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Research Article

# Coupled fixed point results on orthogonal metric spaces with application to nonlinear integral equations

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**Abstract**

In this article, we prove some well-known coupled fixed point theorems in O-complete metric spaces. Also, we present some corollaries related to our study. In addition to this, we give an example showing that our results successfully obtain the existence and uniqueness of the coupled fixed point for O-complete metric spaces, but the results are not valid for complete metric spaces. Finally, we apply our results to examine the existence and uniqueness of a solution of the system of nonlinear integral equations.

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**Keywords.** orthogonal set, coupled fixed point, integral equations

## 1. Introduction

In 1922, Banach [7] stated a fixed point theorem, which is considered a notable results in the history of mathematics, and inspired many other important works. Banach's fixed point theorem has generalizations, enrichements and modifications in a wide variety of forms. Some of them are coupled fixed point theorems that are interesting and difficult to prove. The history of concept of coupled fixed point date back to 1980's. It was introduced by Guo and Lakshmikantham [12]. Afterwards, Bhaskar and Lakshmikantham [8] introduced the notion of the mixed monotone property and prove some coupled fixed point theorems. Since these theorems have important applications in many fields of mathematics, they attracted many authors attention. So, many researchers restated these theorems on different metric spaces as bipolar, modular, partial, cone, e.g. [1, 3, 9, 10, 13–15, 17–20, 23, 24, 26, 27] and many others.

On the other hand, the recently the concept of orthogonal sets (briefly, O-sets) was introduced by Gordji et al. [11]. And they proved Banach's fixed point theorem in that study. In addition this, they discussed the existence of solution of differential equation using their results. For find more details about O-sets and orthogonal metric spaces, the readers are referred to [2, 4–6, 16, 21, 22, 25].

The aim of this article is to present some theorems and corollaries which show existence and uniqueness coupled fixed point in O-complete metric spaces. And, the existence and

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uniqueness of a solution of the system of nonlinear integral equations is investigated as an application of this study.

## 2. Preliminaries

$\mathbb{N}$ and $\mathbb{R}$ denote the set of positive integers and real numbers, respectively.

**Definition 2.1** ([11]). Let $X$ is a nonempty set and $\perp \subseteq X \times X$ be a binary relation. If the relation $\perp$ satisfies the following condition:

$$\exists x_0 \in X : \ (\forall y, \ y \perp x_0) \text{ or } (\forall y, \ x_0 \perp y)$$

then $X$ is called an orthogonal set (briefly, O-set) and $x_0$ is called an orthogonal element. We represent this O-set by $(X, \perp)$.

**Example 2.2** ([11]). Let $X = \mathbb{Z}$. We define $m \perp n$ if there exists $k \in \mathbb{Z}$ such that $m = kn$. It is obvious that $0 \perp n$ for all $n \in \mathbb{Z}$. So, $(X, \perp)$ is an O-set.

**Example 2.3** ([11]). Let $X = [0, \infty)$. We define $x \perp y$ if $xy \in \{x, y\}$. For orthogonal elements $x_0 = 0$ or $x_0 = 1$, $(X, \perp)$ is an O-set.

As seen in the above example, $x_0$ is not necessarily unique.

**Definition 2.4** ([11]). Let $(X, \perp)$ be O-set. A sequence $\{x_i\}_{i \in \mathbb{N}}$ is called an orthogonal sequence (briefly, O-sequence) if

$$(\forall i, x_i \perp x_{i+1}) \text{ or } (\forall i, x_{i+1} \perp x_i).$$

**Definition 2.5** ([11]). The triplet $(X, \perp, d)$ is called orthogonal metric space if $(X, \perp)$ is an O-set and $(X, d)$ is a metric space.

**Definition 2.6** ([11]). Let $(X, \perp, d)$ be an orthogonal metric space. The mapping $f : X \to X$ is called orthogonally continuous (or $\perp$-continuous) in $x \in X$ if we get $f(x_i) \to f(a)$ for each O-sequence $\{x_i\}_{i \in \mathbb{N}}$ in $X$ with $x_i \to x$ as $i \to \infty$. And, the mapping $f$ is called $\perp$-continuous on $X$ if $f$ is $\perp$-continuous for all $x \in X$.

