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Complete Systems of Galileo Invariants of a Motion of Parametric Figure in the Three Dimensional Euclidean Space

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ABSTRACT

Let E^3 be the 3-dimensional Euclidean space and S be a set with at least two elements. The notions of an S-parametric figure and the motion of an S-parametric figure in E^3 are defined. Complete systems of invariants of an S-parametric figure in E^3 for the orthogonal group $O(3, R)$, the special orthogonal group $SO(3, R)$, Euclidean group $MO(3, R)$, the special Euclidean group $MSO(3, R)$ and Galileo groups $Gal_1(3, R)$, $Gal_1^+(3, R)$ are obtained.

Keywords: *Galileo group, invariant, figure, Euclidean geometry.*

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1. Introduction

Complete systems of invariants of S-parametric figures $\gamma(s)$ in the Euclidean geometry have been investigated mainly in the following particular cases: $\gamma(s)$ is an m-tuple, $\gamma(s)$ is a path, $\gamma(s)$ is a curve, $\gamma(s)$ is a surface and $\gamma(s)$ is a vector field. Below we give a short survey of papers in this area. For shortness, an S-parametric figure will be called an S-figure.

Foundations of Euclidean and non-Euclidean geometry are presented in the book [3]. An elementary treatise of the geometry of a triangle, a polygon and a circle are given in the book [13]. In the book [11], plane Euclidean geometry, affine transformations in the Euclidean space, finite group isometries of plane Euclidean geometry, geometry on the sphere are presented. Geometry of sets and measures in Euclidean spaces are considered in the book [12]. The book [4] presents the discovery of non-Euclidean geometry and the subsequent reformulation of the foundations of Euclidean geometry as a suspense story.

Let V be a finite dimensional vector space over a field K and β be a non-degenerate bilinear form on V. Denote by $O(\beta, K)$ the group of all β-orthogonal (that is the form β preserving) transformations of V. Let $MO(\beta, K)$ be the group generated by the group $O(\beta, K)$ and all translations of V. In the paper [5], for the orthogonal group $O(\beta, K)$ in the Euclidean, spherical, hyperbolic and de-Sitter geometries, the orbit of m vectors is characterized by their Gram matrix and an additional subspace. In the book [2, Proposition 9.7.1], for the group $MO(\beta, K)$ in the Euclidean geometry, the orbit of m vectors is characterized by distances between m-vectors. A complete system of relations between elements of this complete system is also given in [2, Theorem 9.7.3.4]. In the paper [9], a complete system of invariants of m-tuples in the two-dimensional Euclidean geometry have obtained. Euclidean invariants of parametric curves appear also in Computer vision theory and in Computational Geometry. General theory of m-point invariants considered in the invariant theory.

Complete systems of global invariants of paths and curves in the Euclidean space are investigated in papers [1, 6, 8]. Complete systems of global invariants of parametric figures in the affine space are investigated in the paper [10]. Galileo invariants are investigated in papers [14, 15, 16, 17].

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Denote by $O(3,R)$ be the group of all the form $\varphi(x,y)$ preserving transformations of E^3, Put $SO(3,R) = \{g \in O(3,R) | det(g) = 1\}$, $MO(3,R) = \{F : E^3 \to E^3 \mid Fx = gx + b, g \in O(3,R), b \in E^3\}$, $MSO(3,R) = \{F : E^3 \to E^3 \mid Fx = gx + b, g \in SO(3,R), b \in E^3\}$.

Denote by $Gal_1(3,R)$ the group of all transformations $F : E^3 \times R \to E^3 \times R$ of the form $F(x,t) = (g(x) + a + tb, t)$, where $a, b \in E^3, g \in O(3,R), t \in R$. Denote by $Gal_1^+(3,R)$ the set of all transformations $F(x,t) = (g(x) + a + tb, t)$, where $a, b \in E^3, g \in SO(3,R), t \in R$.

