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Research Paper

Verification of Karci Algorithm's Efficiency for Maximum Independent Set Problem in Graph Theory

Ali Karci

İnönü University, Faculty of Engineering, Software Engineering Department, Malatya / Türkiye
(ali.karci@inonu.edu.tr; adresverme@gmail.com)

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Abstract— The maximum independent set problem is an NP-complete problem in graph theory. The Karci Algorithm is based on fundamental cut-sets of given graph, and node/nodes with minimum independence values are selected for maximum independent set. In this study, the analytical verification of this algorithm for some special graphs was analysed, and the obtained results were explained. The verification of Karci's Algorithm for maximum independent set was handled in partial.

Keywords : *Maximum Independent Set, Karci Algorithm, NP-Complete*

1. Introduction

The graph concept was introduced to scientific world for the first time due to the studies of Euler on Konigsberg bridge. The graphs are mathematical models to simulate the entities of and their relationships for solving engineering/scientific problems, and modelling computer networks, mathematical equations, object-oriented design, social networks, etc. A graph can be defined as in Definition 1.

Definition 1*: A graph G = (V,E) consists of a set V of vertices and a set E of edges. A graph, which does not consist of parallel and loop edges, called simple graph.*

The main focus of this study is to solve maximum independent set problem in graphs with efficient algorithms. The maximum independent set problem can be defined as in Definition 2.

Definition 2*: G=(V,E) is a simple graph and |V|=v, |E|=m. Assume that* $I\subset V$ *is a set of nodes and if* $\forall v_i, v_j \in I, (v_i, v_j) \notin E$*, I called independent. The set I of maximum size is called maximum independent set (MIS).*

Assume that G=(V,E) is a simple graph where V is a set of nodes (vertices) and E is a set of edges ($E\subseteq VxV$). A node v_i is said to be neighbour of v_j if $(v_i,v_j)\in E$. $I\subseteq V$, $\forall v_i,v_j\in I$, $v_i\notin N(v_j)$ where $N(v_j)$ is the set of nodes which are neighbours of v_j, I is called as independent set for graph G. Assume that $I_2\subseteq V$, $\forall v_i,v_j\in I_2$, $v_i\notin N(v_j)$, if there is no such $I\subset I_2$, I is called maximal independent set. In another word, independent set (stable set, co-clique, anticlique) is a set of nodes in the corresponding graph (so called G), no two of which are adjacent.

The MIS problem is an NP-hard problem, and there are many studies on this problem. Some of these studies can be given as follow in brief. The vertex support algorithm was proposed by Baraji and his/her friends for solving MIS problem (Baraji et al, 2010). Brandstadt and Mosca (Brandstadt and Mosca, 2018) used dynamic programming approach to show that the maximum weight independent set can be solved in polynomial time for claw-free graphs. Laflamme and his/her friends (Laflamme and et al, 2019) tried to show that Kn-free graph and minimal r=r(G,m) where $m\in N$, independent set meets at least m colour classes in a set of size |V| for any balanced r-colouring of the vertices of graph G. Lin et al obtained the number of independent sets and number of maximum independent set for path-tree bipartite graphs (Lin, 2018a), and Oh studied on the number of maximum independent sets for complete rectangular grid graph (Oh, 2017). Wan and his/her friends studied on independent sets and matchings of some special graphs (Wan et al, 2018). Another study is on bipartite permutation graph to obtain the independent sets, maximal independent sets and independent perfect dominating sets (Lin and Chen, 2017). Lin (Lin, 2018b) developed linear-time algorithms for counting independent sets and their two variants, and independent dominating sets, independent perfect dominating sets for distance-hereditary graphs. The intersections of maximum and critical independent sets of a graph concluded in König-Egevary graphs (Jarden et al, 2018). There are limitations on cardinality of independent sets for given graphs without isolated nodes (Sah et al, 2019). The cubic graph of girth at least 5 has got an upper bound on the number of independent sets which was studied by (Peramau and Perkins, 2018). The graph entropy was used to determine the number of independent sets and matchings (Wan et al, 2020).

An acyclic graph does not include cycle and a connected acyclic graph is called tree, otherwise it is called forest (forest is outside of scope of this study). A spanning tree of a connected graph G is a tree of having the all nodes of graph G (Definition 3).

Definition 3: *A spanning tree is a subset of graph G, which has all the vertices covered with minimum possible number of edges without cycle.*

There are recently published papers illustrate that new approaches exist to determine the maximum independent sets and dominating sets in given graphs based on special spanning trees of graphs and fundamental cut-sets corresponding to that special spanning tree of given graph. These studied were done by Karci for the first time (Karci and Karci, 2020; Karci, 2020a; Karci, 2020b; Karci, 2020c), the fundamental cut-sets of the given graph was used in any algorithm for the first time. Section 2 includes the details of Karci algorithms.

