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# On The Projective Planes In Projective Space PG(4,4)

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**Abstract:** In this study, firstly, it is shown that the geometric structure, which is the image under the projection and the Veronesean map of the projective plane over the field GF(4) is the projective plane of order 4 such that the set of points spans PG(4,4) and the lines set consists of conics in PG(4,4) by introducing with coordinates. Then taking points span PG(4,4), the projective plane of order 4 is determined such that every line of the plane is a conic plane in PG(4,4).

# PG(4,4) Projektif Uzayındaki Projektif Düzlemler Üzerine

**Anahtar kelimeler**
Projektif Uzay,
Projektif Düzlem,
Veronesean Dönüşüm,
Konik

**Öz:** Bu çalışmada önce GF(4) cismi üzerindeki projektif düzlemin Veronesean dönüşüm ve izdüşüm altında görüntüsü olan geometrik yapının, noktalar kümesi PG(4,4) projektif uzayını geren noktalardan ve doğrular kümesi PG(4,4) uzayındaki koniklerden oluşan, 4. mertebeden projektif düzlem olduğu koordinatlarının sunumu ile verilmektedir. Daha sonra PG(4,4) uzayını geren noktalar kümesi alınarak, doğruları konik düzlemleri olan 4. mertebeden projektif düzlem koordinatları belirlenmektedir.

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## 1. Introduction

The n-dimensional projective space over the Galois field $GF(q)$ is denoted by $PG(n,q)$. The Veronesean map maps the point of $PG(n,q)$ with coordinates $(x_0, x_1, \dots, x_n)$ onto the point of $PG\left(\frac{n(n+3)}{2}, q\right)$ with coordinates $(x_0^2, x_1^2, \dots, x_n^2, x_0x_1, \dots, x_0x_n, \dots, x_1x_n, \dots, x_{n-1}x_n)$. The image of the Veronesean map is called the quadric Veronesean. The quadric Veronesean has the geometric and combinatorial properties. Many researchers have studied using the quadric Veroneseans in classical algebraic and finite geometry. A characterization of the Veronese spaces by means of their maximal subspaces was presented in [7]. In [6], the combinatorial characterization of Veronesean in *PG(n,q)* was given. In [2], Veronese varieties of degree $d$ over a Galois field were studied and it was shown that the projections of Veronese varieties were some of known caps. Authors in [8] showed that all Veronesean caps in finite projective spaces are projections of quadric Veroneseans and got the stronger result by weaking one of the conditions in [6] . In [9], they presented that the classification all embeddings $\theta: PG(n,q) \to PG(d,q)$, with $d \geq \frac{n(n+3)}{2}$, such that $\theta$ maps the points on each line to a coplanar points and such that the image of $\theta$ generates $PG(d,q)$, gave the characterization on quadric Veroneseans and union of projections and introduced the notation of a generalized Veronesean embedding. In [1], the classification all embeddings $\theta: PG(n,\mathcal{K}) \to PG(d,\mathcal{F})$, with $d \geq \frac{n(n+3)}{2}$ and $\mathcal{K},\mathcal{F}$ skew field, such that $\theta$ maps the points on each line in $PG(d,\mathcal{K})$ to a coplanar points in $PG(d,\mathcal{F})$, and such that the image of $\theta$ generates $PG(d,\mathcal{F})$ is given and to relax the conditions in [9] is studied.

In present study firstly it is introduced that the image under the projection and the Veronesean map of $PG(2,4)$ is the projective plane of order 4 in 4-dimensional projective space. Then, when the points spanning $PG(4,4)$are taken in three conic planes, the projective plane of order 4 is determined such that every line of the plane is a conic plane in $PG(4,4)$. This is handled with introducing coordinates in $PG(2,4)$ and in $PG(4,4)$ and the computations are done with coordinates based on the field $GF(4)$.

