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Solution of Fuzzy Volterra Integral and Fractional Differential Equations via Fixed Point Theorems

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Abstract

In this paper we present fuzzy coupled fixed point results in the turf of complete b-metric spaces via nonlinear F-contraction; in follow we derive some interesting results as byproducts. Further, we apply our results in solving fuzzy Volterra integral equations and Caputo-Hadamard type of fractional differential equations.

Keywords: Fuzzy mapping Fuzzy coupled fixed point b-metric space Fuzzy Volterra integral equations Caputo-Hadamard fractional derivative
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1. Introduction

The notion of F-contraction is defined and discussed by Wardowski[23] in 2012. Ahmad, Piri and Nguyena[2, 20, 19] are others who extended the theory further. Recently, Wardowski[25] introduced a nonlinear form of F-contraction. Bakhtin[7] defined the concept of b-metric spaces so as to study pattern matching problems; significant works in this context are presented by Alqahtani, Czerwik, Kutbi, and Qawaqneh. [4, 11, 17, 21].
The idea of coupled fixed points is initiated by Bhaskar and Lakshmikantham[8] in 2006; notable works in this context are seen in [22]. Heilpern [13] posted a generalization of Nadler's fixed point theorem via fuzzy mappings. Abu, Azam et al. and Lee et al.[1, 5, 18] are some others who presented certain substantial results in the turf of fuzzy mappings. Recently, Zhu[28] extended the concepts of coupled coincidence and common fixed points in this context. Fuzzy integral and fractional differential equations are widely used in modelling many real life problems. Existence theorems for Volterra type integral equations presented in [3, 12] are some important works related to the theory developed here. In 2012, Jarad et al.[14] defined and discussed the notion of Caputo-Hadamard fractional derivatives.

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Recently, Boutiara[9] proved an existence result for a Caputo-Hadamard fractional boundary value problem using Monch's fixed point theorem.
In section 3, we present fuzzy coupled fixed point theorems via nonlinear F-contraction; subsequently, we extract some interesting results as corollaries. In section 4, we apply the theory to solve a system of fuzzy Volterra integral equations; in section 5, we exhibit the existence of solution for a system of Caputo-Hadamard fractional differential equations through the theory developed.

2. Preliminaries

Any function from a nonempty set $\mathtt{X}$ to $[0, 1]$ is called a fuzzy set [27]. As usual, $\mathtt{I}^{\mathtt{X}}$ denotes the family of all fuzzy sets in $\mathtt{X}$. An α-level set of a fuzzy set μ denoted by $[\mu]^\alpha$ is defined as

$$[\mu]^\alpha \quad = \quad \{u : \mu(u) \geq \alpha\} \text{ if } \alpha \in (0, 1].$$

For $\alpha = 0$, the level set is given by

$$[\mu]^0 = \overline{\{u : \mu(u) > 0\}}.$$

Here for any subset $\mathtt{A}$ of $\mathtt{X}$, $\bar{\mathtt{A}}$ denotes its closure.

Definition 2.1. *[17] Let* $(\mathtt{M}, \mathtt{d})$ *be a* $\mathtt{b}$*-metric space and* $\mathtt{C_B(M)}$ *be the class of nonempty, closed and bounded subsets of* $\mathtt{M}$*. For any* $\mathtt{A}, \mathtt{B} \in \mathtt{C_B(M)}$*, define*

$$\mathtt{H(A, B)} = \max\left\{\sup_{\mathtt{u} \in \mathtt{A}} \mathtt{d(u, B)}, \sup_{\mathtt{u'} \in \mathtt{B}} \mathtt{d(u', A)}\right\},$$

where

$$\mathtt{d(u, A)} = \inf_{\mathtt{u'} \in \mathtt{A}} \mathtt{d(u, u')}.$$

Lemma 2.2. *[17] Let* $\mathtt{A}$ *and* $\mathtt{B}$ *be nonempty closed and bounded subsets of a* $\mathtt{b}$*-metric space* $(\mathtt{M}, \mathtt{d})$*. If* $\mathtt{u} \in \mathtt{A}$*, then* $\mathtt{d(u, B)} \leq \mathtt{H(A, B)}$.

Let $\mathtt{X}$ and $\mathtt{Y}$ be two non empty sets, then any mapping $\mathbb{A}$ from $\mathtt{X}$ into $\mathtt{I}^{\mathtt{Y}}$ is called a fuzzy mapping[15].
For any two fuzzy sets μ and ν of $\mathtt{M}$. If there exists an $\alpha \in [0, 1]$ so that $[\mu]^\alpha, [\nu]^\alpha \in \mathtt{C_B(M)}$, then define

$$\mathtt{p}_\alpha(\mu, \nu) = \inf_{u \in [\mu]^\alpha, u' \in [\nu]^\alpha} \mathtt{d}(u, u')$$

and

$$\mathtt{D}_\alpha(\mu, \nu) = \mathtt{H}([\mu]^\alpha, [\nu]^\alpha).$$

If $[\mathtt{A}]^\alpha, [\mathtt{B}]^\alpha \in \mathtt{C_B(M)}$ for all $\alpha \in [0, 1]$, then define

$$\mathtt{p(A, B)} = \sup_{\alpha \in [0,1]} \mathtt{p}_\alpha(\mathtt{A, B})$$

and

$$\mathtt{d}_\infty(\mathtt{A, B}) = \sup_{\alpha \in [0,1]} \mathtt{D}_\alpha(\mathtt{A, B}).$$

For conventional reason, we use $\mathtt{p(u, B)}$ instead of $\mathtt{p(\{u\}, B)}$ unless otherwise stated.
A fuzzy set μ in a metric linear space $\mathtt{M}$ is said to be an approximate quantity if $[\mu]^\alpha$ is compact and convex in $\mathtt{M}$ for each $\alpha \in [0, 1]$ and $\sup_{u \in \mathtt{M}} \mu(u) = 1$. The collection of all such approximate quantities in $\mathtt{M}$ is denoted by $\mathtt{W}^{\mathtt{M}}$.
Let $\mathtt{E}^n$ be the family of functions $\mu : \mathbb{R}^n \to [0, 1]$ that satisfy the following conditions:

1. μ is normal, that is, there exists an $u \in \mathbb{R}^n$ such that $\mu(u) = 1$;
2. μ is fuzzy convex, that is, for $0 \leq \beta \leq 1$, $\mu(\beta u + (1 - \beta)u') \geq \min\{\mu(u), \mu(u')\}$;
3. μ is upper semicontinuous;
4. $[\mu]^0 = \overline{\{u \in \mathbb{R}^n : \mu(u) > 0\}}$ is compact.

As we know that $[\mu]^\alpha = \{u \in \mathbb{R}^n : \mu(u) \geq \alpha\}$, for all $\alpha \in (0,1]$, it is obvious to see that the α-level set $[\mu]^\alpha$ is a nonempty compact convex subset of $\mathbb{R}^n$ for all $\alpha \in [0,1]$.

If we let $\mathtt{D} : \mathtt{E}^n \times \mathtt{E}^n \to [0,\infty)$ as a mapping defined by

$$\mathtt{D}(\mu,\nu) = \sup_{\alpha\in[0,1]} \mathtt{H}\left([\mu]^\alpha, [\nu]^\alpha\right)$$

for all $\mu, \nu \in \mathtt{E}^n$, then $\mathtt{D}$ is a metric on $\mathtt{E}^n$.

Definition 2.3. *[28] Let $\mathbb{A} : \mathtt{X}^2 \to \mathtt{I}^{\mathtt{X}}$ be a fuzzy mapping. An element $(u,u') \in \mathtt{X}^2$ is said to be a fuzzy coupled fixed point of $\mathbb{A}$ if there exists $\alpha \in (0,1]$ so that $u \in [\mathbb{A}(u,u')]^\alpha$ and $u' \in [\mathbb{A}(u',u)]^\alpha$.*

The upcoming class of mappings $\mathcal{F}$ is introduced in [23] by Wardowski. A mapping $F : \mathbb{R}_+ \to \mathbb{R}$ belongs to the set $\mathcal{F}$ if it satisfy the following conditions:

(F1) F is strictly increasing;

(F2) For every sequence $\{t_n\}$ of nonzero nonnegative numbers, if $\lim_{n\to\infty} F(t_n) = -\infty$, then $\lim_{n\to\infty} t_n = 0$;

(F3) there exists $k \in \left(0, \frac{1}{1+\log \mathtt{b}}\right)$ so that $\lim_{t\to 0^+} t^k F(t) = 0$;

(F4) F is lower semi-continuous.

Let Ψ be the class of all mappings $\psi : \mathbb{R}^* \to \mathbb{R}_+$ with $\liminf_{x\to t} \psi(x) > 0$ for any $t \geq 0$.

Definition 2.4. *[9] The left-sided fractional integral of order $\alpha > 0$ of a function $y : (p,q) \to \mathbb{R}$ is given by*

$$I^\alpha_{p^+} y(t) = \frac{1}{\Gamma(\alpha)} \int_p^t \left(\log \frac{t}{s}\right)^{\alpha-1} y(s) \frac{ds}{s}$$

provided the right integral converges.

Definition 2.5. *[9] Let $\alpha = 0$, $J = [p,q]$, $n = [\alpha]+1$. If $y(x) \in AC^n_\delta[p,q]$, where $0 < p < q < \infty$ and*

$$AC^n_\delta(J,\mathbb{R}) = \{h : J \to \mathbb{R} : \delta^{n-1} h(t) \in AC(J,\mathbb{R})\}$$

The left-sided Caputo-type modification of left-Hadamard fractional derivatives of order α is given by

$$^C D^\alpha_{p^+} y(t) = ^C D^\alpha_{p^+} \left(y(t) - \sum_{k=0}^{n-1} \frac{\delta^k y(p)}{k!} \left(\log \frac{t}{s}\right)^k \right)$$

Lemma 2.6. *[9] Let $\alpha \geq 0$, $J = [p,q]$ and $n = [\alpha]+1$. If $y(t) \in AC^n_\delta(J,\mathbb{R})$, then Caputo fractional differential equation $^C D^\alpha_{p^+} y(t) = 0$ has a solution*

$$y(t) = \sum_{k=0}^{n-1} c_k \left(\log \frac{t}{p}\right)^k$$

and

$$I^\alpha_{p^+}(^C D^\alpha_{p^+} y)(t) = y(t) + \sum_{k=0}^{n-1} c_k \left(\log \frac{t}{p}\right)^k$$

where $c_k \in \mathbb{R}$, $k = 1, 2, \cdots, n-1$.