The motivation of this study is to verify that the proposed algorithm by Karci is optimal for special graphs such as their spanning trees are single ring with multiples chords without pendant nodes.

2. Karci Algorithm for Maximum Independent Set

In this study, we will prove that Karci's algorithm is to obtain maximum independent set for given graph, analytically. This algorithm (Karci and Karci, 2020; Karci, 2020a; Karci, 2020b; Karci, 2020c)) is based on a special spanning tree of given graph whose construction is based on breadth first search technique with exceptional. The cut-sets of given graph are used to find the minimum dominating sets and maximum independent sets by Karci for the first time (Karci and Karci, 2020; Karci, 2020a; Karci, 2020b; Karci, 2020c)). This tree is used to construct fundamental cut-sets.

Breadth-first search is a search technique in artificial intelligence for investigation of solution/goal. Breadth-first search consists of searching through a tree one level at a time, and then going to next down level for searching, and so on.

G=(V,E) is a given graph where $V=\{v_1,v_2,\ldots,v_n\}$. The set V is sorted with respect to the node degrees of nodes in V in ascending order. Any node with minimum degree (assume it is v_i) is selected as root node for spanning tree T of given graph G. The node in $N(v_i)$ are added to spanning tree T as children of v_i. The children of v_i are expanded from minimum remaining degree to maximum remaining degree. The obtained tree is called as **Kmin Tree** (Karci Minimum Spanning Tree). The remaining degree is the number of neighbours not included in tree yet, of a node in tree.

Algorithm 1 was developed to construct Kmin tree for given graph. In the case of equality of remaining degrees of nodes, the node near to root has got priority to be selected.

In algorithm 1, one of the minimum degree nodes in the graph is selected as the root node of the Kmin tree and its neighbour nodes are added to a queue (QL and NL are arrays of linked lists for neighbours of selected node, NL is the array of linked lists of neighbours of selected nodes), then the neighbour node degrees are reduced by one. Via Level_Wise_minimum, one of the nodes with the minimum remaining degree from the nodes in the queue is selected as the next node to be expanded, and selected nodes are deleted directly from the queue. If there are more than one node of minimum remaining degrees in the queue, the node selection is made according to queue order. However, if the tree levels of nodes with minimum remaining degrees with the same degrees are same, priority is given to the node near to root in the tree. Algorithm 1 gives two outputs such as AT and NL; AT is adjacency matrix of spanning tree Kmin, and NL is the array of linked lists constituted by using neighbours of nodes added to spanning tree as linked lists.

```
Algorithm 1: Kmin_Tree(G,A,AT,D) // output=AT, NL
Q←V
r←min(D) // D is degree matrix
while Q≠∅
    Q←Q-{r}
    Add(QL,Level, r,N(r)) //QL is array of linked list
    i←1,…,|N(r)|
        Make_List(NL,r,vi)
        A(r,vi) ←0, A(vi,r) ←0
        AT(r,vi) ←1, AT(vi,r) ←1
        Q←Q-{vi}
        ∀vj,vk∈N(r), A(vj,vk) ←0, A(vk,vj) ←0
        D←Compute(A), Level←Level+1
        r←Level_Wise_min(D,QL)
```

After the Kmin spanning tree is constructed, fundamental cut-sets must be obtained by using this spanning tree. Algorithm 2 is used to satisfy this aim. The neighbourhood in Algorithm 2 is determined by using AT matrix of spanning tree Kmin (T=(V,E1)).

```
Algorithm 2: Fundamental_Cut_Sets(G,AT,B,C) //Output=C
TD ← →∑ AT //row-wise summation
i←1,2,…,|V|
    V1←∅, V3←∅
    if TD(i)=1
        C(i,:) ←B(i,:)   //leaf cut-set
    else if AT(i,j)=1 and TD(i)>1 and TD(j)>1
        V1←V1∪{vi}
        V3←V3∪N(vi)
        while V3≠∅
            V1←V1∪{first(V3)}
            V3←V3-{first(V3)}
            V3←V3∪N(first(V3))
        V2←V-V1
        ∀vk,vj∈V, vj∈V1, vk∈V2, C(i,(vk,vj))← B(i,(vk,vj)) //internal cut-set
```

Algorithm 2 gives the cut-set matrix as C by using Kmin spanning tree. There are two types cut-sets such as leaf cut-sets and internal cut-sets.

```
Algorithm 3: Computing_Independence_Value(G,B,C,D)
I←∅, Gr←∅
while V≠(I∪Gr)
    E = →∑ B * C^T + →∑ D // E is a column vector.
    I ← I ∪ {vi |Ind(vi) is minimum in E} //corresponding cut is j.
    Gr←Gr∪N(vi)
    ∀vk∈V, and (vi,vk)∈E1, C(j,:)←0
```

The meaning of $\overrightarrow{\sum}$ is row-wise summation of corresponding matrix.