## 2. Material and Method

In sequel, the basic definitions and theorems required for this study are summarized as the following:

### 2.1 Projective Plane

**Definition 2.1.1.** A projective plane $\mathcal{P}$ is an incidence structure $(\mathcal{N},\mathcal{D},\circ)$ where $\mathcal{N}$ is a set whose elements are called points, $\mathcal{D}$ is a set whose elements are called lines and $\circ \subset \mathcal{N} \times \mathcal{D}$ is an incidence relation such that the following axioms are satisfied:
P1) Any two distinct points are on a unique line,
P2) Any two distinct lines are intersect at a unique point,
P3) There are four points in $\mathcal{P}$ such that no three points of them are collinear, [5].

**Theorem 2.1.2.** The positive integer $n$ that satisfies the following properties is called the order of the finite projective plane $\mathcal{P}$.
i) There are $n + 1$ points on every line in $\mathcal{P}$
ii) There are $n + 1$ lines passing through each point in $\mathcal{P}$
iii) There exist $n^2 + n + 1$ points in $\mathcal{P}$,
iv) There exist $n^2 + n + 1$ lines in $\mathcal{P}$, [5].

**Theorem 2.1.3.** Let $\mathcal{F}$ be any field. A point-line geometry is a triple $(\mathcal{N},\mathcal{D},\circ)$ consisting of the points set $\mathcal{N}$, the lines set $\mathcal{D}$ determined algebraically with the elements of the field $\mathcal{F}$ and the incidence relation $\circ$ . Obviously,

$\mathcal{N} = \{(x_1,x_2,x_3): x_i \epsilon \mathcal{F}, (x_1,x_2,x_3) \neq (0,0,0), (x_1,x_2,x_3) \equiv \lambda(x_1,x_2,x_3), \lambda \epsilon\, \mathcal{F} - \{0\}\}$
$\mathcal{D} = \{[a_1,a_2,a_3]: a_i \epsilon \mathcal{F}, [a_1,a_2,a_3] \neq [0,0,0], [a_1,a_2,a_3] \equiv \mu[a_1,a_2,a_3], \mu \epsilon\, \mathcal{F} - \{0\}\}$,
$\circ: (x_1,x_2,x_3) \circ [a_1,a_2,a_3] \Leftrightarrow a_1x_1 + a_2x_2 + a_3x_3 = 0.$

Any point in $\mathcal{N}$ is represented by a triple $(x_1,x_2,x_3)$ where $x_1,x_2,x_3$ are not all zero. Nonzero multiples of a triple represent the same point. $\mathcal{D}$ has the same properties. This point-line geometry $(\mathcal{N},\mathcal{D},\circ)$ defined by $\mathcal{F}$, is a projective plane and is denoted by $P_2\mathcal{F}$. Let $r$ and $p$ be a positive integer and a prime number, respectively. The projective plane of order $n = p^r$ over the finite Galois field $\mathcal{F} = GF(p^r)$ of $p^r$ elements is denoted by $P_2\mathcal{F} = PG(2,p^r)$ , [5].
A polynomial $p(x) = x^2 + x + 1$ over the field $\mathcal{F} = GF(2)$ is irreducible. The field $GF(4) = GF(2^2) = \{0,1,t,t^2\}$ is the extension of the field $GF(2)$. The set of points in the projective plane $PG(2,4)$ over $GF(4)$ is $\mathcal{N} = \{N_0,N_1,\dots,N_{20}\}$, where

$N_0 = (0,1,0), N_1 = (0,0,1), N_2 = (0,1,1), N_3 = (0,1,t^2), N_4 = (0,1,t), N_5 = (1,1,1), N_6 = (1,0,1), N_7 = (1,t,1),$
$N_8 = (1,t^2,1), N_9 = (1,1,0), N_{10} = (1,1,t^2), N_{11} = (1,1,t), N_{12} = (1,t,t^2), N_{13} = (1,t^2,t), N_{14} = (1,0,0), N_{15} = (1,t,0),$
$N_{16} = (1,t^2,0), N_{17} = (1,0,t^2), N_{18} = (1,0,t), N_{19} = (1,t^2,t^2), N_{20} = (1,t,t).$