The present paper is devoted to solutions of problems of G-equivalence of S-figures and $S \times R$-figures in E^3 for the groups $G = O(3,R), SO(3,R), MO(3,R), MSO(3,R)$ in terms of G-invariants of an S-figure and $G = Gal_1(3,R), Gal_1^+(3,R)$ in terms of G-invariants of an $S \times R$-figure, respectively. Complete systems of invariants of an S-figure and an $S \times R$-figure in E^3 for these groups are obtained. This paper is organized as follows. In Section 1, some known definitions and results are given. They are used in next sections. In Section 2, complete systems of invariants of a parametric figure in E^3 for the groups $O(3,R)$ and $SO(3,R)$ are given. In Section 3, complete systems of invariants of a parametric figure in E^3 for the groups $MO(3,R)$ and $MSO(3,R)$ are obtained. In Section 4, complete systems of invariants of a parametric figure in E^3 for the Galileo groups $Gal_1(3,R)$ and $Gal_1^+(3,R)$ are given.

2. Preliminaries

Let S be a set such that it has at least two elements.

Definition 2.1. A mapping $\gamma : S \to E^3$ will be called an S-parametric figure (S-figure) in E^3.

Denote by $\Phi(S, E^3)$ the set of all mappings of the S to E^3. Let G be a subgroup of the group $MO(3,R)$.

Definition 2.2. Two S-figures $\gamma(s)$ and $\eta(s)$ in E^3 are called G-equivalent if there exists $g \in G$ such that $\eta(s) = g\gamma(s), \forall s \in S$. In this case, we write $\eta(s) = g\gamma(s), \forall s \in S$, or $\gamma(s) \overset{G}{\sim} \eta(s), \forall s \in S$.

Definition 2.3. A subset $C \subseteq \Phi(S, E^3)$ is called G-invariant if $g(u) \in C, \forall u \in C, \forall g \in G$.

Definition 2.4. Let Υ be a set and it has at least two elements and C be a G-invariant subset of $\Phi(S, E^3)$. A mapping $f : C \to \Upsilon$ is called G-invariant on C if $u \in C, v \in C$ and $u \overset{G}{\sim} v$, implies $f(u) = f(v)$.

Definition 2.5. Let C be a G-invariant subset of $\Phi(S, E^3)$. A system $\{f_i(\gamma(s)) | i \in I\}$ of G-invariants functions of S-figures in C will be called a complete system of G-invariant functions on C if equalities $f_i(\gamma(s)) = f_i(\eta(s)), \forall i \in I$, for $\gamma(s) \in C$ and $\eta(s) \in C$ imply $\gamma(s) \overset{G}{\sim} \eta(s), \forall s \in S$.

Let $\gamma(s) \in \Phi(S, E^3)$. Denote by θ the zero vector of E^3. Put $Z(\gamma) = \{s \in S | \gamma(s) = \theta\}$.

Proposition 2.1. *Let G be a subgroup of $O(3,R)$ and $\gamma(s), \eta(s)$ be S-figures such that $\gamma(s) \overset{G}{\sim} \eta(s), \forall s \in S$. Then $Z(\gamma) = Z(\eta)$.*

Proof. It is obvious. □

Proposition 2.2. *Let G be a subgroup of $O(3,R)$ and $\gamma(s), \eta(s) \in \Phi(S, E^3)$ be such that $Z(\gamma) = Z(\eta)$. Then $\gamma(s)$ and $\eta(s)$ are G-equivalent on S if and only if there exists $g \in G$ such that $\eta(s) = g\gamma(s), \forall s \in S \setminus Z(\gamma)$.*

Proof. It is obvious. □

Let $\gamma(s) \in \Phi(S, E^3)$. Denote by $rank(\gamma(s))$ the rank of the system $\{\gamma(s) | s \in S\}$ of vectors in the space E^3.

Proposition 2.3. *Let G be a subgroup of $O(3,R)$. Assume that $\gamma(s)$ and $\eta(s)$ be S-figures in E^3 such that $\gamma(s) \overset{G}{\sim} \eta(s), \forall s \in S$. Then $rank(\gamma(s)) = rank(\eta(s))$.*

Proof. It is obvious. □

3. Complete systems of invariants of an S-figure in E^3 for the groups $O(3,R)$ and $SO(3,R)$

Put $N_3 = \{1,2,3\}$. Denote by W the following 3×3 matrix: $W = \|w_{ij}\|$, where $w_{ij} = 0, \forall i,j \in N_3, i \neq j$, $w_{33} = -1$ and $w_{ii} = 1, \forall i = 1,2$.

