array}$$

the diagram is comutative.

ii. f_2 and f_1 are f_0-equivarant, (f_1,f_0) is a pre-crossed module morphism.

iii. $\varphi^* : C \to C$ is induced from (f_1,f_0).

We will denote this category with $\mathfrak{QM}_{\mathfrak{Lie}}$.

Example 2.2. [10] Let

$$M \times M \xrightarrow{\{-,-\}} L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$$

be a 2-crossed module of Lie algebras. Take

$$C_2 = L/P_2, \quad C_1 = M/P_1, \quad C_0 = N$$

where P_3 is an ideal of M with generators $\langle m_1, \langle m_2, m_3\rangle\rangle$ and $\langle\langle m_1, m_2\rangle, m_3\rangle$ for $m_1, m_2, m_3 \in M$ and P_2 is an ideal of L with generators $\{\langle m_1, m_2\rangle, m_3\}$ and $\{m_1, \langle m_2, m_3\rangle\}$. Then,

$$\begin{array}{ccccc} & & C \otimes C & & \\ & \overset{\omega}{\swarrow} & \downarrow w & & \\ C_2 & \xrightarrow[\delta]{} & C_1 & \xrightarrow[\partial]{} & C_0 \end{array}$$

is an object in $\mathfrak{QM}_{\mathfrak{Lie}}$ with

$$C = \frac{C_1^{cr}}{[C_1^{cr}, C_1^{cr}]}$$

, $\delta : C_2 \longrightarrow C_1$ is given by $\delta(c_2 + P_2) = \partial_2 l + P_2$ and $\partial : C_1 \to C_0$ is given by $\partial(c_1 + P_1) = \partial_1(c_1)$, for all $c_1 \in C_1, c_2 \in C_2$. The quadratic map

$$\omega : C \otimes C \longrightarrow C_2$$

is defined as

$$\omega\left([q_1 c_1] \otimes [q_1 c_1']\right) = q_2\{c_1, c_1'\}$$

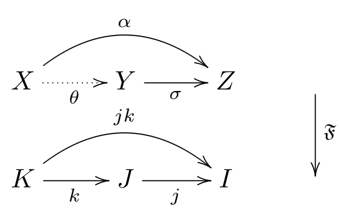
for all $c_1, c_1' \in C_1$. Here $q_1 : M \to C_1$ and $q_2 : L \to C_2$ are quotient maps.

Definition 2.3. [10] Let $\delta : C_1 \to C_0$ be a pre-crossed module, $P_1(\delta) = C_1$ and $P_2(\delta)$ be the Peiffer Lie ideal of C with generators

$$\langle c_0, c_1 \rangle = \delta(c_0) \cdot c_1 - [c_0, c_1]$$

called Peiffer elements for $c_0, c_1 \in C_1$. If $P_3(\delta) = 0$, where $P_3(\delta)$ is the ideal of the Lie algebra C_1 with generators $\langle c_1, c_2, c_3 \rangle$ of length 3 then a pre-crossed module $\delta : C_1 \to C_0$ is called a nil(2)-module.

Definition 2.4. [11] Let $\mathfrak{F} : \mathfrak{C} \to \mathfrak{D}$ be a functor. A morphism $\sigma : Y \to Z$ in $\mathfrak{C}$ over $j := \Phi(\sigma)$ is called cartesian if and only if for all $k : K \to J$ in $\mathfrak{D}$ and $\alpha : X \to Z$ satisfying $\Phi(\alpha) = jk$ there exists a unique morphism $\theta : X \to Y$ satisfying $\mathfrak{F}(\theta) = k$ and $\alpha = \sigma\theta$.



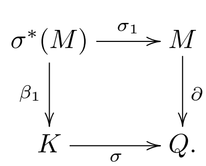
3. Quadratic Modules of Lie Algebras Fibred Over Nil(2)-Modules of Lie Algebra

Proposition 1. The forgetful functor

$$\Phi_1 : \mathfrak{Nil}(2)_{\mathfrak{Lie}} \to \mathfrak{Lie}$$

is fibred.