The lines set of $PG(2,4)$ is $\mathcal{D} = \{D_0,D_1,\dots,D_{20}\}$, where

$D_0 = [1,0,0], D_1 = [1,0,1], D_2 = [0,0,1], D_3 = [1,0,t], D_4 = [1,0,t^2], D_5 = [1,1,0], D_6 = [0,1,0], D_7 = [1,t,0],$
$D_8 = [1,t^2,0], D_9 = [0,1,1], D_{10} = [1,t^2,t], D_{11} = [1,t,t^2], D_{12} = [1,1,1], D_{13} = [1,t,t], D_{14} = [1,t^2,t^2], D_{15} = [1,t^2,1],$
$D_{16} = [1,t,1], D_{17} = [1,1,t^2], D_{18} = [1,1,t], D_{19} = [0,1,t^2], D_{20} = [0,1,t].$

Incidence relation " $\circ$" is as in table 2.1.

**Table 2.1**. Incidence Relation of *PG(2,4)*

| $D_0$ | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ | $D_6$ | $D_7$ | $D_8$ | $D_9$ | $D_{10}$ | $D_{11}$ | $D_{12}$ | $D_{13}$ | $D_{14}$ | $D_{15}$ | $D_{16}$ | $D_{17}$ | $D_{18}$ | $D_{19}$ | $D_{20}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $N_0$ | $N_0$ | $N_0$ | $N_0$ | $N_0$ | $N_1$ | $N_1$ | $N_1$ | $N_1$ | $N_2$ | $N_4$ | $N_3$ | $N_2$ | $N_2$ | $N_2$ | $N_3$ | $N_4$ | $N_4$ | $N_3$ | $N_4$ | $N_3$ |
| $N_1$ | $N_5$ | $N_9$ | $N_{10}$ | $N_{11}$ | $N_5$ | $N_6$ | $N_8$ | $N_7$ | $N_5$ | $N_5$ | $N_5$ | $N_6$ | $N_7$ | $N_8$ | $N_6$ | $N_6$ | $N_7$ | $N_8$ | $N_8$ | $N_7$ |
| $N_2$ | $N_6$ | $N_{14}$ | $N_{12}$ | $N_{13}$ | $N_9$ | $N_{14}$ | $N_{13}$ | $N_{12}$ | $N_{14}$ | $N_{13}$ | $N_{12}$ | $N_9$ | $N_{11}$ | $N_{10}$ | $N_{11}$ | $N_{10}$ | $N_9$ | $N_9$ | $N_{11}$ | $N_{10}$ |
| $N_3$ | $N_7$ | $N_{15}$ | $N_{17}$ | $N_{18}$ | $N_{10}$ | $N_{17}$ | $N_{16}$ | $N_{15}$ | $N_{19}$ | $N_{15}$ | $N_{16}$ | $N_{12}$ | $N_{16}$ | $N_{15}$ | $N_{15}$ | $N_{16}$ | $N_{18}$ | $N_{17}$ | $N_{12}$ | $N_{13}$ |
| $N_4$ | $N_8$ | $N_{16}$ | $N_{19}$ | $N_{20}$ | $N_{11}$ | $N_{18}$ | $N_{19}$ | $N_{20}$ | $N_{20}$ | $N_{17}$ | $N_{18}$ | $N_{13}$ | $N_{17}$ | $N_{18}$ | $N_{19}$ | $N_{20}$ | $N_{19}$ | $N_{20}$ | $N_{14}$ | $N_{14}$ |

## 2.2 The Projective Spaces

Let $V$ be an n+1-dimensional vector space over the field $\mathcal{F}$. For any two nonzero vectors A = $(a_0, a_1, \dots, a_n)$ and B = $(b_0, b_1, \dots, b_n)$ in $V$, A~B $\Leftrightarrow$ $a_i = \lambda b_i$ , i = 1,2, … , n, $\lambda \epsilon \mathcal{F}$ and $\lambda \neq 0$, is the equivalence relation on the nonzero vectors in $V$. The equivalence classes are the 1-dimensional subspaces of the vector space with the origin removed. The set of them is called n-dimensional projective space over $\mathcal{F}$ and is denoted by $PG(n, \mathcal{F})$. The elements of $PG(n, \mathcal{F})$ are called points of projective space. If $\mathcal{F} = \mathrm{GF}(q)$, the projective space of order q is shown by $PG(n, q)$, [3].