3. Fuzzy coupled fixed point theorem

Let us fix some notations here. Let $(\mathtt{M}, \mathsf{d})$ be a complete b-metric space and $\mathbb{A} : \mathtt{M}^2 \to \mathtt{I}^{\mathtt{M}}$ be a fuzzy mapping. For any $\alpha \in (0, 1]$ and $(u, u') \in \mathtt{M}^2$, we denote the image of (u, u') under $\mathbb{A}$ as $\mathbb{A}_{(u,u')}$ and the corresponding α level set by $[\mathbb{A}]^{\alpha}_{(u,u')}$. Meanings of the labels are same, throughout the section, except as otherwise indicated.

Theorem 3.1. *If for each $(u, u') \in \mathtt{M}^2$, there exists $\alpha_{(u,u')} \in (0, 1]$ so that $\emptyset \neq [\mathbb{A}]^{\alpha}_{(u,u')} \in \mathtt{C_B}(\mathtt{M})$ and if there exist mappings $F \in \mathcal{F}$ and $\psi \in \Psi$ with*

$$\psi\left(\mathcal{P}(u, u', v, v')\right) + F\left(\mathsf{b}Q(u, u', v, v')\right) \leq F\left(\frac{\mathcal{R}(u, u', v, v')}{\mathsf{b}}\right), \tag{1}$$

where

$$\begin{aligned}
\mathcal{P}(u, u', v, v') &= d(u, v) + d(u', v');\\
Q(u, u', v, v') &= \mathtt{H}([\mathbb{A}]^{\alpha}_{(u,u')}, [\mathbb{A}]^{\alpha}_{(v,v')}) + \mathtt{H}([\mathbb{A}]^{\alpha}_{(u',u)}, [\mathbb{A}]^{\alpha}_{(v',v)});\\
\mathcal{R}(u, u', v, v') &= \max\{\mathsf{d}(u, v) + \mathsf{d}(u', v'), \mathsf{d}(u, [\mathbb{A}]^{\alpha}_{(u,u')}) + \mathsf{d}(u', [\mathbb{A}]^{\alpha}_{(u',u)}),\\
&\qquad \mathsf{d}(v, [\mathbb{A}]^{\alpha}_{(v,v')}) + \mathsf{d}(v', [\mathbb{A}]^{\alpha}_{(v',v)}), \frac{1}{2}(\mathsf{d}(u, [\mathbb{A}]^{\alpha}_{(v,v')}) + \mathsf{d}(u', [\mathbb{A}]^{\alpha}_{(v',v)})),\\
&\qquad \frac{1}{2}(\mathsf{d}(v, [\mathbb{A}]^{\alpha}_{(u,u')}) + \mathsf{d}(v', [\mathbb{A}]^{\alpha}_{(u',u)}))\}.
\end{aligned}$$

for all $\mathcal{P}(u, u', v, v') > 0$ and $Q(u, u', v, v') > 0$, then $\mathbb{A}$ has a fuzzy coupled fixed point in $\mathtt{M}^2$.

To avoid ambiguity, it should be noted that the choice of α relies on (u, u').

Proof. Let

$$\begin{aligned}
\mathtt{P_n} &= \mathcal{P}\left(u_{n-1}, u'_{n-1}, u_n, u'_n\right)\\
\mathtt{Q_n} &= Q\left(u_{n-1}, u'_{n-1}, u_n, u'_n\right)\\
\mathtt{R_n} &= \mathcal{R}\left(u_{n-1}, u'_{n-1}, u_n, u'_n\right).
\end{aligned}$$

Let $(u_0, u'_0) \in \mathtt{M}^2$, then there exist $\alpha_{(u_0,u'_0)}$ and $\alpha_{(u'_0,u_0)}$ such that $\emptyset \neq [\mathbb{A}]^{\alpha}_{(u_0,u'_0)}$ and $\emptyset \neq [\mathbb{A}]^{\alpha}_{(u'_0,u_0)}$ in $\mathtt{C_B}(\mathtt{M})$; accordingly we can choose $u_1 \in [\mathbb{A}]^{\alpha}_{(u_0,u'_0)}$ and $u'_1 \in [\mathbb{A}]^{\alpha}_{(u'_0,u_0)}$ with

$$\mathsf{d}(u_0, u_1) = \mathsf{d}(u_0, [\mathbb{A}]^{\alpha}_{(u_0,u'_0)})$$

and

$$\mathsf{d}(u'_0, u'_1) = \mathsf{d}(u'_0, [\mathbb{A}]^{\alpha}_{(u'_0,u_0)}).$$

Repeating the process, one can construct a sequence $\{(u_n, u'_n)\}_{n=0}^{\infty}$ so that

$$\mathsf{d}(u_{n-1}, u_n) = \mathsf{d}(u_{n-1}, [\mathbb{A}]^{\alpha}_{(u_{n-1},u'_{n-1})})$$

and

$$\mathsf{d}(u'_{n-1}, u'_n) = \mathsf{d}(u'_{n-1}, [\mathbb{A}]^{\alpha}_{(u'_{n-1},u_{n-1})}),$$

where $u_n \in [\mathbb{A}]^{\alpha}_{(u_{n-1},u'_{n-1})}$ and $u'_n \in [\mathbb{A}]^{\alpha}_{(u'_{n-1},u_{n-1})}$.

If $\mathtt{P_m} = 0$ or $\mathtt{Q_m} = 0$ for some $\mathtt{m} \in \mathbb{N}$, then $u_m \in [\mathbb{A}]^{\alpha}_{(u_m,u'_m)}$ and $u'_m \in [\mathbb{A}]^{\alpha}_{(u'_m,u_m)}$ which implies (u_m, u'_m) is a fuzzy coupled fixed point of $\mathbb{A}$.

On the other side if we assume $\mathtt{P_n} > 0$ and $\mathtt{Q_n} > 0$, for all n. Then from (1), we get

$$\psi(\mathtt{P_n}) + F(\mathsf{b}\mathtt{Q_n}) \leq F\left(\frac{\mathtt{R_n}}{\mathsf{b}}\right).$$

By lemma 2.2, we get $\mathtt{P}_{n+1} \leq \mathtt{Q}_n$. Since F is strictly increasing, we have

$$\begin{aligned} F(\mathtt{P}_{n+1}) &\leq F(\mathtt{Q}_n) \\ &\leq F(\mathtt{bQ}_n) \\ &\leq F\left(\frac{\mathtt{R}_n}{\mathtt{b}}\right) - \psi(\mathtt{P}_n) \\ &\leq F(\mathtt{R}_n) - \psi(\mathtt{P}_n) \end{aligned} \tag{2}$$

Here we claim that $\mathtt{R}_n \leq \mathtt{P}_n$, for all n. It is obvious that $\mathtt{R}_n$ cannot be equal to $\mathtt{P}_{n+1}$ for any n. Therefore by the definition of $\mathtt{R}_n$ the only remaining possibility is

$$\mathtt{R}_n = \frac{1}{2}\mathsf{d}(u_{n-1}, [\mathbb{A}]^{\alpha}_{(u_n,u'_n)}) + \mathsf{d}(u'_{n-1}, [\mathbb{A}]^{\alpha}_{(u'_n,u_n)}),$$

for some n. If we let so, then

$$\begin{aligned} \mathtt{R}_n &\leq \frac{1}{2}(\mathsf{d}(u_{n-1}, u_{n+1}) + \mathsf{d}(u'_{n-1}, u'_{n+1})) \\ &\leq \frac{1}{2}(\mathtt{P}_n + \mathtt{P}_{n+1}), \end{aligned} \tag{3}$$

which in turn implies

$$F(\mathtt{R}_n) \leq F\left(\frac{1}{2}(\mathtt{P}_n + \mathtt{P}_{n+1})\right).$$

Applying the above inequality in (2), we get

$$\begin{aligned} F(\mathtt{P}_{n+1}) &\leq F\left(\frac{1}{2}(\mathtt{P}_n + \mathtt{P}_{n+1})\right) - \psi(\mathtt{P}_n) \\ &\leq F\left(\frac{1}{2}(\mathtt{P}_n + \mathtt{P}_{n+1})\right), \end{aligned}$$

which implies $\mathtt{P}_{n+1} \leq \mathtt{P}_n$; whence from (3), it follows that $\mathtt{R}_n \leq \mathtt{P}_n$ as desired.

Consequently, from (2) it results that

$$F(\mathtt{P}_{n+1}) \leq F(\mathtt{P}_n) - \psi(\mathtt{P}_n) \tag{4}$$

and hence $\{\mathtt{P}_n\}$ must converge to some point $\mathtt{P} \geq 0$. Also since

$$\liminf_{\mathtt{P}_n \to \mathtt{P}} \psi(\mathtt{P}_n) > 0,$$

there exists $\mathtt{c} \in \mathbb{R}_+$ and $\mathtt{N} \in \mathbb{N}$ such that $\psi(\mathtt{P}_n) \geq \mathtt{c}$ for all $\mathtt{n} \geq \mathtt{N}$. Using equation (4) successively, we get

$$\begin{aligned} F(\mathtt{P}_n) &\leq F(\mathtt{P}_{n-1}) - \psi(\mathtt{P}_{n-1}) \\ &\leq F(\mathtt{P}_{n-2}) - \psi(\mathtt{P}_{n-2}) - \psi(\mathtt{P}_{n-1}) \\ &\vdots \qquad\qquad \vdots \\ &\leq F(\mathtt{P}_1) - \sum_{k=1}^{n-1} \psi(\mathtt{P}_k) \\ &< F(\mathtt{P}_1) - \sum_{k=\mathtt{N}}^{n-1} \psi(\mathtt{P}_k) \\ &< F(\mathtt{P}_1) - (\mathtt{n} - \mathtt{N})\mathtt{c}, \ \mathtt{n} \geq \mathtt{N}. \end{aligned} \tag{5}$$

By letting $n \to \infty$, we have $F(\mathtt{P}_n) \to -\infty$ and hence from $(F2)$, it follows that

$$\lim_{n\to\infty} \mathtt{P}_n \to 0.$$

By $(F3)$, there exists $k \in \left(0, \frac{1}{1+\log \mathtt{b}}\right)$ such that

$$\lim_{\mathtt{n}\to\infty} \mathtt{P_n}^k F(\mathtt{P_n}) \to 0.$$

Also by equation (5), we have

$$\begin{aligned} \mathtt{P_n}^k F(\mathtt{P_n}) &< \mathtt{P_n}^k F(\mathtt{P_1}) - \mathtt{P_n}^k(\mathtt{n}-\mathtt{N})\mathtt{W}, \\ &= \mathtt{P_n}^k F(\mathtt{P_1}) - \mathtt{n}\mathtt{P_n}^k + \mathtt{NWP_n}^k, \end{aligned}$$

for all $\mathtt{n} \geq \mathtt{N}$, which implies

$$\mathtt{nP_n}^k < \mathtt{P_n}^k F(\mathtt{P_1}) - \mathtt{P_n}^k F(\mathtt{P_n}) + \mathtt{NWP_n}^k.$$