Below we use the following equality $O(3,R) = SO(3,R) \cup SO(3,R)W$, where $SO(3,R)W = \{g \cdot W | g \in SO(3,R)\}$. Let $< x, y >= x \cdot y = x_1y_1 + x_2y_2 + x_3y_3$ be the scalar product in E^3, where $x = (x_1,x_2,x_3) \in E^3, y = (y_1,y_2,y_3) \in E^3$.

Let $\gamma(s)$ be an S-figure in E^3 such that $rank(\gamma(s)) = 3$. Then there are elements s_1,s_2,s_3 of the set $S \setminus Z(\gamma(s))$ such that vectors $\gamma(s_1),\gamma(s_2),\gamma(s_3)$ are linearly independent in E^3. Let $\gamma(s_i) = (\gamma_{i1}(s_i),\gamma_{i2}(s_i)),\gamma_{i3}(s_i))$ for $i \in N_3$. Denote by $\Lambda(\gamma(s_1),\gamma(s_2),\gamma(s_3))$ the following 3×3-matrix $\begin{pmatrix} \gamma_{11}(s_1) & \gamma_{12}(s_1) & \gamma_{13}(s_1) \\ \gamma_{21}(s_2) & \gamma_{22}(s_2) & \gamma_{23}(s_2) \\ \gamma_{31}(s_3) & \gamma_{32}(s_3) & \gamma_{33}(s_3) \end{pmatrix}$.

Proposition 3.1. *Let $\gamma(s)$ be an S-figure in E^3 such that $rank(\gamma(s)) = 3$. Assume that elements s_1,s_2,s_3 of the set $S \setminus Z(\gamma(s))$ such that vectors $\gamma(s_1),\gamma(s_2),\gamma(s_3)$ are linearly independent in E^3. Then $(\Lambda(\gamma(s_1),\gamma(s_2),\gamma(s_3)))^{-1}$ exists.*

Proof. Since vectors $\gamma(s_1),\gamma(s_2),\gamma(s_3)$ are linearly independent in E^3 we have $det(\Lambda(\gamma(s_1),\gamma(s_2),\gamma(s_3))) \neq 0$. This inequality implies an existence of the matrix $(\Lambda(\gamma(s_1),\gamma(s_2),\gamma(s_3)))^{-1}$. □

Theorem 3.1. *Let $\gamma(s)$ and $\eta(s)$ be S-figures in E^3 such that $rank(\gamma(s)) = 3$. Assume that elements s_1,s_2,s_3 of the set $S \setminus Z(\gamma(s))$ such that vectors $\gamma(s_1),\gamma(s_2),\gamma(s_3)$ are linearly independent in E^3.*

Assume that $\gamma(s) \overset{O(3,R)}{\sim} \eta(s), \forall s \in S$.Then following equalities hold for all $s \in S \setminus Z(\gamma(s))$ and for all $i \in N_3$:

$$\begin{cases} Z(\gamma(s)) = Z(\eta(s)) \\ rank(\gamma(s)) = rank(\eta(s)) \\ \langle\gamma(s_i),\gamma(s)\rangle = \langle\eta(s_i),\eta(s)\rangle. \end{cases} \tag{3.1}$$

Conversely, assume that the equalities (3.1) hold. Then the matrix $(\Lambda(\eta(s_1),\eta(s_2),\eta(s_3)))^{-1}$ exists and the unique $F \in O(3,R)$ exists such that $\eta(s) = F\gamma(s), \forall s \in S$. In this case, F has following form: $F = ((\Lambda(\eta(s_1),\eta(s_2),\eta(s_3)))^{-1} \cdot \Lambda(\gamma(s_1),\gamma(s_2),\gamma(s_3))$.

Proof. ⇒ Assume that $\gamma(s) \overset{O(3,R)}{\sim} \eta(s), \forall s \in S$. Then there exists $F \in O(3,R)$ such that $\eta(s) = F\gamma(s), \forall s \in S$.

By Proposition 2.1, the equality $Z(\gamma(s)) = Z(\eta(s))$ holds. By Proposition 2.3, the equality $rank(\gamma(s)) = 3$ implies the equality $rank(\gamma(s)) = rank(\eta(s)) = 3$.