A k-space (k-dimensional subspace) in $PG(n, \mathcal{F})$ is a set of points all of whose corresponding to vectors determine a $k+1$-dimensional subspace in $V$. i-dimensional subspaces i=0,1,2 are called a point, a line, a plane, respectively.

A k-space $\pi_k$ is the set of points represented by the vector $t_0X_0 + t_1X_1 + \cdots + t_kX_k$ , where $X_0, X_1, \dots, X_k$ are $k+1$ linearly independent vectors and $t_i \epsilon \mathcal{F} - \{0\}$.

## 2.3 The Veronesean Map

The Veronesean map is a map

$$\vartheta: PG(n, \mathcal{F}) \to PG\left(\frac{n(n+3)}{2}, \mathcal{F}\right), n \geq 1,$$

$$P(x_0, x_1, \dots, x_n) \to (y_{00}, y_{11}, \dots, y_{n-1,n}), y_{ij} = x_i x_j, 0 \leq i, j \leq n$$

$\vartheta$ maps the set of points of each line of $PG(n, \mathcal{F})$ to a set of coplanar points of $PG(\frac{n(n+3)}{2}, \mathcal{F})$ and such that the image of $\vartheta$ generates $PG(\frac{n(n+3)}{2}, \mathcal{F})$. The Veronesean map maps a point $P(x_0, x_1, \dots, x_n)$ of $PG(n, \mathcal{F})$ onto the point $P(x_0^2, \dots, x_n^2, x_0x_1, \dots, x_0x_n, \dots, x_{n-1}x_n)$ of $PG(\frac{n(n+3)}{2}, \mathcal{F})$. The set of these points of $PG(\frac{n(n+3)}{2}, \mathcal{F})$ is called the quadric Veronesean and is denoted by $\mathcal{V}_n^{2^n}$, or, for short $\mathcal{V}_n$, [4].

For $n = 1$, the Veronesean $\mathcal{V}_1^2$ is a conic in $PG(2, \mathcal{F})$. For $n = 2$, the Veronesean is a surface $\mathcal{V}_2^4$ of order 4 in $PG(5, \mathcal{F})$. The quadric Veronesean $\mathcal{V}_n$ is a set of $\theta(n)$ points in $PG(\frac{n(n+3)}{2}, \mathcal{F})$ such that no three points are collinear. In particular, for $n = 1$, $|\mathcal{V}_1| = \theta(1) = q + 1$ and for $n = 2$, $|\mathcal{V}_2| = \theta(2) = q^2 + q + 1$, $\mathcal{F} = GF(q)$.

**Theorem 2.3.1.** Let $\pi_s$ be s-dimensional subspace of $PG(n, \mathcal{F})$. The image of $\pi_s$ under the Veronesean map is a quadric Veronesean $\mathcal{V}_s$, which is the complete intersection of $\mathcal{V}_n$ and the space $PG(\frac{s(s+3)}{2}, \mathcal{F})$ containing $\mathcal{V}_s$, [4].

As a particular case, the Veronesean map maps the lines (1-dimensional subspace) in $PG(n, \mathcal{F})$ to conics of $\mathcal{V}_n$.

**Theorem 2.3.2.** Any two points of the quadric Veronesean $\mathcal{V}_n$ are contained in a unique conic of $\mathcal{V}_n$, [4].

**Theorem 2.3.3.** The quadric Veronesean $\mathcal{V}_n$ contains $\Phi(s; n, q)$ quadric Veroneseans $\mathcal{V}_s$, ($\Phi(s; n, q)$ is the number of s-dimensional subspace of $PG(n, q)$) , [4].

## 3. Results

In this section, firstly that the image under the projection and the Veronesean map of $PG(2,4)$ is the projective plane of order 4 in 4-dimensional projective space is shown by giving the computations made with the coordinates in $PG(2,4)$ and in $PG(4,4)$.