3. Verifying the Optimality of Karci Algorithm on Special Graphs

Assume that G=(V,E) is a graph without isolated and pendant nodes and the Kmin spanning tree of G with chords is a ring. Each row of E corresponds to a node and its value corresponds to the effectiveness of related node.

Theorem 1: *Assume that G=(V,E) is graph without pendant nodes. Karci Algorithm obtains maximum independent set in case of |V|=|E|.*

Proof: Assume that G=(V,E) is a graph without pendant node(s) and |V|=n, |E|=n. In this case, G is a ring and K_{min} is serial connected tree. There is only one chord. Figure 1 illustrates K_{min} of G and chord is illustrated on K_{min}.

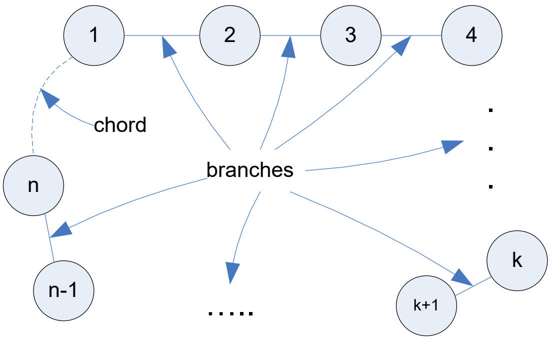

Figure 1. G=(V,E) is a ring and its corresponding K_{min} (chord is illustrated on K_{min}).

Assume that chord c=(1,n), and the remaining edges are included in K_{min}. B is the incidence matrix of G. There are n-1 fundamental cut-sets, and the corresponding cut-set matrix C has n-1 rows (each row is corresponding to a cut-set). Eq.1 illustrates the effectiveness of each node (the arrow on sigma letter demonstrates the row-wise summation).

$$E = \overset{\rightarrow}{\Sigma} \quad B * C^T + \overset{\rightarrow}{\Sigma} \quad D \tag{1}$$

Each row of E is illustrated as Ind(v_i) and this value is called as the effectiveness of corresponding node. Ind(2)= Ind (3)=…= Ind(n-1)=2. Ind(1)= Ind(n)=n. The independent set can be computed in two cases:

Case 1: n is even.

$$I = \left\{\frac{n}{2}, \frac{n}{2} - 2, \frac{n}{2} + 2, \frac{n}{2} - 4, \frac{n}{2} + 4, \dots, \frac{n}{2} - k, \frac{n}{2} + k \right\}$$

and

$$Gr = \left\{\frac{n}{2} - 1, \frac{n}{2} + 1, \frac{n}{2} - 3, \frac{n}{2} + 3, \frac{n}{2} - 5, \frac{n}{2} + 5, \dots, \frac{n}{2} - k - 1, \frac{n}{2} + k + 1 \right\}$$

If $\frac{n}{2}$ is odd, then $\frac{n}{2} + k + 1 = n \Longrightarrow k = \frac{n}{2} - 1$. So, |I|=n/2.

If $\frac{n}{2}$ is even, then $\frac{n}{2} + k = n \Longrightarrow k = \frac{n}{2}$. So, |I|=n/2.

Case 2: n is odd.

$$I = \left\{\left\lceil\frac{n}{2}\right\rceil, \left\lceil\frac{n}{2}\right\rceil - 2, \left\lceil\frac{n}{2}\right\rceil + 2, \left\lceil\frac{n}{2}\right\rceil - 4, \left\lceil\frac{n}{2}\right\rceil + 4, \dots, \left\lceil\frac{n}{2}\right\rceil - k, \left\lceil\frac{n}{2}\right\rceil + k \right\}$$

and

$$Gr = \left\{\left\lceil\frac{n}{2}\right\rceil - 1, \left\lceil\frac{n}{2}\right\rceil + 1, \left\lceil\frac{n}{2}\right\rceil - 3, \left\lceil\frac{n}{2}\right\rceil + 3, \left\lceil\frac{n}{2}\right\rceil - 5, \left\lceil\frac{n}{2}\right\rceil + 5, \dots, \left\lceil\frac{n}{2}\right\rceil - k - 1, \left\lceil\frac{n}{2}\right\rceil + k + 1 \right\}$$

If $\left\lceil\frac{n}{2}\right\rceil$ is odd, then $\left\lceil\frac{n}{2}\right\rceil + k = n \Longrightarrow k = \left\lfloor\frac{n}{2}\right\rfloor$. So, $|I| = \left\lfloor\frac{n}{2}\right\rfloor$.