**Definition 2.7** ([11]). Let $(X, \perp, d)$ be an orthogonal metric space. $X$ is called orthogonally complete (briefly, O-complete) if every Cauchy O-sequence is convergent.

**Remark 2.8** ([11]). Every complete metric space is O-complete and the converse is not true.

From the Definition 3.10. in [11], the following definition can be written.

**Definition 2.9.** Let $(X, \perp)$ be an O-set. A mappings $S : X \times X \to X$ is said to be $\perp$-preserving if $x \perp a$ and $y \perp b$ implies $S(x, y) \perp S(a, b)$.

## 3. Main results

In this section, we investigate some coupled fixed point results in orthogonal metric spaces. These results extend and generalize some new and old well-known coupled fixed point results.

**Theorem 3.1.** *Let $(X, \perp, d)$ be an O-complete metric space (not necessarily complete metric space) and $S : X \times X \to X$ be $\perp$-preserving mapping. If the condition*

$$d(S(x, y), S(a, b)) \leq kd(x, a) + ld(y, b) \tag{3.1}$$

*holds for all $x, y, a, b \in X$ with $x \perp a$ and $y \perp b$ where $k, l \geq 0$ and $k + l < 1$, then $S$ has a unique coupled fixed point.*

***Proof.*** From the definition of orthogonality, we can say that there exist orthogonal elements $x_0, y_0 \in X$ such that

$$(x_0 \perp y \quad \text{for all} \quad y \in X) \quad \text{or} \quad (y \perp x_0 \quad \text{for all} \quad y \in X)$$

and

$$(y_0 \perp y \quad \text{for all} \quad y \in X) \quad \text{or} \quad (y \perp y_0 \quad \text{for all} \quad y \in X).$$

So, we get

$$x_0 \perp S(x_0, y_0) \text{ or } S(x_0, y_0) \perp x_0$$

and

$$y_0 \perp S(y_0, x_0) \text{ or } S(y_0, x_0) \perp y_0$$

for $x_0, y_0 \in X$. We set

$$\begin{aligned} x_1 = S(x_0, y_0) \quad &\text{and} \quad y_1 = S(y_0, x_0) \\ x_2 = S(x_1, y_1) \quad &\text{and} \quad y_2 = S(y_1, x_1) \\ &\vdots \\ x_{i+1} = S(x_i, y_i) \quad &\text{and} \quad y_{i+1} = S(y_i, x_i) \end{aligned}$$

for $i \in \mathbb{N}$. Hence, we get

$$x_0 \perp S(x_0, y_0) = x_1 \quad \text{or} \quad x_1 = S(x_0, y_0) \perp x_0$$

and

$$y_0 \perp S(y_0, x_0) = y_1 \quad \text{or} \quad y_1 = S(y_0, x_0) \perp y_0.$$

Since $S$ is $\perp$-preserving, we have

$$x_1 = S(x_0, y_0) \perp S(x_1, y_1) = x_2 \quad \text{or} \quad x_2 = S(x_1, y_1) \perp S(x_0, y_0) = x_1$$

and

$$y_1 = S(y_0, x_0) \perp S(y_1, x_1) = y_2 \quad \text{or} \quad y_2 = S(y_1, x_1) \perp S(y_0, x_0) = y_1.$$

If we continue in the same way, we get

$$x_i \perp x_{i+1} \quad \text{or} \quad x_{i+1} \perp x_i$$

and

$$y_i \perp y_{i+1} \quad \text{or} \quad y_{i+1} \perp y_i$$

for all $i \in \mathbb{N}$. So, $\{x_i\}_{i \in \mathbb{N}}$ and $\{y_i\}_{i \in \mathbb{N}}$ are O-sequences. Now, we want to see that $\{x_i\}_{i \in \mathbb{N}}$ and $\{y_i\}_{i \in \mathbb{N}}$ are Cauchy O-sequences. From (3.1), we get

$$\begin{aligned} d(x_i, x_{i+1}) &= d(S(x_{i-1}, y_{i-1}), S(x_i, y_i)) \\ &\leq kd(x_{i-1}, x_i) + ld(y_{i-1}, y_i) \end{aligned} \tag{3.2}$$