**Theorem 3.1.** The image under the projection and the Veronesean map of $PG(2,4)$ is the projective plane of order 4 in 4-dimensional projective space over $GF(4)$ such that the set of points spans 4-dimensional projective space and every line of the plane is a conic.

**Proof.** Let the image under the projection and the Veronesean map of $PG(2,4)$ be denoted by $\mathcal{P}$. It will be shown that $\mathcal{P}$ is a projective plane in $PG(4,4)$ such that the points set of $\mathcal{P}$ spans $PG(4,4)$ and every line of $\mathcal{P}$ is a conic. It is well known that the projective plane of order 4 is embedded in the projective space $PG(5,4)$ by the Veronesean Map. This embedding is identified on each point of $PG(2,4)$ with its image under the Veronesean map. For $\mathcal{F} = GF(4)$ and $n = 2$, the Veronesean map is

$$\vartheta: PG(2,4) \to PG(5,4)$$
$$(x, y, z) \to (x^2, y^2, z^2, xy, xz, yz)$$
$$N_i \to \vartheta(N_i) = P_i.$$

The images of points of $PG(2,4)$ under the Veronesean map are as the following:

$P_0 = (0,1,0,0,0,0), P_1 = (0,0,1,0,0,0), P_2 = (0,1,1,0,0,1), P_3 = (0,1,t,0,0,t^2), P_4 = (0,1,t^2,0,0,t), P_5 = (1,1,1,1,1,1), P_6 = (1,0,1,0,1,0), P_7 = (1,t^2,1,t,1,t), P_8 = (1,t,1,t^2,1,t^2), P_9 = (1,1,0,1,0,0), P_{10} = (1,1,t,1,t^2,t^2), P_{11} = (1,1,t^2,1,t,t), P_{12} = (1,t^2,t,t,t^2,1), P_{13} = (1,t,t^2,t^2,t,1), P_{14} = (1,0,0,0,0,0), P_{15} = (1,t^2,0,t,0,0), P_{16} = (1,t,1,t^2,1,t^2), P_{17} = (1,0,t,0,t^2,0), P_{18} = (1,0,t^2,0,t,0), P_{19} = (1,t,t,t^2,t^2,t), P_{20} = (1,t^2,t^2,t,t,t^2).$

The images of lines in $PG(2,4)$ under the Veronesean map are conics. 21 points $P_i$ and 21 conics are in the quadric Veronesean $\mathcal{V}_2^4$ of $PG(5,4)$. For instance, the image of the line $D_0 =< N_0, N_1, N_2, N_3, N_4 >$ of $PG(2,4)$ under $\vartheta$ is the conic with the equation $x_1x_2 + x_5^2 = 0$ in $PG(5,4)$ such that $P_i, i = 0,1, \dots ,4$ are on it. Similarly, the image of the line $D_1 =< N_0, N_5, N_6, N_7, N_8 >$ of $PG(2,4)$ under $\vartheta$ is the conic with the equation $x_0x_1 + x_3^2 = 0$ and the points $P_i, i = 0,5,6,7,8$ on this conic.
Consider a projection of the quadric Veronesean $\mathcal{V}_2^4$ in $PG(5,4)$ onto the hyperplane $x_0 = 0$ from the point $P = (1,1,1,0,0,0)$ which is not contained in $\mathcal{V}_2^4$. If the hyperplane $x_0 = 0$ is denoted by $\pi$, the projection is

$$\beta: \mathcal{V}_2^4 \subset PG(5,4) \to \pi$$
$$P_i = (x_0, x_1, \dots, x_5) \to P_i^{'} = \beta(P_i) = (x_0 + x_1, x_0 + x_2, x_3, x_4, x_5).$$

$\beta$ maps every point $P$ of $\mathcal{V}_2^4$ in $PG(5,4)$ to the intersection point $P'$ of the line $XP$ and π. With $\beta$, $P_i^{'} = \beta(P_i) = P_i, i = 0,1, \dots ,4$ and the remaining points are