If we let $n \to \infty$ in the above inequality, we get

$$\lim_{\mathtt{n}\to\infty} \mathtt{nP_n}^k = 0.$$

Therefore there exists $\mathtt{N_0} \in \mathbb{N}$ such that $\mathtt{nP_n}^k < 1$ for all $\mathtt{n} \geq \mathtt{N_0}$ which implies $\mathtt{P_n} < \frac{1}{\mathtt{n}^{\frac{1}{k}}}$ for all $\mathtt{n} \geq \mathtt{N_0}$. Consequently,

$$\begin{aligned} \mathtt{d}(u_n, u_m) + \mathtt{d}(u'_n, u'_m) &\leq \mathtt{P_{n+1}} + \mathtt{P_{n+2}} + \cdots + \mathtt{P_m} \\ &\leq \sum_{\mathtt{i}=1}^{\infty} \mathtt{P_i} \\ &\leq \sum_{\mathtt{i}=1}^{\infty} \frac{1}{\mathtt{i}^{\frac{1}{k}}} \end{aligned}$$

for all $\mathtt{m} > \mathtt{n} > \mathtt{N_0}$. But since $\sum_{\mathtt{i}=1}^{\infty} \frac{1}{\mathtt{i}^{\frac{1}{k}}}$ is convergent, it results that

$$\lim_{n\to\infty} (\mathtt{d}(u_n, u_m) + \mathtt{d}(u'_n, u'_m)) = 0.$$

Hence $\{u_n\}$ and $\{u'_n\}$ must be Cauchy and have to converge, as $\mathtt{M}$ is complete, let us assume that $u_n \to u$ and $u'_n \to u'$. In follow we claim that

$$\mathtt{d}(u, [\mathbb{A}]^{\alpha}_{(u,u')}) + \mathtt{d}(u', [\mathbb{A}]^{\alpha}_{(u',u)}) = 0.$$

Let us assume that

$$\mathtt{d}(u, [\mathbb{A}]^{\alpha}_{(u,u')}) + \mathtt{d}(u', [\mathbb{A}]^{\alpha}_{(u',u)}) > 0$$

on the contrary. As

$$\mathtt{d}(u_{n+1}, [\mathbb{A}]^{\alpha}_{(u,u')}) \leq \mathtt{H}([\mathbb{A}]^{\alpha}_{(u_n,u'_n)}, [\mathbb{A}]^{\alpha}_{(u,u')})$$

and

$$\mathtt{d}(u'_{n+1}, [\mathbb{A}]^{\alpha}_{(u',u)}) \leq \mathtt{H}([\mathbb{A}]^{\alpha}_{(u'_n,u_n)}, [\mathbb{A}]^{\alpha}_{(u',u)}),$$

we have

$$\mathcal{Q}(u_n, u'_n, u, u') \geq \mathtt{d}(u_{n+1}, [\mathbb{A}]^{\alpha}_{(u,u')}) + \mathtt{d}(u'_{n+1}, [\mathbb{A}]^{\alpha}_{(u',u)})\}.$$

This implies that

$$\lim_{n\to\infty} \mathcal{Q}(u_n, u'_n, u, u') \geq \mathtt{d}(u, [\mathbb{A}]^{\alpha}_{(u,u')}) + \mathtt{d}(u', [\mathbb{A}]^{\alpha}_{(u',u)})$$

and therefore there exists $n_0 \in \mathbb{N}$,

$$\mathcal{Q}(u_n, u'_n, u, u') > 0 \text{ for all } n \geq n_0.$$

As $\mathtt{P_n} > 0$ and $\mathtt{Q_n} > 0 \; \forall \; n$,

$$\mathcal{P}(u_n, u'_n, u, u') > 0 \; \forall \; n.$$

Also since

$$\lim_{n\to\infty} \mathcal{R}(u_n, u'_n, u, u') = \mathtt{d}(u, [\mathbb{A}]^{\alpha}_{(u,u')}) + \mathtt{d}(u', [\mathbb{A}]^{\alpha}_{(u',u)}),$$

using $(F4)$ it results that

$$\lim_{n\to\infty} F(\mathsf{b}Q(u_n,u'_n,u,u')) \geq F(\mathsf{b}(\mathsf{d}(u,[\mathbb{A}]^{\alpha}_{(u,u')})+\mathsf{d}(u',[\mathbb{A}]^{\alpha}_{(u',u)})))$$

and

$$\lim_{n\to\infty} F\left(\frac{\mathcal{R}(u_n,u'_n,u,u')}{b}\right) = F\left(\frac{\mathsf{d}(u,[\mathbb{A}]^{\alpha}_{(u,u')})+\mathsf{d}(u',[\mathbb{A}]^{\alpha}_{(u',u)})}{\mathsf{b}}\right).$$

From contractive condition (1), we get

$$F(\mathcal{R}(u_n,u'_n,u,u')) \geq \psi(\mathcal{P}(u_n,u'_n,u,u'))+F(Q(u_n,u'_n,u,u'))$$

for all $n\geq n_0$. Finally using $(F1)$ we get that
$\liminf_{n\to\infty}\psi(\mathcal{P}(u_n,u'_n,u,u'))$

$$\begin{aligned}
&\leq \liminf_{n\to\infty} F\left(\frac{\mathcal{R}(u_n,u'_n,u,u')}{\mathsf{b}}\right)-\liminf_{n\to\infty} F(\mathsf{b}Q(u_n,u'_n,u,u'))\\
&< 0,
\end{aligned}$$

which is a contradiction. Therefore

$$\mathsf{d}(u,[\mathbb{A}]^{\alpha}_{(u,u')})+\mathsf{d}(u',[\mathbb{A}]^{\alpha}_{(u',u)})=0,$$

which implies that (u,u') is a required fuzzy coupled fixed point of $\mathbb{A}$. □

Example 3.2. *Let* $\mathtt{M}=[0,1]$ *and* $\mathtt{d}:\mathtt{M}^2\to[0,\infty)$ *be the mapping defined as* $\mathsf{d}(u,u')=|u-u'|^2$, *then* $(\mathtt{M},\mathsf{d})$ *is a complete* b*-metric space with coefficient* 2. *Let* $I_{uu'}=\left[\sin\left(\frac{u+u'}{4}\right),\frac{u+u'}{2}\right]$. *Define a mapping* $\mathbb{A}:\mathtt{M}^2\to\mathtt{I}^{\mathtt{M}}$ *by*

$$\mathbb{A}(u,u')(t)=\begin{cases}\frac{u^2+u'^2+1}{3} & \text{if } t\in I_{uu'}\\ 0 & \text{otherwise.}\end{cases}$$

Let $(u,u')\in\mathtt{M}^2$, *then the* α*-level sets of the fuzzy set* $\mathbb{A}(u,u')$ *are given by*

$$[\mathbb{A}]^{\alpha}_{(u,u')}=\begin{cases}I_{uu'} & \text{if } 0\leq\alpha\leq\frac{u^2+u'^2+1}{3}\\ \emptyset & \text{otherwise.}\end{cases}$$

Therefore, for any $(u,u')\in\mathtt{M}^2$, *if we take* $\alpha_{(u,u')}=\frac{u^2+u'^2+1}{3}$, *then* $\alpha_{(u,u')}\in(0,1]$ *and the* α*-level set* $[\mathbb{A}]^{\alpha}_{(u,u')}=I_{uu'}$ *is closed and bounded. Also*

$$\begin{aligned}
\mathcal{P}(u,u',v,v') &= |u-v|^2+|u'-v'|^2;\\
Q(u,u',v,v') &= 2\mathrm{H}(I_{uu'},I_{vv'});\\
\mathcal{R}(u,u',v,v') &= \max\Big\{|u-v|^2+|u'-v'|^2, |u-I_{uu'}|^2+\left|u'-I_{uu'}\right|^2,\\
&\quad |v-I_{vv'}|^2+\left|v'-I_{vv'}\right|^2, \frac{1}{2}\left(|u-I_{vv'}|^2+\left|u'-I_{vv'}\right|^2\right),\\
&\quad \frac{1}{2}\left(|v-I_{uu'}|^2+\left|v'-I_{uu'}\right|^2\right)\Big\}.
\end{aligned}$$

In this plot, if we let

$$F(x)=\ln x \text{ and } \psi(x)=\frac{x+1}{10},$$

then

$$\psi(\mathcal{P}(u,u',v,v'))+F(\mathsf{b}Q(u,u',v,v')) \leq F\left(\frac{R(u,u',v,v')}{\mathsf{b}}\right)$$

for all $\mathcal{P}(u,u',v,v')>0$ *and* $Q(u,u',v,v')>0$. *Hence by Theorem 3.1,* $\mathbb{A}$ *has a fuzzy coupled fixed point. Indeed the elements of the form* $(u,u)\in\mathtt{M}^2$ *are coupled fixed points of* $\mathbb{A}$.

Corollary 3.3. *Let $\mathtt{M}$ be a complete* b*-metric linear space and $\mathbb{A} : \mathtt{M}^2 \to \mathtt{W}^{\mathtt{M}}$ be a mapping. Suppose there exist functions $F \in \mathcal{F}$ and $\psi \in \Psi$ with*

$$\psi\left(\mathcal{P}(u,u',v,v')\right) + F\left(\mathsf{b}\mathcal{Q}(u,u',v,v')\right) \leq F\left(\frac{\mathcal{R}(u,u',v,v')}{\mathsf{b}}\right) \tag{6}$$

where

$$\begin{aligned}
\mathcal{P}(u,u',v,v') &= d(u,v) + d(u',v');\\
\mathcal{Q}(u,u',v,v') &= \mathsf{d}_\infty(\mathbb{A}(u,u'),\mathbb{A}(v,v')) + \mathsf{d}_\infty(\mathbb{A}(u',u),\mathbb{A}(v',v));\\
\mathcal{R}(u,u',v,v') &= \max\Big\{\mathsf{d}(u,v) + \mathsf{d}(u',v'), \mathtt{p}(u,\mathbb{A}(u,u')) + \mathtt{p}(u',\mathbb{A}(u',u)),\\
&\qquad \mathtt{p}(v,\mathbb{A}(v,v')) + \mathtt{p}(v',\mathbb{A}(v',v)), \frac{1}{2}(\mathtt{p}(u,\mathbb{A}(v,v')) + \mathtt{p}(u',\mathbb{A}(v',v))),\\
&\qquad \frac{1}{2}(\mathtt{p}(v,\mathbb{A}(u,u')) + \mathtt{p}(v',\mathbb{A}(u',u)))\Big\},
\end{aligned}$$

for all $\mathcal{P}(u,u',v,v') > 0$ and $\mathcal{Q}(u,u',v,v') > 0$, then there exists $(u_0,u_0') \in M^2$ such that $\{u_0\} \subset \mathbb{A}(u_0,u_0')$ and $\{u_0'\} \subset \mathbb{A}(u_0',u_0)$.