and similarly

$$\begin{aligned} d(y_i, y_{i+1}) &= d(S(y_{i-1}, x_{i-1}), S(y_i, x_i)) \\ &\leq kd(y_{i-1}, y_i) + ld(x_{i-1}, x_i) \end{aligned} \tag{3.3}$$

for all $i \in \mathbb{N}$ and $k + l < 1$. Let $d_i = d(x_i, x_{i+1}) + d(y_i, y_{i+1})$. From (3.2) and (3.3), we get

$$\begin{aligned} d_i &= d(x_i, x_{i+1}) + d(y_i, y_{i+1}) \\ &\leq kd(x_{i-1}, x_i) + ld(y_{i-1}, y_i) + kd(y_{i-1}, y_i) + ld(x_{i-1}, x_i) \\ &= (k + l)(d(x_{i-1}, x_i) + d(y_{i-1}, y_i)) \\ &= (k + l)d_{i-1} \end{aligned}$$

for all $i \in \mathbb{N}$. Repeating this argument, we have

$$0 \leq d_i \leq (k + l)d_{i-1} \leq (k + l)^2 d_{i-2} \leq \cdots \leq (k + l)^i d_0 \tag{3.4}$$

for all $i \in \mathbb{N}$. Let $d_0 = 0$. That is, $d_0 = d(x_0, x_1) + d(y_0, y_1) = 0$. So, we get

$$d(x_0, x_1) = 0 \Rightarrow x_0 = x_1 = S(x_0, y_0)$$

and

$$d(y_0, y_1) = 0 \Rightarrow y_0 = y_1 = S(y_0, x_0).$$

These imply that $S$ has coupled fixed point $(x_0, y_0)$. Let $d_0 > 0$. Then we have

$$d(x_i, x_j) \leq d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) + \cdots + d(x_{j-1}, x_j) \tag{3.5}$$

and

$$d(y_i, y_j) \leq d(y_i, y_{i+1}) + d(y_{i+1}, y_{i+2}) + \cdots + d(y_{j-1}, y_j) \tag{3.6}$$

for any positive integer $j$ and $i$ with $i \leq j$. From (3.4), (3.5) and (3.6), we have

$$\begin{aligned} d(x_i, x_j) + d(y_i, y_j) &\leq d(x_i, x_{i+1}) + d(y_i, y_{i+1}) + d(x_{i+1}, x_{i+2}) + d(y_{i+1}, y_{i+2}) + \\ &\quad \cdots + d(x_{j-1}, x_j) + d(y_{j-1}, y_j) \\ &= d_i + d_{i+1} + \cdots + d_{j-1} \\ &\leq [(k+l)^i + (k+l)^{i+1} + \cdots + (k+l)^{j-1}]d_0 \\ &\leq \frac{(k+l)^i}{1-(k+l)} d_0 \end{aligned}$$

for $i \leq j$. If we take limit as $i, j \to \infty$, since $\frac{(k+l)}{1-(k+l)} < 1$, we can say that $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ are Cauchy O-sequences in $X$. Since $(X, \perp, d)$ is an O-complete metric space, there exists $u, v \in X$ such that $x_i \to u$, $y_i \to v$. Then, there exists $i_0 \in \mathbb{N}$ with

$$d(x_i, u) < \frac{\epsilon}{2} \quad \text{and} \quad d(y_i, v) < \frac{\epsilon}{2} \tag{3.7}$$

for all $i \geq i_0$ and every $\epsilon > 0$. By choice of $u$ and $v$, we have $u \perp x_i$ or $x_i \perp u$ and $v \perp y_i$ or $y_i \perp v$. So, from (3.1) and (3.7), we have

$$\begin{aligned} d(S(u, v), u) &\leq d(S(u, v), x_{i+1}) + d(x_{i+1}, u) \\ &= d(S(u, v), S(x_i, y_i)) + d(x_{i+1}, u) \\ &\leq kd(x_i, u) + ld(y_i, v) + d(x_{i+1}, u) \\ &< (k+l)\frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon \end{aligned}$$

for $k + l < 1$. It follows that $d(S(u, v), u) = 0$ and so $S(u, v) = u$. Similarly, we can show that $S(v, u) = v$. Then $S$ has a coupled fixed point $(u, v)$.

To see the uniqueness of coupled fixed point of $S$, we take another coupled fixed point $(u^*, v^*) \in X \times X$. That is, $S(u^*, v^*) = u^*$ and $S(v^*, u^*) = v^*$.