If $\left\lceil\frac{n}{2}\right\rceil$ is even, then $\left\lceil\frac{n}{2}\right\rceil - k = 1 \Longrightarrow k = \left\lceil\frac{n}{2}\right\rceil - 1$. So, $|I| = \left\lceil\frac{n}{2}\right\rceil - 1$ ■

Theorem 2: *Assume that G=(V,E) is graph without pendant nodes. Karci Algorithm obtains maximum independent set in case of |V|+1=|E| (There are two chords of G on Kmin).*

Proof: In this case, G is a union of two rings with two nodes in common. One ring has got 3 nodes and the other has got n-1 nodes. In order to verify this claim, there will be more |V| cases. This theorem can be proved by using mathematical induction phenomena.

Case 1: The first ring with three nodes is related to K_3. The second ring contains n-1 nodes. The proof of Theorem 1 can be applied to the second ring, since the nodes in the first ring are neighbours. One of them can be selected to independent set (Fig.1 illustrates this case).

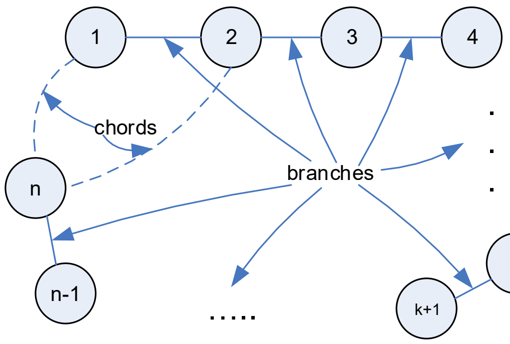

Figure 2: G=(V,E) and its corresponding K_{min} (chords are illustrated on K_{min}).

Case 2: Assume that |V|=n and there are two rings such as R_1 of size 4 and R_2 of size n-2. The verification step must be applied to R_2 at first, and assume that the two common nodes in R_1 and R_2 are v and u.

a) n-2 is even.

$$I_{R2}=\left\{\frac{n-2}{2},\frac{n-2}{2}-2,\frac{n-2}{2}+2,\frac{n-2}{2}-4,\frac{n-2}{2}+4,\ldots,\frac{n-2}{2}-k,\frac{n-2}{2}+k\right\}$$

and

$$Gr=\left\{\frac{n-2}{2}-1,\frac{n-2}{2}+1,\frac{n-2}{2}-3,\frac{n-2}{2}+3,\frac{n-2}{2}-5,\frac{n-2}{2}+5,\ldots,\frac{n-2}{2}-k-1,\frac{n-2}{2}+k+1\right\}$$

If $\frac{n-2}{2}$ is odd, then $\frac{n-2}{2}-k=1\Longrightarrow k=\frac{n-4}{2}$. One of v and u will be element of I, and one node of R_1 except u and v will be element of I. So, $|I|=\frac{n-2}{2}+1=\frac{n}{2}$.

If $\frac{n-2}{2}$ is even, then $\frac{n-2}{2}+k=n-2\Longrightarrow k=\frac{n-2}{2}$. So, $|I|=\frac{n}{2}$.

b) n-2 is odd, and one v and u will be element of independent set ($v\in I$ or $u\in I$, $u,v\notin I$).

$$I_{R2}=\left\{\left\lceil\frac{n-2}{2}\right\rceil,\left\lceil\frac{n-2}{2}\right\rceil-2,\left\lceil\frac{n-2}{2}\right\rceil+2,\left\lceil\frac{n-2}{2}\right\rceil-4,\left\lceil\frac{n-2}{2}\right\rceil+4,\ldots,\left\lceil\frac{n-2}{2}\right\rceil-k,\left\lceil\frac{n-2}{2}\right\rceil+k\right\}$$

for R_2.and Maximum independent set contains I_{R2} and one element of R_1 except u and v.

$$Gr=\left\{\left\lceil\frac{n-2}{2}\right\rceil-1,\left\lceil\frac{n-2}{2}\right\rceil+1,\left\lceil\frac{n-2}{2}\right\rceil-3,\left\lceil\frac{n-2}{2}\right\rceil+3,\left\lceil\frac{n-2}{2}\right\rceil-5,\left\lceil\frac{n-2}{2}\right\rceil+5,\ldots,\left\lceil\frac{n-2}{2}\right\rceil-k-1,\left\lceil\frac{n-2}{2}\right\rceil+k+1\right\}$$

If $\left\lceil\frac{n-2}{2}\right\rceil$ is odd, then $\left\lceil\frac{n-2}{2}\right\rceil+k=n-2\Longrightarrow k=n-2-\left\lceil\frac{n-2}{2}\right\rceil-\frac{1}{2}=\frac{n-3}{2}$. So, $|I|=\left\lfloor\frac{n}{2}\right\rfloor$.

If $\left\lceil\frac{n-2}{2}\right\rceil$ is even, then $\left\lceil\frac{n-2}{2}\right\rceil+k+1=n-2\Longrightarrow k=n-3-\frac{n-2}{2}-\frac{1}{2}=\frac{n-5}{2}$. So, $|I|=\left\lfloor\frac{n}{2}\right\rfloor$.