Proof. From the definition of d_∞ metric, we get

$$\begin{aligned}
\mathtt{H}([\mathbb{A}]^\alpha_{(u,u')}, \mathbb{A}^\alpha_{(v,v')}) &\leq \mathsf{d}_\infty(\mathbb{A}(u,u'),\mathbb{A}(v,v'))\\
\mathtt{H}([\mathbb{A}]^\alpha_{(u',u)}, \mathbb{A}^\alpha_{(v',v)}) &\leq \mathsf{d}_\infty(\mathbb{A}(u',u),\mathbb{A}(v',v))
\end{aligned}$$

$\forall\, u,u',v,v' \in \mathtt{M}$.

Further we know that $[\mathbb{A}]^1_{(u,u')} \subseteq [\mathbb{A}]^\alpha_{(u,u')}$, for all $\alpha \in (0,1]$ and $(u,u') \in \mathtt{M}$. Therefore

$$\mathsf{d}(u,[\mathbb{A}]^\alpha_{(u,u')}) \leq \mathsf{d}(u,[\mathbb{A}]^1_{(u,u')}),$$

which results that $\mathtt{p}(u,\mathbb{A}(u,u')) \leq \mathsf{d}(u,[\mathbb{A}]^\alpha_{(u,u')})$.

Analogously, it can be seen that

$$\begin{aligned}
\mathtt{p}(u',\mathbb{A}(u',u)) &\leq \mathsf{d}(u',[\mathbb{A}]^\alpha_{(u',u)});\\
\mathtt{p}(v,\mathbb{A}(v,v')) &\leq \mathsf{d}(v,[\mathbb{A}]^\alpha_{(v,v')});\\
\mathtt{p}(v',\mathbb{A}(v',v)) &\leq \mathsf{d}(v',[\mathbb{A}]^\alpha_{(v',v)}).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\psi\left(\mathcal{P}(u,u',v,v')\right) + F\left(\mathsf{b}Q(u,u',v,v')\right) &\leq \psi\left(\mathcal{P}(u,u',v,v')\right) + F\left(\mathsf{b}\mathcal{Q}(u,u',v,v')\right)\\
&\leq F\left(\frac{\mathcal{R}(u,u',v,v')}{\mathsf{b}}\right)\\
&\leq F\left(\frac{\mathcal{R}(u,u',v,v')}{\mathsf{b}}\right).
\end{aligned}$$

Thus by Theorem 3.1, there exists $(u_0,u_0') \in \mathtt{M}^2$ such that $u_0 \in [\mathbb{A}]^1_{(u_0,u_0')}$ and $u_0' \in [\mathbb{A}]^1_{(u_0',u_0)}$, which implies $u_0 \subset \mathbb{A}(u_0,u_0')$ and $u_0' \subset \mathbb{A}(u_0',u_0)$. □

Let $\mathbb{A} : \mathtt{M}^2 \to \mathtt{I}^{\mathtt{M}}$ be a fuzzy mapping. If we define $\tilde{\mathbb{A}} : \mathtt{M}^2 \to \mathtt{C}_{\mathtt{B}}(\mathtt{M})$ as

$$\tilde{\mathbb{A}}(u,u') = \{v' \in \mathtt{M} : \mathbb{A}_{(u,u')}(v') = \max_{v \in \mathtt{M}} \mathbb{A}_{(u,u')}(v)\},$$

then we can conclude that (u_0,u_0') is a coupled fixed point of $\tilde{\mathbb{A}}$ if and only if $\mathbb{A}_{(u_0,u_0')}(u_0) \geq \mathbb{A}_{(u_0,u_0')}(v)$ and $\mathbb{A}_{(u_0',u_0)}(u_0') \geq \mathbb{A}_{(u_0',u_0)}(v)$ for all $v \in \mathtt{M}$.

Corollary 3.4. *Let $\emptyset \neq \tilde{\mathbb{A}}(u,u') \in \mathtt{C_B}(\mathtt{M})$. If there exist mappings $F \in \mathcal{F}$ and $\psi \in \Psi$ so that*

$$\psi\left(\mathcal{P}(u,u',v,v')\right) + F\left(\mathsf{b}\mathcal{Q}(u,u',v,v')\right) \quad \leq \quad F\left(\frac{\mathcal{R}(u,u',v,v')}{\mathsf{b}}\right)$$

where

$$\begin{aligned}
\mathcal{P}(u,u',v,v') &= d(u,v) + d(u',v');\\
\mathcal{Q}(u,u',v,v') &= \mathtt{H}(\tilde{\mathbb{A}}(u,u'), \tilde{\mathbb{A}}(v,v')) + \mathtt{H}(\tilde{\mathbb{A}}(u',u), \tilde{\mathbb{A}}(v',v));\\
\mathcal{R}(u,u',v,v') &= \max\{\mathsf{d}(u,v) + \mathsf{d}(u',v'), \mathsf{p}(u,\tilde{\mathbb{A}}(u,u')) + \mathsf{p}(u',\tilde{\mathbb{A}}(u',u)),\\
&\quad \mathsf{p}(v,\tilde{\mathbb{A}}(v,v')) + \mathsf{p}(v',\tilde{\mathbb{A}}(v',v)), \frac{1}{2}(\mathsf{p}(u,\tilde{\mathbb{A}}(v,v')) + \mathsf{p}(u',\tilde{\mathbb{A}}(v',v))),\\
&\quad \frac{1}{2}(\mathsf{p}(v,\tilde{\mathbb{A}}(u,u')) + \mathsf{p}(v',\tilde{\mathbb{A}}(u',u)))\},
\end{aligned}$$

for all $\mathcal{P}(u,u',v,v') > 0$ and $\mathcal{Q}(u,u',v,v') > 0$, then there exists a point $(u_0,u_0') \in \mathtt{M}^2$ such that

$$\mathbb{A}_{(u_0,u_0')}(u_0) \geq \mathbb{A}_{(u_0,u_0')}(v,v')$$

and

$$\mathbb{A}_{(u_0',u_0)}(u_0') \geq \mathbb{A}_{(u_0',u_0)}(v,v')$$

for all $(v,v') \in \mathtt{M}^2$.

Proof. By Theorem 3.1, there exits a coupled fixed point $(u_0,u_0') \in \mathtt{M}^2$. Hence $\mathbb{A}_{(u_0,u_0')}(u_0) \geq \mathbb{A}_{(u_0,u_0')}(v,v')$ and $\mathbb{A}_{(u_0',u_0)}(u_0') \geq \mathbb{A}_{(u_0',u_0)}(v)$ for all $v \in \mathtt{M}$ as desired. □

Corollary 3.5. *Let $\mathtt{G} : \mathtt{M}^2 \to \mathtt{C_B}(\mathtt{M})$ be a mapping. Suppose there exist functions $F \in \mathcal{F}$ and $\psi \in \Psi$ with*

$$\psi\left(\mathcal{P}(u,u',v,v')\right) + F\left(\mathsf{b}\mathcal{Q}(u,u',v,v')\right) \quad \leq \quad F\left(\frac{\mathcal{R}(u,u',v,v')}{\mathsf{b}}\right) \tag{7}$$

where

$$\begin{aligned}
\mathcal{P}(u,u',v,v') &= \mathsf{d}(u,v) + \mathsf{d}(u',v');\\
\mathcal{Q}(u,u',v,v') &= \mathtt{H}(\mathtt{G}(u,u'), \mathtt{G}(v,v')) + \mathtt{H}(\mathtt{G}(u',u), \mathtt{G}(v',v));\\
\mathcal{R}(u,u',v,v') &= \max\{\mathsf{d}(u,v) + \mathsf{d}(u',v'), \mathsf{d}(u,\mathtt{G}(u,u')) + \mathsf{d}(u',\mathtt{G}(u',u)),\\
&\quad \mathsf{d}(v,\mathtt{G}(v,v')) + \mathsf{d}(v',\mathtt{G}(v',v)), \frac{1}{2}(\mathsf{d}(u,\mathtt{G}(v,v')) + \mathsf{d}(u',\mathtt{G}(v',v))),\\
&\quad \frac{1}{2}(\mathsf{d}(v,\mathtt{G}(u,u')) + \mathsf{d}(v',\mathtt{G}(u',u)))\},
\end{aligned}$$

for all $\mathcal{P}(u,u',v,v') > 0$ and $\mathcal{Q}(u,u',v,v') > 0$, then $\mathtt{G}$ has a coupled fixed point in $\mathtt{M}^2$.

Proof. Suppose we let $\mathbb{A} : \mathtt{M}^2 \to \mathtt{I}^{\mathtt{M}}$ as

$$\mathbb{A}(u,u')(t) = \begin{cases} \alpha(u,u') & \text{if } t \in \mathtt{G}(u,u') \\ 0 & \text{otherwise,} \end{cases}$$

where α is a mapping from $\mathtt{M}^2$ to $(0,1]$, then $\mathbb{A}$ satisfies all the needs of Theorem 3.1, as

$$[\mathbb{A}]^{\alpha}_{(u,u')} = \{t : \mathbb{A}(u,u')(t) \geq \alpha(u,u')\} = \mathtt{G}(u,u'),$$

for all $u,u' \in \mathtt{M}$. Hence by Theorem 3.1, we get $(u_0,u_0') \in \mathtt{M}^2$ with $u_0 \in [\mathbb{A}]^{\alpha}_{(u_0,u_0')}$ and $u_0' \in [\mathbb{A}]^{\alpha}_{(u_0',u_0)}$ which implies (u_0,u_0') is a coupled fixed point of $\mathtt{G}$. □

Corollary 3.6. *Let* $\mathtt{f} : \mathtt{M}^2 \to \mathtt{M}$ *be a mapping. If there exist functions* $F \in \mathcal{F}$ *and* $\psi \in \Psi$ *so that*

$$\psi\left(\mathcal{P}(u,u',v,v')\right) + F\left(\mathsf{b}Q(u,u',v,v')\right) \leq F\left(\frac{\mathcal{R}(u,u',v,v')}{\mathsf{b}}\right)$$

where

$$\begin{aligned}
\mathcal{P}(u,u',v,v') &= \mathtt{d}(u,v) + \mathtt{d}(u',v');\\
Q(u,u',v,v') &= \mathtt{d}(\mathtt{f}(u,u'),\mathtt{f}(v,v')) + \mathtt{d}(\mathtt{f}(u',u),\mathtt{f}(v',v));\\
\mathcal{R}(u,u',v,v') &= \max\Big\{\mathtt{d}(u,v) + \mathtt{d}(u',v'), \mathtt{d}(u,\mathtt{f}(u,u')) + \mathtt{d}(u',\mathtt{f}(u',u)),\\
&\qquad \mathtt{d}(v,\mathtt{f}(v,v')) + \mathtt{d}(v',\mathtt{f}(v',v)), \frac{1}{2}(\mathtt{d}(u,\mathtt{f}(v,v')) + \mathtt{d}(u',\mathtt{f}(v',v))),\\
&\qquad \frac{1}{2}(\mathtt{d}(v,\mathtt{f}(u,u')) + \mathtt{d}(v',\mathtt{f}(u',u)))\Big\},
\end{aligned}$$

$\forall\, \mathcal{P}(u,u',v,v') > 0$ *and* $Q(u,u',v,v') > 0$. *Then* $\mathtt{f}$ *has a unique coupled fixed point in* $\mathtt{M}^2$.