It is known that the function $\langle\gamma(s_i),\gamma(s)\rangle$ is $O(3,R)$-invariant for all $i \in N_3, \forall s \in S$. Hence the equivalence $\gamma(s) \overset{O(3,R)}{\sim} \eta(s), \forall s \in S$, implies following equalities

$$\langle\gamma(s_i),\gamma(s)\rangle = \langle\eta(s_i),\eta(s)\rangle, \forall s \in S, \forall i \in N_3.$$

Hence equalities (3.1) hold.

⇐ Conversely, assume that the equalities (3.1) hold. Let $s \in S \setminus Z(\gamma(s))$. Consider vectors $\gamma(s)$ and $\eta(s)$ as column vectors:

$$\gamma(s) = \begin{pmatrix} \gamma_1(s) \\ \gamma_2(s) \\ \gamma_3(s) \end{pmatrix}, \eta(s) = \begin{pmatrix} \eta_1(s) \\ \eta_2(s) \\ \eta_3(s) \end{pmatrix}.$$

Denote by $\gamma(s)^\top$ the transpose of the vector $\gamma(s)$. Let $\eta(s)^\top$ be the transpose of $\eta(s)$. Consider the multiplication of matrices $\gamma(s_i)^\top$ and $\gamma(s)$,where $i \in N_3$,

$$\gamma(s_i)^\top \cdot \gamma(s) = (\gamma_1(s_i),\gamma_2(s_i),\gamma_3(s_i)) \cdot \begin{pmatrix} \gamma_1(s) \\ \gamma_2(s) \\ \gamma_3(s) \end{pmatrix} = \langle\gamma(s_i),\gamma(s)\rangle, \forall s \in T.$$

Using these equalities, we obtain following equalities for the multiplication of matrices $\Lambda(\gamma(s_1),\gamma(s_2),\gamma(s_3))$ and $\gamma(s)$:

$$\Lambda(\gamma(s_1),\gamma(s_2),\gamma(s_3)) \cdot \gamma(s) = \begin{pmatrix} \langle\gamma(s_1),\gamma(s)\rangle \\ \langle\gamma(s_2),\gamma(s)\rangle \\ \langle\gamma(s_3),\gamma(s)\rangle \end{pmatrix}. \tag{3.2}$$

Similar equalities we obtain for the multiplication of matrices $\Lambda(\eta(s_1),\eta(s_2),\eta(s_3))$ and $\eta(s)$:

$$\Lambda(\eta(s_1),\eta(s_2),\eta(s_3))\cdot\eta(s)=\begin{pmatrix}\langle\eta(s_1),\eta(s)\rangle\\ \langle\eta(s_2),\eta(s)\rangle\\ \langle\eta(s_3),\eta(s)\rangle\end{pmatrix}. \quad (3.3)$$

Using equalities $\langle\gamma(s_i),\gamma(s)\rangle=\langle\eta(s_i),\eta(s)\rangle,\forall s\in S\setminus Z(\gamma(s)),\forall i\in N_3$ in Theorem 3.1 and equalities (3.2), (3.3) we obtain following equalities:

$$\Lambda(\gamma(s_1),\gamma(s_2),\gamma(s_3))\cdot\gamma(s)=\Lambda(\eta(s_1),\eta(s_2),\eta(s_3))\cdot\eta(s),\forall s\in S\setminus Z(\gamma(s)). \quad (3.4)$$

The equality $Z(\gamma(s))=Z(\eta(s))$ in (3.1) implies following equalities

$$\gamma_i(s)=\eta_i(s)=0,\forall s\in Z(\gamma(s)),\forall i\in N_3. \quad (3.5)$$

Equalities (3.4) and (3.5) imply following equalities

$$\Lambda(\eta(s_1),\eta(s_2),\eta(s_3))\cdot\eta(s)=\Lambda(\gamma(s_1),\gamma(s_2),\gamma(s_3))\cdot\gamma(s),\forall s\in S. \quad (3.6)$$

To continue the proof, let us give the following lemma.