$P_5^{'} = (0,0,1,1,1),\ P_6^{'} = (1,0,0,1,0),\ P_7^{'} = (t,0,t,1,t), P_8^{'} = (1,0,1,t,1),\ P_9^{'} = (0,1,1,0,0), P_{10}^{'} = (0,t^2,1,t^2,t^2), P'_{11} = (0,t,1,t,t), P'_{12} = (t,t^2,t,t^2,1), P'_{13} = (t^2,t,t^2,t,1), P'_{14} = (1,1,0,0,0), P'_{15} = (t,1,t,0,0), P'_{16} = (t^2,1,t^2,0,0), P'_{17} = (1,t^2,0,t^2,0), P'_{18} = (1,t,0,t,0), P'_{19} = (t^2,t^2,t^2,t^2,t), P'_{20} = (t,t,t,t,t^2).$

The points $P_i^{'}$ belong to the geometric structure $\mathcal{P}$. Also $P_0^{'}, P_1^{'}, P_2^{'}, P_5^{'}$ and $P_6^{'}$ are linear independent and generate $PG(4,4)$. The projections of conics ( Veronesean $\mathcal{V}_1$) which are the image of lines of $PG(2,4)$ are conics in π. For instance, the image of the line $D_0 =< N_0, N_1, N_2, N_3, N_4 >$ of $PG(2,4)$ under $\beta\vartheta$ is the conics with the equation $y_0y_1 + y_4^2 = 0$ in π such that $P_i^{'}, i = 0,1, \dots ,4$ are on it. Also the points $P_i^{'}, i = 0,1, \dots ,4$ on this conic determine the plane $\{(y_0, y_1, y_2, y_3, y_4): y_2 = y_3 = 0, y_i \in GF(4)\}$. Similarly, the image of the line $D_1$ in $PG(2,4)$ under $\beta\vartheta$ is the conics with the equation ${y_2}^2 + {y_3}^2 + y_0y_3 = 0$ and the points $P_i^{'}, i = 0,5,6,7,8$ on this conic determine the plane $\{(y_0, y_1, y_2, y_3, y_4): y_1 = 0, y_2 = y_4, y_i \in GF(4)\}$. $\mathcal{P}$ contains 21 points $P_i{'}$ and 21 conics of $PG(4,4)$. $\beta\vartheta$ is a injective map from $PG(2,4)$ onto $\mathcal{P} = \beta\vartheta(PG(2,4))$ and preserves the incidence structure of $PG(2,4)$. Therefore $\mathcal{P}$ is isomorphic to $PG(2,4)$. Every conic consists of five points $P_i{'}$ such that no three are collinear. Each two of these points are contained in unique one of these 21 conics and each two of these conics intersect in unique one of these 21 points. Thus, $\mathcal{P}$ is the projective plane of order 4 in $PG(4,4)$.■

In the final part of the study, when the points spanning $PG(4,4)$ are taken in three conic planes, it is shown that the projective plane of order 4 is determined such that every line of the plane is a conic plane in $PG(4,4)$. This is handled with introducing coordinates in $PG(2,4)$ and in $PG(4,4)$ and the computations are made with coordinates based on the field $GF(4)$.

**Theorem 3.2.** Let the points spanning $PG(4,4)$ be taken in three conic planes determined by the conic equations $x_4^2 = x_0x_1$, $x_2^2 + x_3^2 + x_0x_3 = 0$ and $x_2^2 + x_3^2 + x_1x_2 = 0$. Then the projective plane of order 4 is obtained in $PG(4,4)$ such that every line of the plane is a conic plane.

**Proof.** We want to find a projective plane of order 4 in $PG(4,4)$. All the points in this projective plane are supposed to generate $PG(4,4)$ and every line of the plane is a conic plane. So, we must find 21 points and 21 conic planes in $PG(4,4)$ such that each conic plane contains exactly 5 points of 21 points, and two conic planes meet exactly in one point of 21 points. We use the notation for points, lines and incidence relation of $PG(2,4)$ in computations to obtain the projective plane.

It is known that all points and all lines of a projective plane are identified with three distinct nonconcurrent lines. So, three conic planes corresponding to given three conics are already contained in the projective plane.