Proof. If we let $\mathtt{G}(u,u') = \{\mathtt{f}(u,u')\}$, then by corollary 3.5 $\mathtt{f}$ has to possess a coupled fixed point. Suppose (u,u') and $(v,v') \in \mathtt{M}^2$ are two distinct coupled fixed points of $\mathtt{f}$, then by contractive condition (7), we have

$$\psi\left(\mathcal{P}(u,u',v,v')\right) + F\left(\mathsf{b}Q(u,u',v,v')\right) \leq F\left(\frac{\mathcal{R}(u,u',v,v')}{\mathsf{b}}\right).$$

Therefore

$$F(\mathtt{d}(u,v) + \mathtt{d}(u',v')) \geq \psi\left(\mathtt{d}(u,v) + \mathtt{d}(u',v')\right) + F\left(\mathtt{d}(u,v) + \mathtt{d}(u',v')\right),$$

which is not possible and hence $\mathtt{f}$ has to possess a unique coupled fixed point. □

Example 3.7. *Let* $\mathtt{M} = [0,1]$ *and* $\mathtt{d} : \mathtt{M}^2 \to [0,\infty)$ *be the mapping defined by* $\mathtt{d}(u,u') = |u-u'|^2$. *Then clearly* $(\mathtt{M},\mathtt{d})$ *is a complete* b*-metric space with coefficient* 2. *Define a mapping* $\mathtt{f} : \mathtt{M}^2 \to \mathtt{M}$ *by* $\mathtt{f}(u,u') = \frac{\sin^2 u}{4}$. *Also, we have*

$$\begin{aligned}
\mathcal{P}(u,u',v,v') &= |u-v|^2 + |u'-v'|^2;\\
Q(u,u',v,v') &= \left|\frac{\sin^2 u}{4} - \frac{\sin^2 v}{4}\right|^2 + \left|\frac{\sin^2 u'}{4} - \frac{\sin^2 v'}{4}\right|^2;\\
\mathcal{R}(u,u',v,v') &= \max\left\{|u-v|^2 + |u'-v'|^2, \left|u - \frac{\sin u}{4}\right|^2 + \left|u' - \frac{\sin^2 u'}{4}\right|^2,\right.\\
&\qquad \left|v - \frac{\sin^2 v}{4}\right|^2 + \left|v' - \frac{\sin^2 v'}{4}\right|^2, \frac{1}{2}\left|u - \frac{\sin^2 v}{4}\right|^2 + \left|u' - \frac{\sin^2 v'}{4}\right|^2,\\
&\qquad \left.\frac{1}{2}\left|v - \frac{\sin u}{4}\right|^2 + \left|v' - \frac{\sin^2 u'}{4}\right|^2\right\}.
\end{aligned}$$

In this scenario, if we let

$$F(x) = \frac{-1}{\sqrt{x}} \text{ and } \psi(x) = \frac{2x^2+1}{10},$$

then we have

$$\psi(\mathcal{P}(u,u',v,v')) + F\left(\mathsf{b}Q(u,u',v,v')\right) \leq F\left(\frac{\mathcal{R}(u,u',v,v')}{\mathsf{b}}\right),$$

for all $\mathcal{P}(u,u',v,v') > 0$ *and* $Q(u,u',v,v') > 0$. *Thus by corollary 3.6 , the mapping* $\mathtt{f}$ *has a unique coupled fixed point and it is easy to note that* $(0,0)$ *is the required one.*

In follow, we present some natural extensions of Theorem 3.1.

Theorem 3.8. *Let*

$$\begin{aligned}\mathcal{R}'(u,u',v,v') &= \beta_1(\mathtt{d}(u,v)+\mathtt{d}(u',v'))+\beta_2(\mathtt{d}(u,[\mathbb{A}]^{\alpha}_{(u,u')})+\mathtt{d}(u',[\mathbb{A}]^{\alpha}_{(u',u)}))\\ &\quad+\beta_3(\mathtt{d}(v,[\mathbb{A}]^{\alpha}_{(v,v')})+\mathtt{d}(v',[\mathbb{A}]^{\alpha}_{(v',v)}))\\ &\quad+\beta_4(\mathtt{d}(u,[\mathbb{A}]^{\alpha}_{(v,v')})+\mathtt{d}(u',[\mathbb{A}]^{\alpha}_{(v',v)}))\\ &\quad+\beta_5(\mathtt{d}(v,[\mathbb{A}]^{\alpha}_{(u,u')})+\mathtt{d}(v',[\mathbb{A}]^{\alpha}_{(u',u)})),\end{aligned}$$

where $\beta_1+\beta_2+\beta_3+\beta_4+\beta_5<1$. *In Theorem 3.1 replace as* $\mathcal{R}(u,u',v,v')$ *by* $\mathcal{R}'(u,u',v,v')$*, then* $\mathbb{A}$ *has a fuzzy coupled fixed point in* $\mathtt{M}^2$.

Proof. By proceeding as in Theorem 3.1, we get

$$\begin{aligned}F(\mathtt{P}_{\mathtt{n+1}}) &\le F(\mathtt{R}_{\mathtt{n}})-\psi(\mathtt{P}_{\mathtt{n}})\\ &\le F\left((\beta_1+\beta_2+\beta_4)\mathtt{P}_{\mathtt{n}}+\beta_3\mathtt{P}_{\mathtt{n+1}}\right)-\psi(\mathtt{P}_{\mathtt{n}})\\ &\le F\left((\beta_1+\beta_2+\beta_4)\mathtt{P}_{\mathtt{n}}+\beta_3\mathtt{P}_{\mathtt{n+1}}\right).\end{aligned} \tag{8}$$

Since F is strictly increasing, we get

$$\mathtt{P}_{\mathtt{n+1}} \le (\beta_1+\beta_2+\beta_4)\mathtt{P}_{\mathtt{n}}+\beta_3\mathtt{P}_{\mathtt{n+1}}.$$

which implies

$$\mathtt{P}_{\mathtt{n+1}} < \frac{(\beta_1+\beta_2+\beta_4)}{1-\beta_3}\mathtt{P}_{\mathtt{n}} < \mathtt{P}_{\mathtt{n}}.$$

Applying the above inequality in (8), we get

$$\begin{aligned}F(\mathtt{P}_{\mathtt{n+1}}) &\le F((\beta_1+\beta_2+\beta_3+\beta_4)\mathtt{P}_{\mathtt{n}})-\psi(\mathtt{P}_{\mathtt{n}})\\ &\le F(\mathtt{P}_{\mathtt{n}})-\psi(\mathtt{P}_{\mathtt{n}})\\ &< F(\mathtt{P}_{\mathtt{n}}).\end{aligned}$$

From here one can easily derive the remaining proof by retracing the steps followed in Theorem 3.1. □

Example 3.9. *Let* $\mathtt{M}=[0,1]$ *and* $\mathtt{d}:\mathtt{M}^2\to[0,\infty)$ *be the mapping defined by* $\mathtt{d}(u,u')=|u-u'|^2$*, then* $(\mathtt{M},\mathtt{d})$ *is a complete* b*-metric space with coefficient* 2*. Let* $I_{uu'}=\left[\frac{u+u'}{4},\frac{u+u'}{2}\right]$*. Define a mapping* $\mathbb{A}:\mathtt{M}^2\to\mathtt{I}^{\mathtt{M}}$ *by*

$$\mathbb{A}(u,u')(t)=\begin{cases}\frac{u+u'+1}{3} & \text{if } t\in I_{uu'}\\ 0 & \text{otherwise.}\end{cases}$$

Let $(u,u')\in\mathtt{M}^2$*. Then the* α*-level sets of the fuzzy set* $\mathbb{A}(u,u')$ *are given by*

$$[\mathbb{A}]^{\alpha}_{(u,u')}=\begin{cases}I_{uu'} & \text{if } 0\le\alpha\le\frac{u+u'+1}{3}\\ \emptyset & \text{otherwise.}\end{cases}$$

Consequently, for any $(u,u')\in\mathtt{M}^2$*, if we take* $\alpha_{(u,u')}=\frac{u+u'+1}{3}$*, then* $\alpha_{(u,u')}\in(0,1]$ *and the* α*-level set* $[\mathbb{A}]^{\alpha}_{(u,u')}=I_{uu'}$ *is closed and bounded. Also*

$$\begin{aligned}\mathcal{P}(u,u',v,v') &= |u-v|^2+|u'-v'|^2;\\ \mathcal{Q}(u,u',v,v') &= 2\mathtt{H}(I_{uu'},I_{vv'});\\ \mathcal{R}'(u,u',v,v') &= 0.2\left(|u-v|^2+|u'-v'|^2\right)+0.2\left(|u-I_{uu'}|^2+\left|u'-I_{uu'}\right|^2\right)\\ &\quad+0.2\left(|v-I_{vv'}|^2+\left|v'-I_{vv'}\right|^2\right)+0.1\left(|u-I_{vv'}|^2+\left|u'-I_{vv'}\right|^2\right)\\ &\quad+0.1\left(|v-I_{uu'}|^2+\left|v'-I_{uu'}\right|^2\right),\end{aligned}$$

In this plot, if we let

$$F(x) = \ln x \text{ and } \psi(x) = \frac{x+1}{10},$$

then

$$\psi(\mathcal{P}(u,u',v,v')) + F\left(\mathsf{b}Q(u,u',v,v')\right) \quad \leq \quad F\left(\frac{R'(u,u',v,v')}{\mathsf{b}}\right),$$

for all $\mathcal{P}(u,u',v,v') > 0$ *and* $Q(u,u',v,v') > 0$*. Hence by Theorem 3.8,* $\mathbb{A}$ *has a fuzzy coupled fixed point. Indeed the elements of the form* $(u,u) \in \mathtt{M}^2$ *are coupled fixed points of* $\mathbb{A}$.