Lemma 3.1. *Let $\eta(s)$ be an S-figure in E^3 such that equalities (3.4) hold. Then $\Lambda(\eta(s_1),\eta(s_2),\eta(s_3))\neq 0$ and $\Lambda(\eta(s_1),\eta(s_2),\eta(s_3))^{-1}$ exists.*

Proof. By Proposition 3.1, $\Lambda(\gamma(s_1),\gamma(s_2),\gamma(s_3))\neq 0$. Assume that $\Lambda(\eta(s_1),\eta(s_2),\eta(s_3))=0$. Then $\Lambda(\eta(s_1),\eta(s_2),\eta(s_3))\cdot\eta(s)=0,\forall s\in S$. This equalities and the equalities (3.6) imply equalities $\Lambda(\gamma(s_1),\gamma(s_2),\gamma(s_3))\cdot\gamma(s)=0,\forall s\in S$. Since $\Lambda(\gamma(s_1),\gamma(s_2),\gamma(s_3))\neq 0$, the equalities $\Lambda(\gamma(s_1),\gamma(s_2),\gamma(s_3))\cdot\gamma(s)=0,\forall s\in S$, imply following equalities $\gamma(s)=0,\forall s\in S$. But these equalities contradicts to the our supposition $rank(\gamma(s))=3$ in Theorem 3.1. Hence $\Lambda(\eta(s_1),\eta(s_2),\eta(s_3))\neq 0$ and $\Lambda(\eta(s_1),\eta(s_2),\eta(s_3))^{-1}$ exists. The lemma is proved. □

Using the existence of the matrix $\Lambda(\eta(s_1),\eta(s_2),\eta(s_3))^{-1}$ and the equalities (3.6), we obtain following equalities

$$\eta(s)=\Lambda(\eta(s_1),\eta(s_2),\eta(s_3))^{-1}\cdot\Lambda(\gamma(s_1),\gamma(s_2),\gamma(s_3))\cdot\gamma(s),\forall s\in S. \quad (3.7)$$

We prove that the matrix $\Lambda(\eta(s_1),\eta(s_2),\eta(s_3))^{-1}\cdot\Lambda(\gamma(s_1),\gamma(s_2),\gamma(s_3))$ is orthogonal. For shortness, denote this matrix by P. Then the equalities (3.7) has the form:

$$\eta(s)=P\cdot\gamma(s),\forall s\in S. \quad (3.8)$$

Using the equalities (3.1) and (3.8), we obtain following equalities:

$$\langle P\cdot\gamma(s_i),P\cdot\gamma(s)\rangle=\langle\eta(s_i),\eta(s)\rangle=\langle\gamma(s_i),\gamma(s)\rangle,\forall s\in S\setminus Z(\gamma(s)),\forall i\in N_3.$$

Hence

$$\langle P\cdot\gamma(s_i),P\cdot\gamma(s)\rangle=\langle\gamma(s_i),\gamma(s)\rangle,\forall s\in S\setminus Z(\gamma(s)),\forall i\in N_3. \quad (3.9)$$

Let $x\in E^3$ be an arbitrary element. Since the system of vectors $\gamma(s_i),i=1,2,3$ is a basis in E^3, there exist numbers $r_i\in R,i=1,2,3$ such that $x=r_1\gamma(s_1)+r_2\gamma(s_2)+r_3\gamma(s_3)$. The equalities (3.9) imply following equalities:

$$\langle P\cdot r_i\gamma(s_i),P\cdot\gamma(s)\rangle=\langle r_i\gamma(s_i),\gamma(s)\rangle,\forall s\in S\setminus Z(\gamma(s)),\forall i\in N_3.$$

These equalities and the equality $x=r_1\gamma(s_1)+r_2\gamma(s_2)+r_3\gamma(s_3)$ imply following equalities:

$$\langle Px,P\cdot\gamma(s)\rangle=\langle x,\gamma(s)\rangle,\forall s\in S\setminus Z(\gamma(s)),\forall x\in E^3.$$

These equalities imply following equality:

$$\langle Px,P\cdot\gamma(s_i)\rangle=\langle x,\gamma(s_i)\rangle,\forall i\in N_3,\forall x\in E^3. \quad (3.10)$$

Let $y \in E^3$ be an arbitrary element. Since the system of vectors $\gamma(s_i), i = 1,2,3$ is a basis in E^3, there exist numbers $a_i \in R, i = 1,2,3$ such that $y = a_1\gamma(s_1) + a_2\gamma(s_2) + a_3\gamma(s_3)$. This equality and the equalities (3.10) imply following equalities:

$$\langle Px, Py\rangle = \langle x, y\rangle, \forall x \in E^3, \forall y \in E^3.$$

This equality means that $P = \Lambda(\eta(s_1), \eta(s_2), \eta(s_3))^{-1} \cdot \Lambda(\gamma(s_1), \gamma(s_2), \gamma(s_3))$ is an orthogonal matrix.