Case 3: Assume that |V|=n and there are two rings such as R_1 of size n-k+1 and R_2 of size k+1. The verification step must be applied to R_2 at first, and assume that the two common nodes in R_1 and R_2 are v=1 and u=k+1. Fig.3, depicts this case. The verification process takes place for rings R_1 and R_2 as the situation used in the first two steps in this theorem.

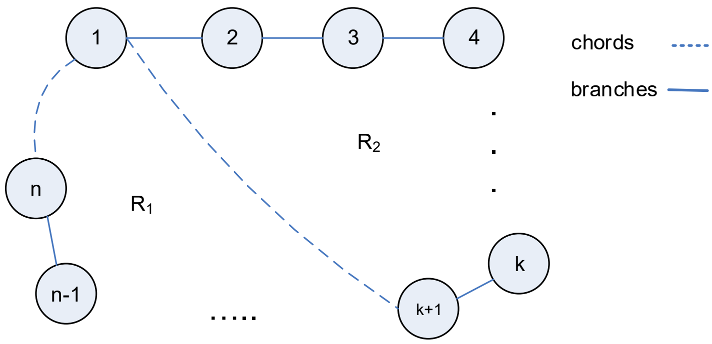

Figure 3: Kmin with two chords and there are two rings such as R_1 and R_2 with common nodes 1, k+1.

Theorem 3: *Assume that G=(V,E) is graph without pendant nodes. Karci Algorithm obtains maximum independent set in case of |V|+k=|E| (There are three chords of G on Kmin).*

Proof: Assume that |V|=n and there are three. If Kmin is a serial connected graph, the following cases can be taken in consideration.

a) n is even.

$$I = \left\{\frac{n}{2}, \frac{n}{2} - 2, \frac{n}{2} + 2, \frac{n}{2} - 4, \frac{n}{2} + 4, \ldots, \frac{n}{2} - k, \frac{n}{2} + k\right\}$$

and

$$Gr = \left\{\frac{n}{2} - 1, \frac{n}{2} + 1, \frac{n}{2} - 3, \frac{n}{2} + 3, \frac{n}{2} - 5, \frac{n}{2} + 5, \ldots, \frac{n}{2} - k - 1, \frac{n}{2} + k + 1\right\}$$

If $\frac{n}{2}$ is odd, then $\frac{n}{2} - k = 1 \Rightarrow k = \frac{n}{2} - 1$. One of v and u will be element of I, and one node of R_1 except u and v will be element of I. So, $|I| = \frac{n}{2}$.
If $\frac{n}{2}$ is even, then $\frac{n}{2} + k = n \Rightarrow k = \frac{n}{2}$. So, $|I| = \frac{n}{2}$.

b) n is odd.

$$I = \left\{\left\lceil\frac{n}{2}\right\rceil, \left\lceil\frac{n}{2}\right\rceil - 2, \left\lceil\frac{n}{2}\right\rceil + 2, \left\lceil\frac{n}{2}\right\rceil - 4, \left\lceil\frac{n}{2}\right\rceil + 4, \ldots, \left\lceil\frac{n}{2}\right\rceil - k, \left\lceil\frac{n}{2}\right\rceil + k\right\}$$

and

$$Gr = \left\{\left\lceil\frac{n}{2}\right\rceil - 1, \left\lceil\frac{n}{2}\right\rceil + 1, \left\lceil\frac{n}{2}\right\rceil - 3, \left\lceil\frac{n}{2}\right\rceil + 3, \left\lceil\frac{n}{2}\right\rceil - 5, \left\lceil\frac{n}{2}\right\rceil + 5, \ldots, \left\lceil\frac{n}{2}\right\rceil - k - 1, \left\lceil\frac{n}{2}\right\rceil + k + 1\right\}$$

If $\left\lceil\frac{n}{2}\right\rceil$ is odd, then $\left\lceil\frac{n}{2}\right\rceil + k = n \Rightarrow k = \left\lceil\frac{n}{2}\right\rceil$. So, $|I| = \left\lceil\frac{n}{2}\right\rceil$.
If $\left\lceil\frac{n}{2}\right\rceil$ is even, then $\left\lceil\frac{n}{2}\right\rceil - k = 1 \Rightarrow k = \left\lceil\frac{n}{2}\right\rceil - 1$. So, $|I| = \left\lceil\frac{n}{2}\right\rceil - 1$.

The proof of this theorem illustrated that if Kmin is a serial connected graph, the results of Karci Algorithm is same. Kmin is serial connected graph ■

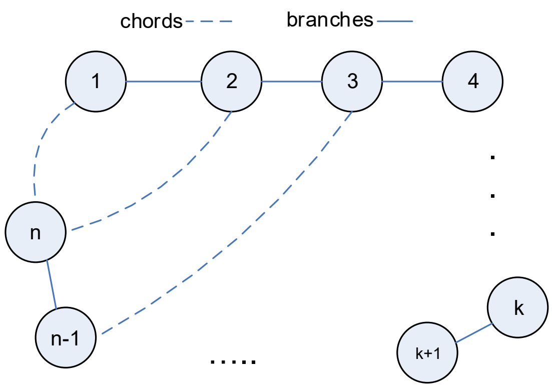

Figure 4: G=(V,E) and its corresponding K_{min} (chords are illustrated on K_{min}).