Note. Results analogous to the corollaries of Theorem 3.1 can be derived for Theorem 3.8 also. However, we present the statement of one specific result that establishes the unique existence of a coupled fixed point for a function $\mathtt{f} : \mathtt{M}^2 \to \mathtt{M}$ in follow.

Corollary 3.10. *Let* $\mathtt{f} : \mathtt{M}^2 \to \mathtt{M}$ *be a mapping. If there exist functions* $F \in \mathcal{F}$ *and* $\psi \in \Psi$ *so that*

$$\psi\left(\mathcal{P}(u,u',v,v')\right) + F\left(\mathsf{b}Q(u,u',v,v')\right) \quad \leq \quad F\left(\frac{\mathcal{R}'(u,u',v,v')}{\mathsf{b}}\right)$$

where

$$\begin{aligned}
\mathcal{P}(u,u',v,v') &= \mathtt{d}(u,v) + \mathtt{d}(u',v');\\
Q(u,u',v,v') &= \mathtt{d}(\mathtt{f}(u,u'),\mathtt{f}(v,v')) + \mathtt{d}(\mathtt{f}(u',u),\mathtt{f}(v',v));\\
\mathcal{R}'(u,u',v,v') &= \beta_1\mathtt{d}(u,v) + \mathtt{d}(u',v') + \beta_2\mathtt{d}(u,\mathtt{f}(u,u')) + \mathtt{d}(u',\mathtt{f}(u',u))\\
&\qquad +\beta_3\mathtt{d}(v,\mathtt{f}(v,v')) + \mathtt{d}(v',\mathtt{f}(v',v))\\
&\qquad +\beta_4(\mathtt{d}(u,\mathtt{f}(v,v')) + \mathtt{d}(u',\mathtt{f}(v',v)))\\
&\qquad +\beta_5(\mathtt{d}(v,\mathtt{f}(u,u')) + \mathtt{d}(v',\mathtt{f}(u',u)))
\end{aligned}$$

for all $\mathcal{P}(u,u',v,v') > 0$ *and* $Q(u,u',v,v') > 0$*, then* $\mathtt{f}$ *has a unique coupled fixed point in* $\mathtt{M}^2$.

Example 3.11. *Let* $\mathtt{M} = [0,1]$ *and* $\mathtt{d} : \mathtt{M}^2 \to [0,\infty)$ *be the mapping defined as* $\mathtt{d}(u,u') = |u-u'|^2$*. Then clearly* $(\mathtt{M},\mathtt{d})$ *is a complete* b*-metric space with coefficient* 2*. Define a mapping* $\mathtt{f} : \mathtt{M}^2 \to \mathtt{I}^{\mathtt{M}}$ *by* $\mathtt{f}(u,u') = \frac{1}{2}\tan\frac{u}{2}$*. Also, we have*

$$\begin{aligned}
\mathcal{P}(u,u',v,v') &= |u-v|^2 + |u'-v'|^2;\\
Q(u,u',v,v') &= \frac{1}{2}\left|\tan\frac{u}{2} - \tan\frac{v}{2}\right|^2 + \frac{1}{2}\left|\tan\frac{u'}{2} - \tan\frac{v'}{2}\right|^2;\\
\mathcal{R}'(u,u',v,v') &= \beta_1\left(|u-v|^2 + |u'-v'|^2\right)\\
&\quad +\beta_2\left(\left|u - \frac{1}{2}\tan\frac{u}{2}\right|^2 + \left|u' - \frac{1}{2}\tan\frac{u'}{2}\right|^2\right)\\
&\quad +\beta_3\left(\left|v - \frac{1}{2}\tan\frac{v}{2}\right|^2 + \left|v' - \frac{1}{2}\tan\frac{v'}{2}\right|^2\right)\\
&\quad +\beta_4\left(\left|u - \frac{1}{2}\tan\frac{v}{2}\right|^2 + \left|u' - \frac{1}{2}\tan\frac{v'}{2}\right|^2\right)\\
&\quad +\beta_5\left(\left|v - \frac{1}{2}\tan\frac{u}{2}\right|^2 + \left|v' - \frac{1}{2}\tan\frac{u'}{2}\right|^2\right).
\end{aligned}$$

If we let $F(x) = \ln x + x$*;* $\psi(x) = \frac{x^2+1}{10}$*;* $\beta_1 = 0.3$*;* $\beta_2 = \beta_3 = \beta_4 = \beta_5 = 0.1$ *then we have*

$$\psi(\mathcal{P}(u,u',v,v')) + F\left(\mathsf{b}Q(u,u',v,v')\right) \leq F\left(\frac{\mathcal{R}'(u,u',v,v')}{\mathsf{b}}\right)$$

for all $\mathcal{P}(u,u',v,v') > 0$ *and* $Q(u,u',v,v') > 0$*. Thus by corollary 3.10, the mapping* $\mathtt{f}$ *has a unique coupled fixed point and it is easy to note that* $(0,0)$ *is the required one.*

4. Application on fuzzy Volterra integral equations

Let $\mathtt{M} = \mathbb{C}([0,1], \mathtt{E}^n)$ be the class of all continuous fuzzy mappings from $[0,1]$ to $\mathtt{E}^n$ and let $\mathtt{d} : \mathtt{M}^2 \to [0,\infty)$ be the mapping defined as

$$\mathtt{d}(\vartheta,\xi) = \sup_{t\in[0,1]} \mathtt{D}(\vartheta_t, \xi_t),$$

where ϑ_t and ξ_t are the images t under ϑ and ξ respectively, then $\mathtt{M}$ is complete.

Let $\tau, \vartheta, \xi : [0,1] \to \mathtt{E}^n$, $\Upsilon : [0,1] \times \mathtt{E}^n \times \mathtt{E}^n \to \mathtt{E}^n$ be continuous fuzzy functions. Let

$$\mathcal{N} = \{(t,s) : 0 \le s \le t \le 1\},$$

and $\kappa : \mathcal{N} \to \mathbb{R}$ be continuous such that $\sup_{t\in[0,1]} \int_0^t |\kappa(t,s)| ds \le 1$.

Consider the following system of fuzzy Volterra integral equations

$$\begin{aligned} \vartheta_t &= \tau_t + \int_0^t \kappa(t,s)\Upsilon(s,\vartheta_s,\xi_s)ds \\ \xi_t &= \tau_t + \int_0^t \kappa(t,s)\Upsilon(s,\xi_s,\vartheta_s)ds, \ t \in [0,1] \end{aligned} \tag{9}$$

Theorem 4.1. *Let $\mathfrak{S} : \mathtt{E}^n \times \mathtt{E}^n \to \mathtt{E}^n$ be a function defined by*

$$\mathfrak{S}(\mu,\nu) = \tau_t + \int_0^t \kappa(t,s)\Upsilon(s,\mu,\nu)ds.$$

If there exists $\theta > 0$ so that

$$\mathtt{D}(\Upsilon(t,\mu_1,\nu_1), \Upsilon(t,\mu_2,\nu_2)) \le \frac{e^{-\theta}}{2} R(\mu_1,\nu_1,\mu_2,\nu_2)$$

where

$$\begin{aligned} R(\mu_1,\nu_1,\mu_2,\nu_2) = \max\{&\mathtt{D}(\mu_1,\mu_2) + \mathtt{D}(\nu_1,\nu_2), \mathtt{D}(\mu_1,\mathfrak{S}(\mu_1,\nu_1)) + \mathtt{D}(\nu_1,\mathfrak{S}(\nu_1,\mu_1)), \\ &\mathtt{D}(\mu_2,\mathfrak{S}(\mu_2,\nu_2)) + \mathtt{D}(\nu_2,\mathfrak{S}(\nu_2,\mu_2))\} \end{aligned} \tag{10}$$

for all μ_1,ν_1,μ_2 and ν_2 in $\mathtt{E}^n$, then the system of fuzzy Volterra integral equations (9) has a solution.

Proof. Let $\Gamma : \mathtt{M}^2 \to \mathtt{I}^{\mathtt{M}}$ be the fuzzy mapping defined by

$$\Gamma(\vartheta,\xi)(\iota) = \begin{cases} \rho(\vartheta,\xi) & \text{if } \iota(t) = \mathfrak{S}(\vartheta_t,\xi_t) \\ 0 & \text{otherwise,} \end{cases}$$

where $\rho : \mathtt{M}^2 \to (0,1]$, then

$$\begin{aligned} [\Upsilon]^{\alpha}_{(\vartheta,\xi)} &= \{\iota \in \mathtt{M} : \Gamma(\vartheta,\xi)(\iota) \ge \rho(\vartheta,\xi)\} \\ &= \{\mathfrak{S}(\vartheta_t,\xi_t)\}, \end{aligned}$$

for all $\vartheta, \xi \in \mathtt{M}$ and therefore

$$
\begin{aligned}
&\mathtt{H}([\Upsilon]^{\alpha}_{(\vartheta_1,\xi_1)}, [\Upsilon]^{\alpha}_{(\vartheta_2,\xi_2)})\\
&\le \sup_{t\in[0,1]} \mathtt{D}(\mathfrak{S}(\vartheta_{1_t}, \xi_{1_t}), \mathfrak{S}(\vartheta_{2_t}, \xi_{2_t}))\\
&\le \sup_{t\in[0,1]} \mathtt{D}\left(\int_0^t \kappa(t,s)\Upsilon(s,\vartheta_{1_s},\xi_{1_s})ds, \int_0^t \kappa(t,s)\Upsilon(s,\vartheta_{2_s},\xi_{2_s})ds\right)\\
&\le \sup_{t\in[0,1]} \int_0^t \mathtt{D}(\kappa(t,s)\Upsilon(s,\vartheta_{1_s},\xi_{1_s}), \kappa(t,s)\Upsilon(s,\vartheta_{2_s},\xi_{2_s}))ds\\
&\le \sup_{t\in[0,1]} \int_0^t |\kappa(t,s)|\mathtt{D}(\Upsilon(s,\vartheta_{1_s},\xi_{1_s}), \Upsilon(s,\vartheta_{2_s},\xi_{2_s}))ds\\
&\le \sup_{t\in[0,1]} \mathtt{D}(\Upsilon(t,\vartheta_{1_t},\xi_{1_t}), \Upsilon(t,\vartheta_{2_t},\xi_{2_t}))\int_0^t |\kappa(t,s)|ds\\
&\le \sup_{t\in[0,1]} \frac{e^{-\theta}}{2} R(\vartheta_{1_t},\xi_{1_t},\vartheta_{2_t},\xi_{2_t})\\
&\le \frac{e^{-\theta}}{2} R(\vartheta_1,\xi_1,\vartheta_2,\xi_2).
\end{aligned}
$$