If the conic plane determined by the conic with the equation $x_4^2 = x_0x_1$ is denoted by $D_0$, the points belong to $D_0$ in $PG(4,4)$ as following coordinates:

$N_0 = (1,0,0,0,0)$, $N_1 = (0,1,0,0,0)$, $N_2 = (1,1,0,0,1)$, $N_3 = (1,t,0,0,t^2)$. And, the conic plane $D_0$ is the set $\{(y_0,y_1,y_2,y_3,y_4): y_2 = y_3 = 0, y_i \in GF(4)\}$.

If the conic plane determined by the conic with the equation $x_2^2 + x_3^2 + x_0x_3 = 0$ is denoted by $D_1$, the points belong to $D_1$ in $PG(4,4)$ as following coordinates:

$N_0 = (1,0,0,0,0)$, $N_5 = (0,0,1,1,1)$, $N_6 = (1,0,0,1,0)$, $N_7 = (t,0,t,1,t)$, $N_8 = (t^2,0,t^2,1,t^2)$. And, the conic plane $D_1$ is the set $\{(y_0,y_1,y_2,y_3,y_4): y_1 = 0, y_2 = y_4, y_i \in GF(4)\}$.

If the conic plane determined by the conic with the equation $x_2^2 + x_3^2 + x_1x_2 = 0$ is denoted by $D_5$, the points belong to $D_5$ in $PG(4,4)$ as following coordinates:

$N_1 = (0,1,0,0,0)$, $N_5 = (0,0,1,1,1)$, $N_9 = (0,1,1,0,0)$, $N_{10} = (0,t^2,1,t^2,t^2)$, $N_{11} = (0,t,1,t,t)$. And, the conic plane $D_5$ is the set $\{(y_0,y_1,y_2,y_3,y_4): y_0 = 0, y_3 = y_4, y_i \in GF(4)\}$.

So, the lines (the conic planes) $D_0$, $D_1$ and $D_5$ are in the plane and 11 points of the plane are directly obtained by this lines. Note that the intersection points of the pairs of these conic planes are $N_0, N_1, N_5$. The remaining points and conic planes in the projective plane will be determined such that the conic plane contains exactly five points of 21 points and the two conic planes intersect at one of the 21 points.