We prove the uniqueness of a a matrix $F \in O(3, R)$ such that $\eta(s) = F\gamma(s), \forall s \in S$. Assume that a matrix $F \in O(3, R)$ exists such that $\eta(s) = F\gamma(s), \forall s \in S$. These equalities and the equalities (3.8) imply following equalities:

$$F \cdot \gamma(s) = P \cdot \gamma(s), \forall s \in S.$$

These equalities imply following equalities

$$F \cdot \gamma(s_i) = P \cdot \gamma(s_i), \forall i \in N_3. \tag{3.11}$$

Let $x \in E^3$ be an arbitrary element. Since the system of vectors $\gamma(s_i), i = 1,2,3$, is a basis in E^3, there exist numbers $r_i \in R, i = 1,2,3$, such that $x = r_1\gamma(s_1) + r_2\gamma(s_2) + r_3\gamma(s_3)$. The equalities (3.11) imply following equalities: $Fx = Px, \forall x \in E^3$. This equalities imply that $F = P$. Hence the unique matrix F exists such that $\eta(s) = F\gamma(s), \forall s \in S$. The equality $F = P$ implies that $F = \Lambda(\eta(s_1), \eta(s_2), \eta(s_3))^{-1} \cdot \Lambda(\gamma(s_1), \gamma(s_2), \gamma(s_3))$. The uniqueness of F is proved. □

Let $F \in O(3, R)$. Then it is known that $det(F) = 1$ or $detF = -1$.

Corollary 3.1. *Let $\gamma(s)$ and $\eta(s)$ be S-figures in E^3 such that $rank(\gamma(s)) = 3$. Assume that elements s_1, s_2, s_3 of the set $S \setminus Z(\gamma(s))$ such that vectors $\gamma(s_1), \gamma(s_2), \gamma(s_3)$ are linearly independent in E^3.*

Assume that there exists $F \in O(3, R)$ such that $det(F) = 1$ and $\eta(s) = F\gamma(s), \forall s \in S$.Then the following equalities hold for all $s \in S \setminus Z(\gamma(s))$ and for all $i \in N_3$:

$$\begin{cases} Z(\gamma(s)) = Z(\eta(s)) \\ rank(\gamma(s)) = rank(\eta(s)) \\ \langle\gamma(s_i), \gamma(s)\rangle = \langle\eta(s_i), \eta(s)\rangle \\ det(\Lambda(\gamma(s_1), \gamma(s_2), \gamma(s_3))) = det(\Lambda(\eta(s_1), \eta(s_2), \eta(s_3))) \end{cases} \tag{3.12}$$

Conversely, assume that the equalities (3.12) hold. Then the unique $F \in O(3, R)$ exists such that $det(F) = 1$ and $\eta(s) = F\gamma(s), \forall s \in S$. In this case, F has the following form:
$F = ((\Lambda(\eta(s_1), \eta(s_2), \eta(s_3)))^{-1} \cdot \Lambda(\gamma(s_1), \gamma(s_2), \gamma(s_3))$.

Proof. ⇒ Assume that there exists $F \in O(3, R)$ such that $det(F) = 1$ and $\eta(s) = F\gamma(s), \forall s \in S$. By Theorem 3.1, the last equality implies equalities (3.1). The equality $det(F) = 1$ and the equality $\eta(s) = F\gamma(s), \forall s \in S$ imply the following equality $det(\Lambda(\eta(s_1), \eta(s_2), \eta(s_3))) = det(\Lambda(F\gamma(s_1), F\gamma(s_2), F\gamma(s_3))) = det(F)det(\Lambda(\gamma(s_1), \gamma(s_2), \gamma(s_3))) =$
$det(\Lambda(\gamma(s_1), \gamma(s_2), \gamma(s_3)))$. Hence $det(\Lambda(\eta(s_1), \eta(s_2), \eta(s_3))) = det(\Lambda(\gamma(s_1), \gamma(s_2), \gamma(s_3)))$. This equality and equalities (3.1) imply the equalities (3.12).