Theorem 4: *Assume that G=(V,E) is a simple graph where |V|=n,* $|E| = \frac{n(n-1)}{2} - 1$. *Karci Algorithm obtains maximum independent set.*

Proof: The proof was handled based on graph seen in Fig.5 and $(v_1,v_n)\notin E$. Assume that G=(V,E) where |V|=n and |E|== $\frac{n(n-1)}{2} - 1$. The independence value of each node is denoted as Ind(v). So,

$Ind(v_1)$=n-2+n-2=2n-4
$Ind(v_3)$=…=$Ind(v_{n-1})$=n-1+n-4+1+2+n-1=3n-3
$Ind(v_2)$=n-3+n-2+1+n-1=3n-5
$Ind(v_n)$=n-2+n-3+n-3+n-2=4n-10.

The node v_1 has minimum independence value, and so, Maximum Independent Set I={v_1}, and N(v_1)={v_2,v_3,v_4,…,v_{n-1}}. Removing v_1 with its neighbours from graph, v_n will be a pendant node. Thus, I={v_1,v_n} ■

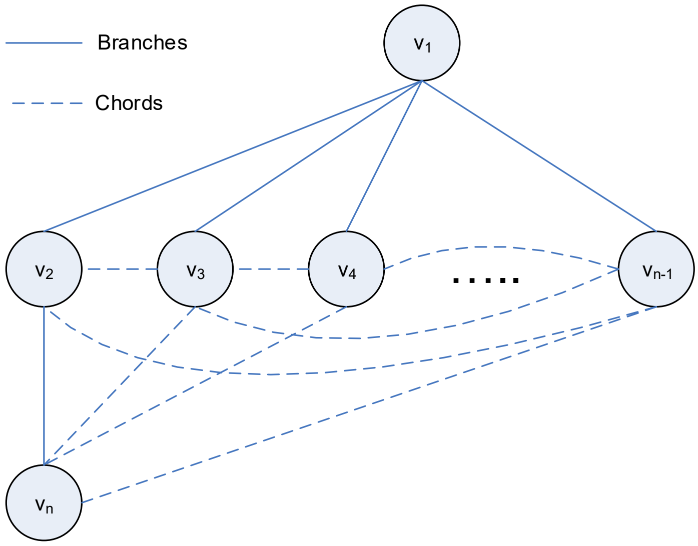

Figure 5: K_{min} of G=(V,E) is a simple graph where |V|=n, $|E| = \frac{n(n-1)}{2} - 1$.

Theorem 5: *Assume that G=(V,E) is a simple graph where |V|=n,* $|E| = \frac{n(n-1)}{2} - 2$*. Karci Algorithm obtains maximum independent set.*

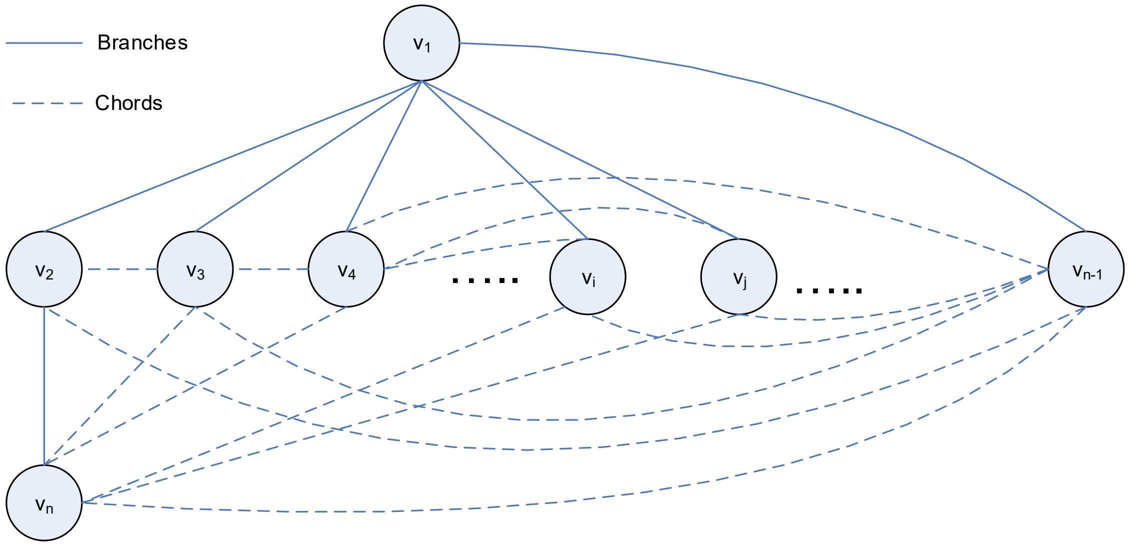

Figure 6: K_{min} of G=(V,E) is a simple graph where |V|=n, $|E| = \frac{n(n-1)}{2} - 2$.