Proof. Constructing a pullback object in $\mathfrak{Nil}(2)_{\mathfrak{Lie}}$ with a homomorphism of Lie algebras has as a necessary condition for proving Φ_1 is fibred. Let $\partial : M \to Q$ be an object in nil(2)-module of Lie algebras and $\sigma : K \to Q$ be a Lie algebra morphism. Let us define $\sigma^*(M)$ as the sub-Lie algebra of $K \times M$ of elements (k, m) such that $\sigma(k) = \partial(m)$. Let $\sigma_1 : (k, m) \mapsto m$ and $\beta_1 : (k, m) \mapsto k$. Then we have a commutative diagram

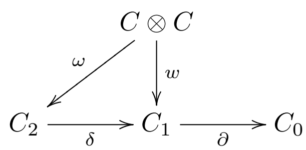


β_1 is a nil(2)-module of Lie algebras with base K by defining the action of $k' \in K$ on $(k, m) \in \sigma^*(M)$ as $k' \cdot (k, m) = (k' \cdot k, \sigma(k') \cdot m)$. β_1 is the *pullback* of $\partial : M \to Q$ along σ. In this pullback diagram, (σ_1, σ) is also *cartesian* morphism over $\Phi_1(\sigma_1, \sigma) = \sigma$ in $\mathfrak{Nil}(2)_{\mathfrak{Lie}}$. Thus Φ_1 is a fibration of categories. $\square$

There exits a forgetful functor;

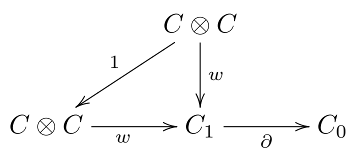
$$\Phi : \mathfrak{QM}_{\mathfrak{Lie}} \to \mathfrak{Nil}(2)_{\mathfrak{Lie}}$$

which maps a quadratic modules of Lie algebra



to its base $(C_1 \xrightarrow{\partial} C_0)$.

Example 3.1. A nil(2)-module of Lie algebras $\partial : C_1 \to C_0$ yields a quadratic module of Lie algebras given by

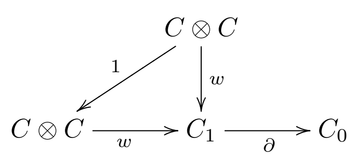


Since $\partial : C_1 \to C_0$ is a nil(2)-module of Lie algebras and $w\{c_1 \otimes c_1'\} = 1$ for $c_1 \otimes c_1' \in C \otimes C$ the conditions in definition.2.1 are satisfied.

Therefore there exits a functor from nil(2)-module of Lie algebras to quadratic module of Lie algebras. We denote this functor as

$$D : \mathfrak{Nil}(2)_{\mathfrak{Lie}} \to \mathfrak{QM}_{\mathfrak{Lie}}.$$

This functor maps a nil(2)-module of Lie algebras, $\partial : C_1 \to C_0$, to a quadratic module of Lie algebras given as

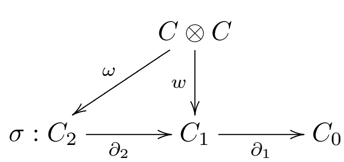


Proposition 2. The forgetful functor

$$\Phi : \mathfrak{QM}_{\mathfrak{Lie}} \to \mathfrak{Nil}(2)_{\mathfrak{Lie}}$$

is fibred and has a left adjoint.

Proof. The left adjoint functor is given in example3.1. We construct the pullback object in $\mathfrak{QM}_{\mathfrak{Lie}}]$ to prove that Φ is fibred. Let



be an object in $\mathfrak{QM}_{\mathfrak{Lie}}$ and $u := (u_1, u_0)$ be a morphism in $\mathfrak{Nil}(2)_{\mathfrak{Lie}}$ illustrated as:

$$\begin{array}{ccc} C_1' & \xrightarrow{u_1} & C_1 \\ {\scriptstyle \partial_1'}\downarrow & & \downarrow{\scriptstyle \partial_1} \\ C_0' & \xrightarrow[u_0]{} & C_0. \end{array}$$

The base nil(2)-module of Lie algebras of the candidate pullback quadratic module of Lie algebras should be $\partial_1' : C_1' \to C_0'$. Let us define

$$u^*(C_2) = \{(c_1', c_2) : r \in ker\partial_1', u_1(c_1') = \partial_2(c_2)\} \subset C_1' \times C_2.$$