(i) If $u \perp u^*$ or $u^* \perp u$ and $v \perp v^*$ or $v^* \perp v$, from (3.1), we get

$$d(u, u^*) = d(S(u, v), S(u^*, v^*)) \leq kd(u, u^*) + ld(v, v^*)$$

$$d(v, v^*) = d(S(v, u), S(v^*, u^*)) \leq kd(v, v^*) + ld(u, u^*)$$

and therefore

$$d(u, u^*) + d(v, v^*) \leq (k+l)(d(u, u^*) + d(v, v^*)).$$

Since $k + l < 1$, we get $d(u, u^*) + d(v, v^*) = 0$ and so $u = u^*$, $v = v^*$.

(ii) If not, for the chosen orthogonal elements $x_0, y_0 \in X$ in the first of proof, we have

$$(x_0 \perp u, \quad x_0 \perp u^*) \quad \text{or} \quad (u \perp x_0, \quad u^* \perp x_0)$$

and

$$(y_0 \perp v, \quad y_0 \perp v^*) \quad \text{or} \quad (v \perp y_0, \quad v^* \perp y_0).$$

Therefore, from (3.1), we get

$$\begin{aligned} d(u,u^*) &= d((S(u,v),S(u^*,v^*)) \\ &\leq d(S(u,v),S(x_0,y_0)) + d(S(x_0,y_0),S(u^*,v^*)) \\ &\leq kd(x_0,u) + ld(y_0,v) + kd(x_0,u^*) + ld(y_0,v^*). \end{aligned}$$

As $i \to \infty$, we obtain that $d(u,u^*) = 0$. So, we have $u = u^*$. Similarly, we obtain that

$$\begin{aligned} d(v,v^*) &= d((S(v,u),S(v^*,u^*)) \\ &\leq d(S(v,u),S(y_0,x_0)) + d(S(y_0,x_0),S(v^*,u^*)) \\ &\leq kd(y_0,v) + ld(x_0,u) + kd(y_0,v^*) + ld(x_0,u^*). \end{aligned}$$

As $i \to \infty$, we get $d(v,v^*) = 0$. So, we have $v = v^*$.

This meaning that $(u,v) = (u^*,v^*)$. Then, $S$ has a unique coupled fixed point in $X$. □

The corollary that can be easily obtained by taking equal the constants in Theorem 3.1 is given below.

**Corollary 3.2.** *Let $(X,\perp,d)$ be an O-complete metric space (not necessarily complete metric space) and $S : X \times X \to X$ be $\perp$-preserving mapping. If the condition*

$$d(S(x,y),S(a,b)) \leq \frac{k}{2}(d(x,a) + d(y,b)) \tag{3.8}$$

*holds for all $x,y,a,b \in X$ with $x\perp a$ and $y\perp b$ where $0 \leq k < 1$, then $S$ has a unique coupled fixed point.*

The following theorem is coupled fixed point theorem of generalized Kannan type mapping in orthogonal metric spaces.

**Theorem 3.3.** *Let $(X,\perp,d)$ be an O-complete metric space (not necessarily complete metric space) and $S : X \times X \to X$ be $\perp$-preserving mapping. If the condition*

$$d(S(x,y),S(a,b)) \leq kd(S(x,y),x) + ld(S(a,b),a) \tag{3.9}$$

*holds for all $x,y,a,b \in X$ with $x\perp a$ and $y\perp b$ where $k,l \geq 0$ and $k + l < 1$, then $S$ has a unique coupled fixed point.*

***Proof.*** We consider O-sequences $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ which have the same properties in the proof of Theorem 3.1. Then we say that $x_{i+1} = S(x_i,y_i)$, $y_{i+1} = S(y_i,x_i)$ and

$$x_i \perp x_{i+1} \quad \text{or} \quad x_{i+1} \perp x_i,$$

$$y_i \perp y_{i+1} \quad \text{or} \quad y_{i+1} \perp y_i$$

for all $i \in \mathbb{N}$. Let $\frac{k}{1-l} = \alpha$ and $\frac{1}{1-k} = \beta$. From (3.9), we have

$$\begin{aligned} d(x_i,x_{i+1}) &= d(S(x_{i-1},y_{i-1}),S(x_i,y_i)) \\ &\leq d(S(x_{i-1},y_{i-1}),x_{i-1}) + ld(S(x_i,y_i),x_i) \\ &= kd(x_i,x_{i-1}) + ld(x_{i+1},x_i). \end{aligned}$$