Proof: The proof was handled based on graph seen in Fig.6 and $(v_1,v_n),(v_i,v_j)\notin E$. Assume that G=(V,E) where |V|=n and |E|== $\frac{n(n-1)}{2} - 2$. The independence value of each node is denoted as Ind(v). So,

$Ind(v_1)$=n-2+n-2=2n-4
If $v_k \neq v_1,v_2,v_i,v_j,v_n$, then
$Ind(v_k)$=n-1+n-1+n-4+1+2=3n-3
$Ind(v_2)$=n-3+n-2+n-1+1=3n-5
$Ind(v_i)=Ind(v_j)$=n-2+n-2+n-4+2=3n-8
$Ind(v_n)$=n-2+n-3+n-3+n-2=4n-10.

The node v_1 has minimum independence value, and so, Maximum Independent Set I={v_1}, and $N(v_1)=\{v_2,v_3,v_4,\ldots,v_{n-1}\}$. Removing v_1 with its neighbours from graph, v_n will be an isolated node. Thus, I={v_1,v_n}■

Theorem 6: *Assume that G=(V,E) is a simple graph where |V|=n,* $|E| = \frac{n(n-1)}{2} - 2$ *and* $N(v_1)$=n-3. *Karci Algorithm obtains maximum independent set.*

Proof: The proof was handled based on graph seen in Fig.7 and $(v_1,v_n),(v_1,v_{n-1})\notin E$. Assume that G=(V,E) where |V|=n and |E|= $= \frac{n(n-1)}{2} - 2$. The independence value of each node is denoted as Ind(v). So,

$Ind(v_1)$=n-3+n-3=2n-6
If $v_k \neq v_1,v_2,v_{n-1},v_n$, then
$Ind(v_k)$=n-1+n-1+n-3+2=3n-3
$Ind(v_2)$=n-3+n-1+n-4+2=3n-6
$Ind(v_n)=Ind(v_{n-1})$=n-2+n-2+n-4+1+n-4+n-4=5n-15.

The node v_1 has minimum independence value, and so, Maximum Independent Set I={v_1}, and $N(v_1)=\{v_2,v_3,v_4,\ldots,v_{n-2}\}$. Removing v_1 with its neighbours from graph, v_n will be a pendant node. Thus, I={v_1,v_n} or I={v_1,v_{n-1}} ■

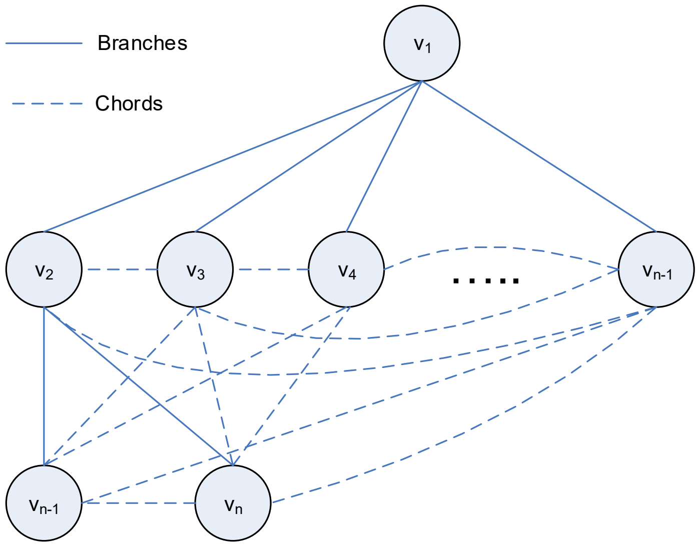

Figure 7: K_{min} of G=(V,E) is a simple graph where |V|=n, $|E| = \frac{n(n-1)}{2} - 2$, $N(v_1)$=n-3.

Theorem 7: Assume that G=(V,E) is connected graph without pendant nodes. Karci Algorithm obtains maximum independent set in case of not serial connected Kmin tree.

Proof: Assume that G=(V,E) is a connected graph such as |V|=n and |E|=m, and corresponding K_{min} tree is T. The minimum degree in G is δ(G) and maximum degree in G is Δ(G). The independence values are illustrated as $Ind(v_i)$ and the minimum effectiveness value is $Ind(v_i)=2\delta(G)$. There are n-1 fundamental cut-sets. One of the node with minimum independence value is selected for independent set firstly. So, the remaining cut-sets number is n-1- δ(T) and the remaining node number is n-1-δ(T). If n-1-δ(T)>0, the node selection process will take place again.