The conic plane $D_{12}$ is spanned by the points $N_2, N_6, N_9$ and denoted by $< N_2, N_6, N_9 >$. For every point $(y_0,y_1,y_2,y_3,y_4)$ in $D_{12}$, the equalities $y_0 = y_3 + y_4, y_1 = y_2 + y_4$ are valid. The lines $D_{12}$ is $\{(y_0,y_1,y_2,y_3,y_4): y_0 = y_3 + y_4, y_1 = y_2 + y_4, y_i \in GF(4)\}$. For every point $(y_0,y_1,y_2,y_3,y_4)$ in the plane $D_{20} =< N_3, N_7, N_{10} >$, the equalities $y_3 = t^2y_2, y_4 = y_0 + y_1$ are obtained. Because the point $N_{13}$ belongs to $D_{12}$ and $D_{20}$, the coordinates of $N_{13}$ must provide the conditions on $D_{12}$ and $D_{20}$. From here, $D_{12}$ and $D_{20}$ have exactly a common point $N_{13} = (1,t^2,1,t^2,t)$. For the planes $D_{14} =< N_2, N_8, N_{10} >$ and $D_{15} =< N_3, N_6, N_{11} >$, the equalities $y_2 = y_0t + y_1 + y_4t^2, y_4 = y_0 + y_1t + y_4t^2$ and $y_3 = y_0 + (y_1 + y_2)t^2, y_4 = y_1t + y_2$ are valid, respectively. The intersection point of the planes $D_{14}$ and $D_{15}$ is $N_{15} = (x_0, x_1, x_0, x_0t + x_1t^2, x_0 + x_1t)$. $N_{15} = (k,1,k,kt + t^2, k + t)$ where $k = \frac{x_0}{x_1}, x_1 \neq 0$. For the planes $D_2 =< N_0, N_9, N_{15} >$ and $D_{20} =< N_3, N_7, N_{10} >$, the equalities $y_3 = 0, y_4 = 0$ and $y_2 = y_3t, y_4 = y_0 + y_1$ are valid, respectively. The intersection point of these planes is $\mathrm{N}_{14} = (\mathrm{kt} + 1, \mathrm{k}, \mathrm{t}(\mathrm{t} +$ k,t+k,t2t+k. *D0* and *D10* intersect at the point *N4=(k2t+1+kt2,kt2k+1,0,0,tk2t+1+kt2)*. Since $D19=<N4, N8,N11>$ contains the point $N_4$, the equalities $k^2 + tk = 0$ and $(k^4 + k)t + k^3 + 1 = 0$ are valid. Then $k$ equals to $t$. If the value of k is written at the points, $N_{15} = (t,1,t,0,0)$, $N_{14} = (t,t,0,0,0)$, $N_4 = (1,t^2,0,0,t)$ are obtained. The intersection point of the planes $D_8 =< N_1, N_7, N_{15} >$ and $D_{12} =< N_2, N_6, N_9 >$ is $N_{12} = D_8 \wedge D_{12} = (t,t^2,t,t^2,1)$. Since the equality $N_{12} = t^2N_4 + tN_8 + t^2N_{11}$ is valid, the point $N_{12}$ is in the plane $D_{19}$ which is spanned by the points $N_4, N_8, N_{11}$. Since the planes $D_2 =< N_0, N_9, N_{15} >$, $D_7 =< N_1, N_8, N_{13} >$ and $D_{11} =< N_3, N_5, N_{12} >$ intersect the point $N_{16}$, its coordinates are found as $(t^2,1,t^2,0,0)$. Similarly, the planes $D_3 =< N_0, N_{10}, N_{12} >$, $D_6 =< N_1, N_6, N_{14} >$ and $D_{10} =< N_4, N_5, N_{13} >$ intersect the point $N_{17} = (1,t^2,0,t^2,0)$. The point $N_{18} = (1,t,0,t,0)$ is obtained from the intersection of the planes $D_4 =< N_0, N_{11}, N_{13} >$, $D_6 =< N_1, N_6, N_{14} >$ and $D_{11} =< N_3, N_5, N_{12} >$. The points $N_{19} = (t^2,t^2,t^2,t^2,t^2)$ and $N_{20} = (1,1,1,1,t)$ are found by the intersections of $D_3 =< N_0, N_{10}, N_{12} >$, $D_7 =< N_1, N_8, N_{13} >$ and $D_4 =< N_0, N_{11}, N_{13} >$, $D_8 =< N_1, N_7, N_{15} >$, respectively.

Let $\{N_i : i = 0,1,\dots,20\}$ be the set of points of $S = (\mathcal{N}', \mathcal{D}', \circ')$, with line set $\{D_i : i = 0,1,\dots,20\}$ such that the points of $\mathcal{N}'$ span $PG(4,4)$ and $D_i$ is a conic plane. Any two distinct elements $N_i$ and $N_j$ of the points set belong to unique member $D_k$ of the lines set $\mathcal{D}'$. Any two distinct elements $D_i$ and $D_j$ of the lines set have unique point $N_k$ of the points set $\mathcal{N}'$ in common. Each element $N_i$ of $\mathcal{N}'$ is contained five conic planes and each element $D_i$ of $\mathcal{D}'$ contains five points of $\mathcal{N}'$. The incidence structure $S$ formed by $\mathcal{N}'$ and $\mathcal{D}'$ is a projective plane of order 4 such that $S$ generates $PG(4,4)$ and $S$ is isomorfic to $PG(2,4)$. ■

## 4. Conclusion

In this study, firstly it is shown computationally that the image under the projection and the Veronesean map of $PG(2,4)$ is the projective plane of order 4 in 4-dimensional projective space. Then taking points span *PG(4,4)* on three distinct conic plane such that the pair of these planes intersects a point, the projective plane of order 4 in $PG(4,4)$ is determined. This computations are handled with coordinates in $PG(2,4)$ and in $PG(4,4)$.

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