Then we get

$$d(x_i,x_{i+1}) \leq \alpha d(x_{i-1},x_i)$$

with $\alpha < 1$. Repeating this process, we get

$$d(x_i,x_{i+1}) \leq \alpha^i d(x_0,x_1)$$

for all $i \in \mathbb{N}$. Then we obtain that

$$\begin{aligned} d(x_i, x_j) &\leq d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) + \cdots + d(x_{j-1}, x_j) \\ &\leq [\alpha^i + \alpha^{i+1} + \cdots + \alpha^{j-1}] d(x_0, x_1) \\ &\leq \frac{\alpha^i}{1-\alpha} d(x_0, x_1) \end{aligned}$$

for any positive integer $i$ and $j$ with $i \leq j$. If we take limit as $i, j \to \infty$, since $\alpha < 1$, then $\{x_i\}_{i\in\mathbb{N}}$ is a Cauchy O-sequences. Similarly, we easily show that $\{y_i\}_{i\in\mathbb{N}}$ is a Cauchy O-sequences in $X$. Since $(X, \perp, d)$ is an O-complete metric space, there exists $u, v \in X$ such that $x_i \to u$, $y_i \to v$. By choice of $u$ and $v$, we have $u \perp x_i$ or $x_i \perp u$ and $v \perp y_i$ or $y_i \perp v$. So, from (3.9), we get

$$\begin{aligned} d(S(u,v), u) &\leq d(S(u,v), x_{i+1}) + d(x_{i+1}, u) \\ &= d(S(u,v), S(x_i, y_i)) + d(x_{i+1}, u) \\ &\leq kd(S(u,v), u) + ld(S(x_i, y_i), x_i) + d(x_{i+1}, u) \end{aligned}$$

which implies

$$\begin{aligned} d(S(u,v), u) &\leq \alpha d(x_{i+1}, x_i) + \beta d(x_{i+1}, u) \\ &\leq \alpha(d(x_{i+1}, u) + d(u, x_i)) + \beta d(x_{i+1}, u) \end{aligned}$$

for $\alpha < 1$ and $\beta < 1$. Letting $i \to \infty$, then we get $d(S(u,v), u) = 0$ and so $S(u,v) = u$. Similarly, we obtain that $S(v,u) = v$. Then $(u,v)$ is a coupled fixed point of $S$.

Now, we see the uniqueness of coupled fixed point of $S$. We take another coupled fixed point $(u^*, v^*) \in X \times X$. That is, $S(u^*, v^*) = u^*$ and $S(v^*, u^*) = v^*$.

(i) If $u \perp u^*$ or $u^* \perp u$ and $v \perp v^*$ or $v^* \perp v$, from (3.9), we get

$$\begin{aligned} d(u, u^*) &= d(S(u,v), S(u^*, v^*)) \\ &\leq kd(S(u,v), u) + ld(S(u^*, v^*), u^*) \\ &= kd(u,u) + ld(u^*, u^*) \\ &= 0. \end{aligned}$$

Hence we get $u = u^*$. Similarly, we get $v = v^*$.

(ii) If not, for the chosen orthogonal elements $x_0, y_0 \in X$ in the first of proof, we get

$$(x_0 \perp u, \quad x_0 \perp u^*) \quad \text{or} \quad (u \perp x_0, \quad u^* \perp x_0)$$

and

$$(y_0 \perp v, \quad y_0 \perp v^*) \quad \text{or} \quad (v \perp y_0, \quad v^* \perp y_0).$$

Therefore, from (3.9), we get

$$\begin{aligned} d(u, u^*) &= d((S(u,v), S(u^*, v^*)) \\ &\leq d(S(u,v), S(x_0, y_0)) + d(S(x_0, y_0), S(u^*, v^*)) \\ &\leq kd(S(u,v), u) + ld(S(x_0, y_0), x_0) + kd(S(x_0, y_0), x_0) + ld(S(u^*, v^*), u^*) \\ &= (k+l) d(x_1, x_0) \\ &\leq (k+l)(d(x_1, u) + d(u, x_0)). \end{aligned}$$

If we take limit as $i \to \infty$, we get $d(u, u^*) = 0$. Hence we have $u = u^*$. Similarly, we get $v = v^*$.