The remaining independence values satisfy the following inequality.

$$\frac{\big(n-1-\delta(T)\big)\delta(G)}{n-1} \leq \forall v_i \in V, Ind(v_i) \leq \frac{(n-1-\delta(T))(\Delta(G)+\delta(G))}{n-1}$$

The selected node with its neighbours in T are removed from K_{min}, and after that independence values are computed with respect to the modified K_{min} tree, the node selection process will take place with respect to the following equation (Assume that the maximum independent set is I and the selected node in the first step is v_1, I={v_1}, $\mathcal{N}=N(v_1)$).

$I=I\cup\{v_i \mid \min(Ind(v_i)), \forall v_i\in V\}$ and $\mathcal{N}=\mathcal{N}\cup N(v_i)$.

v_i is also removed from K_{min} tree with incident edges. This process carries on until $I \cup \mathcal{N} = V$. At each step the node with minimum independence value is selected, so, it has minimum node in G=(V,E)

4. Conclusions

Karci's Algorithm for maximum independent set is a polynomial algorithm, and so, its time complexity will be a polynomial not exponential. The proofs of algorithm to obtain the maximum independent set for given dense/sparse graphs were obtained in this study. The obtained results are analytical results, not just computational results. Due to this case, this study was regarded as partial proof not complete proof.

References

Karci, A.,"New Algorithms for Minimum Dominating Set in Any Graphs", Anatolian Science, Journal of Computer Science, Vol:5, Issue:2, pp:62-70, 2020a.

Karci, A.," Finding Innovative and Efficient Solutions to NP-Hard and NP-Complete Problems in Graph Theory", Anatolian Science, Journal of Computer Science, Vol:5, Issue:2, pp:137-143, 2020b.

Karci, A.," Efficient Algorithms for Determining the Maximum Independent Sets in Graphs", Anatolian Science, Journal of Computer Science, Vol:5, Issue:2, pp:144-149, 2020c.

Karci, A., Karci, Ş.,"Determination of Effective Nodes in Graphs", International Conference on Science, Engineering & Technology, Mecca, Saudi Arabia, pp:25-28, 2020.

Karci, Ş., Ari, A., Karci, A.," Pençesiz çizgelerde maksimum-yakın bağımsız küme ve üst sınırları için yeni algoritma", Journal of the Faculty of Engineering and Architecture of Gazi University (accepted).

Baraji, S., Swaminathan, V., Kannan, K.,"A simple algorithm to optimize maximum independent set", Advanced Modeling and Optimization, vol:12, Issue:1, pp:107-118, 2010.

Brandstadt, A., Mosca, R., "Maximum weight independent set for Lclaw-free graphs in polynomial time", Discrete Applied Mathematics, Vol:237, pp:57-64, 2018.

Lin, M.-S.,"Counting independent sets and maximal independent sets in some subclasses of bipartite graphs", Discrete Applied Mathematics, Vol:251, pp:236-244, 2018a.

Laflamme, C., Aranda, A., Soukup, D.T., Woodrow, R.,"Balanced independent sets in graphs omitting large cliques", Journal of Combinatorial Theory, Series B, Vol:137, pp:1-9, 2019.

Oh, S., "Maximal independent sets on a grid graph", Discrete Mathematics, Vol:340, pp:2762-2768, 2017.

Wan, P., Tu, J., Zhang, S., Li, B., "Computing the numbers of independent sets and matchings of all sizes for graphs with bounded treewidth", Applied Mathematics and Computation, Vol:332, pp:42-47, 2018.

Lin, M.-S., Chen, C.-M.,"Linear-time algorithms for counting independent sets in bipartite permutation graphs", Information Processing Letters, Vol:122, pp:1-7, 2017.

Lin. M.-S., "Simple linear-time algorithms for counting independent sets in distance-hereditary graphs", Discrete Applied Mathematics, Vol: 239, pp:144-153, 2018b.

Sah, A., Sawhhney, M., Stoner, D., Zhao, Y.,"The number of independent sets in an irregular graph", Journal of Combinatorial Theory, Series B, Vol:138, pp:172-195, 2019.

Jarden, A., Levit, V.E., Mandrescu, E.,"Critical and maximum independent sets of a graph", Discrete Applied Mathematics, Vol: 247, pp:127-134, 2018.

Peramau, G., Perkins, W.,"Counting independent sets in cubic graphs of given girth", Journal of Combinatorial Theory, Series B, Vol:133, pp:2018.

Wan, P., Chen, X., Tu, J., Dehmer, M., Zhang, S., Emmert-Streib, F.,"On graph entropy measures based on the number of independent sets and matchings", Information Sciences, Vol:516, pp:491-504, 2020.