Then we get the following commutative diagram

$$\begin{array}{ccc} u^*(C_2) & \xrightarrow{\varphi} & C_2 \\ {\scriptstyle \partial_2'}\downarrow & & \downarrow{\scriptstyle \partial_2} \\ C_1' & \xrightarrow[u_1]{} & C_1 \\ {\scriptstyle \partial_1'}\downarrow & & \downarrow{\scriptstyle \partial_1} \\ C_0' & \xrightarrow[u_0]{} & C_0 \end{array}$$

where $\varphi(c_1', c_2) = c_2$ and $\partial_2'(c_1', c_2) = c_1'$ for $(c_1', c_2) \in u^*(C_2)$. Let us define the quadratic map as

$$\begin{array}{rcl} \omega' : C' \otimes C' & \to & C_2 \\ \omega'(\{c_1'\} \otimes \{d_1'\}) & \mapsto & (\langle c_1', d_1' \rangle, \omega\{u_1(c_1')\} \otimes \{u_1(d_1')\}). \end{array}$$

and the action of C_0' on $u^*(C_2)$ as

$$c_0' \cdot (c_1', c_2) = (c_0' \cdot c_1', u_0(c_0') \cdot c_2)$$

for $c_0' \in C_0'$, $(c_1', c_2) \in u^*(C_2)$ and $c_1', d_1' \in C_1'$. Then being ∂_1' is a pre-crossed module and

$$u_1(c_0' \cdot c_1') = u_0(c_0') \cdot u_1(c_1') = u_0(c_0') \cdot \partial_2(c_2) = \partial_2(u_0(c_0') \cdot c_2)$$

$c_1' \in ker\partial_1'$ implies $c_0' \cdot c_1' \in \ker \partial_1'$ and $(c_0' \cdot c_1', u_0(c_0') \cdot c_2) \in u^*(C_2)$. With these data, we claim that

$$\begin{array}{ccccc} & & C' \otimes C' & & \\ & \swarrow^{\omega'} & \downarrow{\scriptstyle w'} & & \\ u^*(C_2) & \xrightarrow[\partial_2']{} & C_1' & \xrightarrow[\partial_1']{} & C_0' \end{array}$$

is an object in $\mathfrak{QM}_{\mathfrak{Lie}}$.

QML1. $\partial_1' : C_1' \to C_0'$ is a nil(2)-module of Lie algebras. Since $c_1' \in ker\partial_1'$ we get

$$\partial_1'\partial_2'(c_1', c_2) = \partial_1'(c_1') = 1$$

for $(c_1', c_2) \in u^*(C_2)$ that is the bottom row

$$u^*(L) \xrightarrow{\partial_2'} C_1' \xrightarrow{\partial_1'} C_0'$$

is a complex of Lie algebras.

QML2. For $c_1', d_1' \in C_1'$

$$\begin{aligned}
\partial_2' w'([c_1'] \otimes [d_1']) &= \partial_2'(\langle c_1', d_1' \rangle, w\,[u_1(c_1')] \otimes [u_1(d_1')]) \\
&= \langle c_1', d_1' \rangle \\
&= \partial_1'(c_1') \cdot d_1' - [c_1', d_1']
\end{aligned}$$

QML3. For $(c_1', c_2) \in u^*(C_2)$ and $d_1' \in C_1'$:

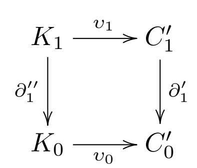
$$\begin{aligned}
& w'([\partial_2'(c_1', c_2)] \otimes [d_1'] + [d_1'] \otimes [\partial_2'(c_1', c_2)]) = w'([c_1'] \otimes [d_1'] + [d_1'] \otimes [c_1']) \\
= \; & (\langle c_1', d_1' \rangle, w\,[u_1(c_1')] \otimes [u_1(d_1')] + \langle d_1', c_1' \rangle, w\,[u_1(d_1')] \otimes [u_1(c_1')]) \\
= \; & (\langle c_1', d_1' \rangle + \langle d_1', c_1' \rangle, w\,[\partial_2(c_2)] \otimes [u_1(d_1')] + [u_1(d_1')] \otimes [\partial_2(c_2)] \\
= \; & (\partial_1'(c_1') \cdot d_1' - [c_1', d_1'] + \partial_1'(d_1') \cdot c_1' - [d_1', c_1'], u_1(d_1') \cdot c_2) \\
= \; & (\partial_1'(c_1') \cdot c_1', u_1(d_1') \cdot c_2) \qquad\qquad (\text{since } c_1 \in ker\partial_1') \\
= \; & \partial_1'(c_1') \cdot (c_1', c_2)
\end{aligned}$$

QML4. For $(c_1', c_2), (d_1', c_2') \in u^*(C_2)$:

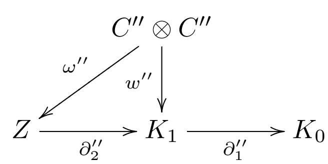
$$\begin{aligned}
w'([\partial_2'(c_1', c_2)] \otimes [\partial_2'(d_1', c_2')]) &= w'([c_1'] \otimes [d_1']) \\
&= (\langle c_1', d_1' \rangle, w\,[u_1(c_1')] \otimes w\,[u_1(d_1')]) \\
&= (\partial_1'(c_1') \cdot d_1' - [c_1', d_1'], w\,[\partial_2(c_2')] \otimes w\,[\partial_2(c_2')] \\
&= ([c_1', d_1'], [c_2, c_2']) \\
&= [(c_1', c_2), (d_1', c_2')]
\end{aligned}$$

From the assumption (u_1, u_0) is a pre-crossed module morphism and from the definition of $u^*(C_2)$ we have $u_1(c_1') = \partial_2(c_2)$. Then it is clear that (φ, u_1, u_0) is a morphism in $\mathfrak{QM}_{\mathfrak{Lie}}$.

Next we will show that (φ, u_1, u_0) is the cartesian morphism over $\Phi(\varphi, u_1, u_0) = (u_1, u_0)$ in $\mathfrak{QM}_{\mathfrak{Lie}}$. Let $(\upsilon_1, \upsilon_0) : \partial_1'' \to \partial_1'$ be morphism in $\mathfrak{Nil}(2)_{\mathfrak{Lie}}$ illustrated as:



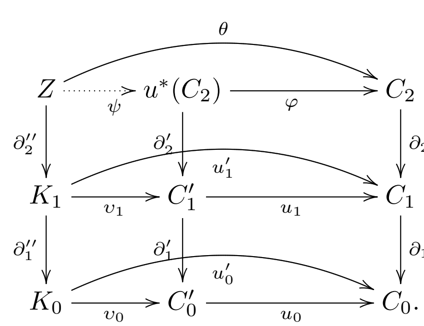
Let



be an object and

$$(\theta, u_1', u_0') : (\omega'', \partial_2'', \partial_1'') \to (\omega, \partial_2, \partial_1)$$

be a morphism in $\mathfrak{QM}_{\mathfrak{Lie}}$ satisfying $u_0\upsilon_0 = u_0'$ and $u_1\upsilon_1 = u_1'$.



The unique morphism $(\psi, \upsilon_1, \upsilon_0)$ in $\mathfrak{QM}_{\mathfrak{Lie}}$ is defined as $\psi(z) = (\upsilon_1\partial_2''(z), \theta(z))$, for $z \in Z$ which implies (φ, u_1, u_0) is cartesian over $\Phi(\varphi, u_1, u_0) = u := (u_1, u_0)$ in $\mathfrak{QM}_{\mathfrak{Lie}}$. The quadratic module of Lie algebras

$$\begin{array}{ccccc} & & C' \otimes C' & & \\ & \swarrow^{\omega'} & \downarrow^{w'} & & \\ u^*(C_2) & \xrightarrow[\partial_2']{} & C_1' & \xrightarrow[\partial_1']{} & C_0' \end{array}$$

is called the *pullback* of

$$\begin{array}{ccccc} & & C \otimes C & & \\ & \swarrow^{\omega} & \downarrow^{w} & & \\ \sigma : C_2 & \xrightarrow[\partial_2]{} & C_1 & \xrightarrow[\partial_1]{} & C_0 \end{array}$$

with a morphism $u := (u_1, u_0)$ between nil(2)-module of Lie algebras.

4. Conclusion

In this work we show that the functor $\Phi : \mathfrak{QM}_{\mathfrak{Lie}} \to \mathfrak{Nil}(2)_{\mathfrak{Lie}}$ is fibred and has a left adjoint $D : \mathfrak{Nil}(2)_{\mathfrak{Lie}} \to \mathfrak{QM}_{\mathfrak{Lie}}$. Since the category of 2-crossed modules, quadratic modules and crossed squares over Lie algebras are equivalent categories analagous constructions can also be obtained for these two categories. □

5. Acknowledgments

The authors would like to thank the reviewers and editors of Journal of Universal Mathematics.

Funding

No funding agency in the governmental, commercial, or not-for-profit sectors provided a specific grant for this study.

The Declaration of Conflict of Interest/ Common Interest

The author(s) declared that no conflict of interest or common interest.

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