This meaning that $(u,v) = (u^*, v^*)$. Therefore, $S$ has a unique coupled fixed point in $X$. □

The corollary that can be easily obtained by taking equal the constants in Theorem 3.3 is given below.

**Corollary 3.4.** *Let $(X, \perp, d)$ be an O-complete metric space (not necessarily complete metric space) and $S : X \times X \to X$ be $\perp$-preserving mapping. If the condition*

$$d(S(x,y), S(a,b)) \leq \frac{k}{2}\left(d(S(x,y),x) + d(S(a,b),a)\right)$$

*holds for all $x, y, a, b \in X$ with $x \perp a$ and $y \perp b$ where $0 \leq k < 1$, then $S$ has a unique coupled fixed point.*

The following theorem is coupled fixed point theorem of generalized Chatterjea type mapping in orthogonal metric spaces.

**Theorem 3.5.** *Let $(X, \perp, d)$ be an O-complete metric space (not necessarily complete metric space) and $S : X \times X \to X$ be $\perp$-preserving mapping. If the condition*

$$d(S(x,y), S(a,b)) \leq kd(S(x,y),a) + ld(S(a,b),x) \tag{3.10}$$

*holds for all $x, y, a, b \in X$ with $x \perp a$ and $y \perp b$ where $k, l \geq 0$ and $k + l < 1$, then $S$ has a unique coupled fixed point.*

***Proof.*** We choose the O-sequences $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ like in the proof Theorem 3.1. Then we say that $x_{i+1} = S(x_i, y_i)$, $y_{i+1} = S(y_i, x_i)$ and

$$x_i \perp x_{i+1} \quad \text{or} \quad x_{i+1} \perp x_i,$$

$$y_i \perp y_{i+1} \quad \text{or} \quad y_{i+1} \perp y_i$$

for all $i \in \mathbb{N}$. From (3.10), we have

$$\begin{aligned} d(x_i, x_{i+1}) &= d(S(x_{i-1}, y_{i-1}), S(x_i, y_i)) \\ &\leq d(S(x_{i-1}, y_{i-1}), x_i) + ld(S(x_i, y_i), x_{i-1}) \\ &= kd(x_i, x_i) + ld(x_{i+1}, x_{i-1}) \\ &\leq ld(x_{i+1}, x_i) + ld(x_i, x_{i-1}). \end{aligned}$$

This implies that

$$d(x_i, x_{i+1}) \leq \frac{l}{1-l} d(x_{i-1}, x_i)$$

with $\frac{l}{1-l} < 1$. Then, the proof continues similarly to the proof of Theorem 3.3. Thus, $\{x_i\}_{i\in\mathbb{N}}$ is a Cauchy 0-sequence. From O-completeness of $X$, there exists $u, v \in X$ such that $x_i \to u$, $y_i \to v$. By choice of $u$ and $v$, we get $u \perp x_i$ or $x_i \perp v$ and $v \perp y_i$ or $y_i \perp v$. So, from (3.10), we get

$$\begin{aligned} d(S(u,v),u) &\leq d(S(u,v), x_{i+1}) + d(x_{i+1}, u) \\ &= d(S(u,v), S(x_i, y_i)) + d(x_{i+1}, u) \\ &\leq kd(S(u,v), x_i) + ld(S(x_i, y_i), u) + d(x_{i+1}, u) \\ &\leq kd(S(u,v), u) + kd(u, x_i) + ld(S(x_i, y_i), u) + (l+1)d(x_{i+1}, u). \end{aligned}$$

If we take limit as $i \to \infty$, then we get

$$d(S(u,v),u) \leq kd(S(u,v),u).$$

Since $k < 1$, it follows that $d(S(u,v),u) = 0 \Rightarrow S(u,v) = u$. Similarly, we can show that $S(v,u) = v$. Then $(u,v)$ is a coupled fixed point of $S$. The proof of the uniqueness of coupled fixed point can be easily obtained similarly to the other results. Then, $S$ has a unique coupled fixed point in $X$.