Analogously, one can prove that

$$\mathtt{H}([\Upsilon]^{\alpha}_{(\xi_1,\vartheta_1)}, [\Upsilon]^{\alpha}_{(\xi_2,\vartheta_2)}) \le \frac{e^{-\theta}}{2} R(\vartheta_1,\xi_1,\vartheta_2,\xi_2).$$

Adding the above two inequalities, we get

$$\mathtt{H}([\Upsilon]^{\alpha}_{(\vartheta_1,\xi_1)}, [\Upsilon]^{\alpha}_{(\vartheta_2,\xi_2)}) + \mathtt{H}([\Upsilon]^{\alpha}_{(\xi_1,\vartheta_1)}, [\Upsilon]^{\alpha}_{(\xi_2,\vartheta_2)}) \le e^{-\theta} R(\vartheta_1,\xi_1,\vartheta_2,\xi_2).$$

Suppose we let $F(x) = \ln x$ and $\psi(x) = \theta$, then by Theorem 3.1, the fuzzy mapping Γ has a fuzzy coupled fixed point; consequently, the system of fuzzy Volterra integral equations (9) has a fuzzy solution as desired. □

Example 4.2. *Let*

$$
\begin{aligned}
\vartheta_t &= \tau_t + \int_0^t ts^2 \frac{t^2\vartheta_t + t\xi_t}{3} ds\\
\xi_t &= \tau_t + \int_0^t ts^2 \frac{t^2\xi_t + t\vartheta_t}{3} ds,
\end{aligned}
$$

be the system of fuzzy Volterra integral equations,with kernel $\kappa(t,s) = ts^2$, then we have

$$
\begin{aligned}
\sup_{t\in[0,1]} \int_0^t |\kappa(t,s)|ds &= \sup_{t\in[0,1]} \int_0^t |ts^2|ds\\
&= \sup_{t\in[0,1]} \frac{t^4}{3}\\
&\le \frac{1}{3}
\end{aligned}
$$

Let $\Upsilon : [0,1] \times \mathtt{E}^1 \times \mathtt{E}^1 \to \mathtt{E}^1$ and $\mathfrak{S} : \mathtt{E}^1 \times \mathtt{E}^1 \to \mathtt{E}^1$ be the functions defined as

$$\Upsilon(t,\mu,\nu) = \frac{t^2\mu + t\nu}{3}$$

and

$$\Im(\mu,\nu) = \tau_t + \int_0^t ts^2\frac{s^2\mu + s\nu}{3}ds.$$

For any two given fuzzy sets μ_1 and ν_1, let us denote that the α-level sets of the fuzzy sets μ_1, ν_1 and $\int_0^t \mu_1 ds$ by

$$[\mu_1]^\alpha = [\mu_{1l}^\alpha, \mu_{1u}^\alpha];$$

$$[\nu_1]^\alpha = [\nu_{1l}^\alpha, \nu_{1u}^\alpha];$$

$$\left[\int_0^t \mu_1 ds\right]^\alpha = [\int_0^t \mu_{1l}^\alpha ds, \int_0^t \mu_{1u}^\alpha ds];$$

In follow, let us compute the terms, that are needed to validate whether the constructed $\Im$ and Υ satisfy the sufficient condition in Theorem 4.1:

$$\begin{aligned}
&\mathtt{D}(\Upsilon(t,\mu_1,\nu_1),\Upsilon(t,\mu_2,\nu_2))\\
&= \sup_{\alpha\in[0,1]} \mathtt{H}\left([\Upsilon(t,\mu_1,\nu_1)]^\alpha, [\Upsilon(t,\mu_2,\nu_2)]^\alpha\right)\\
&= \sup_{\alpha\in[0,1]} \mathtt{H}\left(\left[\frac{t^2\mu_{1l}^\alpha + t\nu_{1l}^\alpha}{3}, \frac{t^2\mu_{1u}^\alpha + t\nu_{1u}^\alpha}{3}\right], \left[\frac{t^2\mu_{2l}^\alpha + t\nu_{2l}^\alpha}{3}, \frac{t^2\mu_{2u}^\alpha + t\nu_{2u}^\alpha}{3}\right]\right)\\
&= \frac{1}{3}\sup_{\alpha\in[0,1]} \max\left\{|t^2\mu_{1l}^\alpha + t\nu_{1l}^\alpha - t^2\mu_{2l}^\alpha - t\nu_{2l}^\alpha|, |t^2\mu_{1u}^\alpha + t\nu_{1u}^\alpha - t^2\mu_{2u}^\alpha - t\nu_{2u}^\alpha|\right\}\\
&= \frac{1}{3}\sup_{\alpha\in[0,1]} \max\left\{|t^2(\mu_{1l}^\alpha - \mu_{2l}^\alpha) + t(\nu_{1l}^\alpha - \nu_{2l}^\alpha)|, |t^2(\mu_{1u}^\alpha - \mu_{2u}^\alpha) + t(\nu_{1u}^\alpha - \nu_{2u}^\alpha)|\right\};
\end{aligned}$$

$$\begin{aligned}
\mathtt{D}(\mu_1,\mu_2) &= \sup_{\alpha\in[0,1]} \mathtt{H}([\mu_1]^\alpha, [\mu_2]^\alpha)\\
&= \sup_{\alpha\in[0,1]} \mathtt{H}([\mu_{1l}^\alpha, \mu_{1u}^\alpha], [\mu_{2l}^\alpha, \mu_{2u}^\alpha])\\
&= \sup_{\alpha\in[0,1]} \max\{|\mu_{1l}^\alpha - \mu_{2l}^\alpha|, |\mu_{1u}^\alpha - \mu_{2u}^\alpha|\};
\end{aligned}$$

$$\mathtt{D}(\nu_1,\nu_2) = \sup_{\alpha\in[0,1]} \max\{|\nu_{1l}^\alpha - \nu_{2l}^\alpha|, |\nu_{1u}^\alpha - \nu_{2u}^\alpha|\};$$

$$\begin{aligned}
\mathtt{D}(\mu_1,\Im(\mu_1,\nu_1)) &= \sup_{\alpha\in[0,1]} \mathtt{H}([\mu_1]^\alpha, [\Im(\mu_1,\nu_1)]^\alpha)\\
&= \sup_{\alpha\in[0,1]} \max\left\{|\mu_{1l}^\alpha - \tau_{tl}^\alpha - \int_0^t\left(\frac{ts^4\mu_{1l}^\alpha + ts^3\nu_{1l}^\alpha}{3}\right)ds|,\right.\\
&\qquad \left.|\mu_{1l}^\alpha - \tau_{tu}^\alpha - \int_0^t\left(\frac{ts^4\mu_{1u}^\alpha + ts^3\nu_{1u}^\alpha}{3}\right)ds|\right\}
\end{aligned}$$

$$\mathtt{D}(\nu_1, \mathfrak{S}(\nu_1,\mu_1)) = \sup_{\alpha\in[0,1]} \max\left\{ |\nu^\alpha_{1l} - \tau^\alpha_{tl} - \int_0^t \left(\frac{ts^4\nu^\alpha_{1l} + ts^3\mu^\alpha_{1l}}{3} \right) ds|, \right.$$
$$\left. |\nu^\alpha_{1l} - \tau^\alpha_{tu} - \int_0^t \left(\frac{ts^4\nu^\alpha_{1u} + ts^3\mu^\alpha_{1u}}{3} \right) ds| \right\}$$

$$\mathtt{D}(\mu_2, \mathfrak{S}(\mu_2,\nu_2)) = \sup_{\alpha\in[0,1]} \max\left\{ |\mu^\alpha_{2l} - \tau^\alpha_{tl} - \int_0^t \left(\frac{ts^4\mu^\alpha_{2l} + ts^3\nu^\alpha_{2l}}{3} \right) ds|, \right.$$
$$\left. |\mu^\alpha_{2l} - \tau^\alpha_{tu} - \int_0^t \left(\frac{ts^4\mu^\alpha_{2u} + ts^3\nu^\alpha_{2u}}{3} \right) ds| \right\}$$

$$\mathtt{D}(\nu_2, \mathfrak{S}(\nu_2,\mu_2)) = \sup_{\alpha\in[0,1]} \max\left\{ |\nu^\alpha_{2l} - \tau^\alpha_{tl} - \int_0^t \left(\frac{ts^4\nu^\alpha_{2l} + ts^3\mu^\alpha_{2l}}{3} \right) ds|, \right.$$
$$\left. |\nu^\alpha_{2l} - \tau^\alpha_{tu} - \int_0^t \left(\frac{ts^4\nu^\alpha_{2u} + ts^3\mu^\alpha_{2u}}{3} \right) ds| \right\}$$

Sequentially, if we let $\theta = \frac{1}{3}$, then it is easy to verify that the condition (10) *holds; and therefore by Theorem* 4.1*, we conclude that the system of fuzzy initial value problem has a solution.*

5. Application on fractional differential equations

Let $\varphi : [1, L] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function, $L > 1$, $I = (0, 1]$ and $I_0 = (0, 1)$. Let

$$^C D^\xi p(t) = \varphi(t, p(t), q(t)); \tag{11}$$
$$^C D^\xi q(t) = \varphi(t, q(t), p(t)) \tag{12}$$

be the system fractional order differential equations with integral boundary conditions

$$p(1) - q(1) = 0; \tag{13}$$
$$\alpha(p(L) - q(L)) = \beta(I^\omega(p(\eta) + q(\eta))), \tag{14}$$

where $^C D^\xi$ is the Caputo-Hadamard fractional derivative, $t \in [1, L]$, $\xi \in I_0$ and $\omega \in I$, $1 < \eta < L$ and $\alpha, \beta \in \mathbb{R}$.

Lemma 5.1. *Let $\tau, \kappa : [1, L] \to \mathbb{R}$ be continuous functions, then a pair $(p(t), q(t))$ is a solution of the system of fractional integral equations*

$$p(t) = I^\xi\tau(t) + \frac{\beta}{\Delta}\left(I^{\xi+\omega}(\tau(\eta) + \kappa(\eta))\right) - \frac{\alpha}{\Delta}\left(I^\xi(\tau(L) + \kappa(L))\right); \tag{15}$$
$$q(t) = I^\xi\kappa(t) + \frac{\beta}{\Delta}\left(I^{\xi+\omega}\tau(\eta) + \kappa(\eta)\right) - \frac{\alpha}{\Delta}\left(I^\xi(\tau(L) + \kappa(L))\right), \tag{16}$$

where $\xi \in I_0$ and $\omega \in I$, $1 < \eta < L$, $\alpha, \beta \in \mathbb{R}$ and

$$\Delta = 2\left(\alpha + \beta\frac{(\log\eta)^\omega}{\Gamma(\omega+1)}\right)$$

if and only if $(p(t), q(t))$ *is a solution of the fractional integral boundary value problem*

$$
\begin{aligned}
{}^{C}D^{\xi} p(t) &= \tau(t); &(17)\\
{}^{C}D^{\xi} q(t) &= \kappa(t), &(18)
\end{aligned}
$$

with the boundary conditions (13) *and* (14).