□

The corollary that can be easily obtained by taking equal the constants in Theorem 3.5 is given below.

**Corollary 3.6.** *Let $(X, \perp, d)$ be an O-complete metric space (not necessarily complete metric space) and $S : X \times X \to X$ be $\perp$-preserving mapping. If the condition*

$$d(S(x,y), S(a,b)) \leq \frac{k}{2}(d(S(x,y), a) + d(S(a,b), x))$$

*holds for all $x, y, a, b \in X$ with $x \perp a$ and $y \perp b$ where $0 \leq k < 1$, then $S$ has a unique coupled fixed point.*

**Example 3.7.** Let $X = \{0, 1, 2, \cdots\}$ and define $x \perp y$ if $0 < y - x$. So, $(X, \perp)$ is an O-set. We consider Euclidian metric $d$ on $X$. $(X, \perp, d)$ is an O-complete metric space. Let $S : X \times X \to X$ be a mapping defined by

$$S(x,y) = \begin{cases} \frac{x+y}{3}, & x < y \\ 0, & otherwise \end{cases}$$

for $x, y \in X$. It is obvious that $S$ is $\perp$-preserving on $X$. Let $x \perp a$ and $y \perp b$. We consider the following four cases:

Case 1: If $x < y$ and $a < b$, then $S(x,y) = \frac{x+y}{3}$ and $S(a,b) = \frac{a+b}{3}$ for all $x, y, a, b \in X$.

Case 2: If $x < y$ and $a \geq b$, then $S(x,y) = \frac{x+y}{3}$ and $S(a,b) = 0$ for all $x, y, a, b \in X$.

Case 3: If $x > y$ and $a < b$, then $S(x,y) = 0$ and $S(a,b) = \frac{a+b}{3}$ for all $x, y, a, b \in X$.

Case 4: If $x > y$ and $a \geq b$, then $S(x,y) = 0$ and $S(a,b) = 0$ for all $x, y, a, b \in X$.

For these four cases, the condition

$$|S(x,y) - S(a,b)| \leq \frac{k}{2}(|x-a| + |y-b|). \tag{3.11}$$

is satisfied for $0 \leq k < 1$ and all $x, y, a, b \in X$. From Corollary 3.2, $S$ has a unique fixed point $(0,0)$. If $(X, \perp)$ is not 0-set, then the condition (3.11) is not satisfied. To show this, we take four point such as $x = 1$, $y = 2$, $a = 1$ and $b = 0$. For each $0 \leq k < 1$, we get

$$|S(1,2) - S(1,0)| = 1 > \frac{k}{2}(|1-2| + |1-0|) = k.$$

On the otherhand, in this example, if we take the mapping $S : X \times X \to X$ as $S(x,y) = \frac{x+y}{2}$ for 0-set $X$, then the condition

$$|S(x,y) - S(a,b)| \leq \frac{1}{2}(|x-a| + |y-b|).$$

holds for $k = 1$. So, $(0,0)$ and $(1,1)$ are two coupled fixed points of $S$. This meaning that the coupled fixed point of $S$ is not unique. In this case, conditions $k < 1$ and $k + 1$ in Corollary 3.2 and Theorem 3.1, respectively, are the most favorable conditions to ensure the uniqueness of the coupled fixed point.

## 4. Application to nonlinear integral equations

In this section, using Theorem 3.1, we show that there exists a unique solution of the following system of the integral equations

$$\begin{aligned} x(t) &= \int_0^T f(t, x(s), y(s))ds \\ y(t) &= \int_0^T f(t, y(s), x(s))ds \end{aligned} \tag{4.1}$$

where $T > 0$, $t \in [0,T]$ and $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The class of $\mathbb{R}$-valued continuous functions on the interval $[0,T]$ is denoted by $C([0,T],\mathbb{R})$.

**Theorem 4.1.** *Let $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a mapping. We suppose that the following conditions hold:*

(i) *$f$ is a continuous mapping,*

(ii) *there exist $k, l \geq 0$ with $k + l < 1$ such that*

$$0 \leq f(t,a,b) - f(t,x,y) \leq \frac{1}{T}(k(a-x) + l(b-y))$$

*for all $x, y, a, b \in \mathbb{R}$, $x, y, a, b \geq 0$ with $a - x \geq 0$, $b - y \geq 0$ and for all $t \in [0,T]$.*

*Then the system of integral equations* (4.1) *has a unique solution.*

***Proof.*** $X = \{x \in C([0,T],\mathbb{R}) : x(t) \geq 0, \ \forall t \in [0,T]\}$. We consider the orthogonality relationship in $X$ by

$$x \perp y \Leftrightarrow y(t) - x(t) \geq 0, \ \forall t \in [0,T].$$

We take an arbitrary $t$ and define

$$d(x,y) = \sup_{t \in [0,T]} |x(t) - y(t)|$$

for all $x, y \in X$. We can easily say that $(X,d)$ is a metric space. We want to show the 0-completeness of $X$. We consider a Cauchy O-sequence $\{x_i\}_{i \in \mathbb{N}} \subseteq X$. It is easily say that $\{x_i\}_{i \in \mathbb{N}}$ is convergent to a point $u \in C([0,T],\mathbb{R})$. Then, we show that $u \in X$. We take arbitrary $t \in [0,T]$. From definition of $\perp$, we can say that $x_i \perp x_{i+1}$ for each $i \in \mathbb{N}$. Since $x_i(t) \geq 0$ for all $i \in \mathbb{N}$, this sequence converges to $u(t)$. This implies that $u(t) \geq 0$. Since $t \in [0,T]$ is arbitrary, $u \geq 0$ and so $u \in X$. Now, we define a mapping $S : X \times X \to X$ by

$$S(x,y)(t) = \int_0^T f(t, x(s), y(s))ds$$

for each $t \in [0,T]$, $x, y \in X$. The fixed point of $S$ is the solution of (4.1). Firstly, we obtain that $S$ is $\perp$-preserving. For all $x, y, a, b \in X$ with $x \perp a$, $y \perp b$ and $t \in [0,T]$, from (ii), we get

$$0 \leq f(t, a(s), b(s)) - f(t, x(s), y(s))$$

which implies

$$f(t, x(s), y(s)) \leq f(t, a(s), b(s)).$$

So, we get

$$\begin{aligned} S(x,y)(t) &= \int_0^T f(t, x(s), y(s))ds \\ &\leq \int_0^T f(t, a(s), b(s))ds \\ &= S(a,b)(t). \end{aligned}$$

It follows that $S(a,b)(t) - S(x,y)(t) \geq 0$. So, we get $S(x,y) \perp S(a,b)$. From condition (ii), for all $x, y, a, b \in X$ with $x \perp a$, $y \perp b$ and $t \in [0,T]$, we get

$$\begin{aligned} |S(a,b)(t) - S(x,y)(t)| &= \left| \int_0^T f(t,a(s),b(s))ds - \int_0^T f(t,x(s),y(s))ds \right| \\ &= \int_0^T |f(t,a(s),b(s)) - f(t,x(s),y(s))|\, ds \\ &\leq \frac{1}{T} \int_0^T (k|a(s) - x(s)| + l|b(s) - y(s)|)ds \\ &\leq \frac{1}{T} \int_0^T (k \sup_{r\in[0,T]} |a(r) - x(r)| + l \sup_{r\in[0,T]} |b(r) - y(r)|)ds \\ &= k \sup_{r\in[0,T]} |a(r) - x(r)| + l \sup_{r\in[0,T]} |b(r) - y(r)|. \end{aligned}$$

This meaning that

$$\sup_{r\in[0,T]} |S(a,b)(t) - S(x,y)(t)| \leq k \sup_{r\in[0,T]} |a(r) - x(r)| + l \sup_{r\in[0,T]} |b(r) - y(r)|.$$

Then, for $x \perp a$, $y \perp b$ and $k + l < 1$, we get

$$d(S(x,y), S(a,b)) \leq kd(x,a) + ld(y,b).$$

Therefore, from Theorem 3.1, (4.1) has a unique solution. □

## 5. Conclusions

In this paper, some coupled fixed point theorems, which extend and generalize new and old well-known coupled fixed point results, are obtained in orthogonal metric spaces and some related results are given. Also, an application in the system of nonlinear integral equations is presented, which demonstrate the validity of the hypotheses and degree of utility of the proposed results for orthogonal metric spaces.

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