Proof. Using lemma 2.6, we can reduce the considered system of fractional differential equations as the following system of integral equations

$$
\begin{aligned}
p(t) &= I^{\xi}\tau(t) + c_1;\\
q(t) &= I^{\xi}\kappa(t) + c_2.
\end{aligned}
$$

In order to proof the inference of the lemma, we have to compute c_1 and c_2; from (13), it is evident that

$$c_1 = c_2.$$

Also from (14), we get

$$\alpha(I^{\xi}\tau(L) + c_1 + I^{\xi}\kappa(L) + c_2) = \beta\left(I^{\xi+\omega}(\tau(\eta) + \kappa(\eta)) + \frac{(\log\eta)^{\omega}}{\Gamma(\omega+1)}(c_1 + c_2)\right)$$

and therefore

$$c_1 = -\frac{\alpha}{\Delta}(I^{\xi}(\tau(L) + \kappa(L))) + \frac{\beta}{\Delta}(I^{\xi}(\tau(\eta) + \kappa(\eta))),$$

as desired. □

Theorem 5.2. *Let*

$$\zeta = \frac{(\log L)^{\xi}}{\Gamma(\xi+1)} + \frac{2|\beta|(\log\eta)^{\xi+\omega}}{|\Delta|\Gamma(\xi+\omega+1)} + \frac{2|\alpha|(\log L)^{\xi}}{|\Delta|\Gamma(\xi+1)},$$

where

$$\Delta = 2\left(\alpha + \beta\frac{(\log\eta)^{\omega}}{\Gamma(\alpha+1)}\right).$$

If the system of fractional differential equations (11) *and* (12) *satisfy the condition*

$$|\varphi(t, p(t), q(t)) - \varphi(t, r(t), s(t))| \leq \frac{\epsilon e^{-\gamma}}{\zeta}(|p(t) - r(t)| + |q(t) - s(t)|)$$

for all $t \in [1, L]$, $\gamma > 0$ *and* $\epsilon < 1$, *then the system has a solution.*

Proof. First let us set some notations for our convenience.

$$
\begin{aligned}
{}^{t}I^{\xi}_{\varphi(p,q)} &= I^{\xi}\varphi(t, p(t), q(t));\\
{}^{t}|\varphi|^{(p,q)}_{(r,s)} &= |\varphi(t, p(t), q(t)) - \varphi(t, r(t), s(t))|.
\end{aligned}
$$

From lemma 5.1, the solution of the system of fractional differential equations (11) and (12) added with the boundary conditions (13) and (14) is equal to the solution of the system of fractional integral equations given by

$$
\begin{aligned}
p(t) &= {}^{t}I^{\xi}_{\varphi(p,q)} + \frac{\beta}{\Delta}\left({}^{\eta}I^{\xi+\omega}_{\varphi(p,q)+\varphi(q,p)}\right) - \frac{\alpha}{\Delta}\left({}^{L}I^{\xi}_{\varphi(p,q)+\varphi(q,p)}\right); &(19)\\
q(t) &= {}^{t}I^{\xi}_{\varphi(q,p)} + \frac{\beta}{\Delta}\left({}^{\eta}I^{\xi+\omega}_{\varphi(p,q)+\varphi(q,p)}\right) - \frac{\alpha}{\Delta}\left({}^{L}I^{\xi}_{\varphi(p,q)+\varphi(q,p)}\right). &(20)
\end{aligned}
$$

Let $\mathcal{B} = C([1, L], \mathbb{R})$ be a complete metric space with the metric

$$\mathrm{d}(p(t), q(t)) = \sup_{t\in[1,L]} |p(t) - q(t)|.$$

Let $\mathfrak{B} : \mathcal{B}^2 \to \mathcal{B}$ be a function defined by

$$\mathfrak{B}(p(t), q(t)) = {}^{t}I^{\xi}_{\varphi(p,q)} + \frac{\beta}{\Delta}\left({}^{\eta}I^{\xi+\omega}_{\varphi(p,q)+\varphi(q,p)}\right) - \frac{\alpha}{\Delta}\left({}^{L}I^{\xi}_{\varphi(p,q)+\varphi(q,p)}\right).$$

In order to prove that the system of fractional integral equations (19) and (20) to possess a solution, it is enough to prove the existence of a coupled fixed point for the function $\mathfrak{B}$. More specifically, it is sufficient to show that the function $\mathfrak{B}$ satisfies the hypothesis of Corollary 3.10 for some $F \in \mathcal{F}$, $\psi \in \Psi$ and β_i, $i = 1$ to 5.

For, let $F(x) = \ln x$, $\psi(x) = \gamma$, $\beta_1 = \epsilon$ and $\beta_i = 0$, $i = 2$ to 5 and let $p(t), q(t), r(t)$ and $s(t)$ be elements in $\mathcal{B}$, then $|\mathfrak{B}(p(t), q(t)) - \mathfrak{B}(r(t), s(t))|$

$$\begin{aligned}
&\leq \left|{}^{t}I^{\xi}_{\varphi(p,q)-\varphi(r,s)} + \frac{\beta}{\Delta}\left({}^{\eta}I^{\xi+\omega}_{\varphi(p,q)-\varphi(r,s)+\varphi(q,p)-\varphi(s,r)}\right) - \frac{\alpha}{\Delta}\left({}^{L}I^{\xi}_{\varphi(p,q)-\varphi(r,s)+\varphi(q,p)-\varphi(s,r)}\right)\right| \\
&\leq \frac{1}{\Gamma(\xi)}\int_1^t \left(\log\frac{t}{u}\right)^{\xi-1} {}^{u}|\varphi|^{(p,q)}_{(r,s)}\frac{du}{u} \\
&\qquad + \frac{|\beta|}{|\Delta|\Gamma(\xi+\omega)}\int_1^{\eta}\left(\log\frac{\eta}{u}\right)^{\xi+\omega-1}\left({}^{u}|\varphi|^{(p,q)}_{(r,s)} + {}^{u}|\varphi|^{(q,p)}_{(s,r)}\right)\frac{du}{u} \\
&\qquad + \frac{|\alpha|}{|\Delta|\Gamma(\xi)}\int_1^{L}\left(\log\frac{L}{u}\right)^{\xi-1}\left({}^{u}|\varphi|^{(p,q)}_{(r,s)} + {}^{u}|\varphi|^{(q,p)}_{(s,r)}\right)\frac{du}{u} \\
&\leq \frac{1}{\Gamma(\xi)}{}^{t}|\varphi|^{(p,q)}_{(r,s)}\int_1^t \left(\log\frac{t}{u}\right)^{\xi-1}\frac{du}{u} \\
&\qquad + \frac{|\beta|}{|\Delta|\Gamma(\xi+\omega)}\left({}^{u}|\varphi|^{(p,q)}_{(r,s)} + {}^{t}|\varphi|^{(q,p)}_{(s,r)}\right)\int_1^{\eta}\left(\log\frac{\eta}{u}\right)^{\xi+\omega-1}\frac{du}{u} \\
&\qquad + \frac{|\alpha|}{|\Delta|\Gamma(\xi)}\left({}^{u}|\varphi|^{(r,s)}_{(p,q)} + {}^{u}|\varphi|^{(s,r)}_{(q,p)}\right)\int_1^{L}\left(\log\frac{L}{u}\right)^{\xi-1}\frac{du}{u} \\
&\leq \frac{\epsilon e^{-\gamma}}{2\zeta}(|p(t)-r(t)| + |q(t)-s(t)|)\left(\frac{(\log t)^{\xi}}{\Gamma(\xi+1)} + \frac{2|\beta|(\log\eta)^{\xi+\omega}}{|\Delta|\Gamma(\xi+\omega+1)} + \frac{2|\alpha|(\log L)^{\xi}}{|\Delta|\Gamma(\xi+1)}\right).
\end{aligned}$$

Therefore

$$\mathrm{d}(\mathfrak{B}(p(t), q(t)), \mathfrak{B}(r(t), s(t))) \leq \frac{\epsilon e^{-\gamma}}{2}(d(p(t), r(t)) + d(q(t), s(t))).$$

In a similar way, we can show that

$$\mathrm{d}(\mathfrak{B}(q(t), p(t)), \mathfrak{B}(s(t), r(t))) \leq \frac{\epsilon e^{-\gamma}}{2}(d(p(t), r(t)) + d(q(t), s(t))).$$

Consequently, by adding the above two inequalities, we get

$$\begin{aligned}
\epsilon e^{-\gamma}(d(p(t), r(t)) + d(q(t), s(t))) \geq\ & \mathrm{d}(\mathfrak{B}(p(t), q(t)), \mathfrak{B}(r(t), s(t))) \\
& + \mathrm{d}(\mathfrak{B}(q(t), p(t)), \mathfrak{B}(s(t), r(t))).
\end{aligned}$$

as desired. □

Example 5.3. *Let*

$$\begin{aligned}
{}^{c}D^{0.75}p(t) &= t^2 + \frac{t^3}{32}(p(t) + q(t)); \\
{}^{c}D^{0.5}q(t) &= t^2 + \frac{t^3}{32}(q(t) + p(t));
\end{aligned}$$

be the coupled system fractional order differential equations with integral boundary conditions

$$p(1) = q(1) = 0;$$

$$6(p(2) - q(2)) = -8(I^{0.5}(p(1.25) + q(1.25)));$$

Then we have

$$\begin{aligned}
&|\phi(t, p(t), q(t)) - \phi(t, r(t), s(t))| \\
&= |t^2 + \frac{t^3}{32}(p(t) + q(t)) - (t^2 + \frac{t^3}{32}(q(t) + p(t)))| \\
&\leq |\frac{t^3}{32}|(|p(t) - r(t)| + |q(t) - s(t)|) \\
&\leq \frac{1}{4}(|p(t) - r(t)| + |q(t) - s(t)|).
\end{aligned}$$

Calculating Δ and ζ, we get $\Delta = 11.9895$, $\zeta = 1.834484$. Consequently, if we let $\epsilon = 0.647568$, $\gamma = 0.345$, then we have

$$\frac{\epsilon e^{-\gamma}}{2\zeta} = 0.25 < 1$$

which asserts the existence of the solution for the system considered.

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