⇐ Assume that the equalities (3.12) hold. These equalities imply equalities (3.1). Then, by Theorem 3.1, there exists $F \in O(3, R)$ such that $\eta(s) = F\gamma(s), \forall s \in S$. This equality and the equality $det(\Lambda(\gamma(s_1), \gamma(s_2), \gamma(s_3))) = det(\Lambda(\eta(s_1), \eta(s_2), \eta(s_3)))$ in the equalities (3.12) imply following equality $det(\Lambda(\gamma(s_1), \gamma(s_2), \gamma(s_3))) = det(\Lambda(F\gamma(s_1), F\gamma(s_2), F\gamma(s_3))) = det(F)det(\Lambda(\gamma(s_1), \gamma(s_2), \gamma(s_3)))$. Since vectors $\gamma(s_1), \gamma(s_2), \gamma(s_3)$ are linearly independent, following inequality $det(\Lambda(\gamma(s_1), \gamma(s_2), \gamma(s_3))) \neq 0$ holds. This inequality, the equality
$det(\Lambda(\gamma(s_1), \gamma(s_2), \gamma(s_3))) = det(\Lambda(\eta(s_1), \eta(s_2), \eta(s_3)))$ in the equalities (3.12) and the equality $det(\Lambda(\gamma(s_1), \gamma(s_2), \gamma(s_3))) = det(F)det(\Lambda(\gamma(s_1), \gamma(s_2), \gamma(s_3)))$ imply the equality $det(F) = 1$. By Theorem 3.1, F has the following form:$F = ((\Lambda(\eta(s_1), \eta(s_2), \eta(s_3)))^{-1} \cdot \Lambda(\gamma(s_1), \gamma(s_2), \gamma(s_n))$. □

Corollary 3.2. *Let $\gamma(s)$ and $\eta(s)$ be S-figures in E^3 such that $rank(\gamma(s)) = 3$. Assume that elements s_1, s_2, s_3 of the set $S \setminus Z(\gamma(s))$ such that vectors $\gamma(s_1), \gamma(s_2), \gamma(s_3)$ are linearly independent in E^3.*

Assume that there exists $F \in O(3, R)$ such that $det(F) = -1$ and $\eta(s) = F\gamma(s), \forall s \in S$.Then the following equalities hold for all $s \in S \setminus Z(\gamma(s))$ and for all $i \in N_3$:

$$\begin{cases} Z(\gamma(s)) = Z(\eta(s)) \\ rank(\gamma(s)) = rank(\eta(s)) \\ \langle\gamma(s_i), \gamma(s)\rangle = \langle\eta(s_i), \eta(s)\rangle \\ det(\Lambda(\gamma(s_1), \gamma(s_2), \gamma(s_3))) = -det(\Lambda(\eta(s_1), \eta(s_2), \eta(s_3))) \end{cases} \tag{3.13}$$

Conversely, assume that the equalities (3.13) hold. Then the unique $F \in O(3,R)$ exists such that $det(F) = -1$ and $\eta(s) = F\gamma(s), \forall s \in S$. In this case, F has the following form: $F = ((\Lambda(\eta(s_1), \eta(s_2), \eta(s_3)))^{-1} \cdot \Lambda(\gamma(s_1), \gamma(s_2), \gamma(s_3))$.

Proof. A proof is similar to the proof of Corollary 3.1 and it is omitted. □

Let $\gamma(s)$ an S-figure in E^3 such that $rank(\gamma(s)) = m$, where $m = 1$ or $m = 2$. In these cases, analogues of Theorem 3.1, Corollary 3.1 and Corollary 3.2 easy have obtained. They are omitted.

4. Complete systems of invariants of a S-figure in E^3 for the groups $MO(3,R)$ and $MSO(3,R)$

Denote by $Tr(3,R)$ the group of all translations of E^3. Let $G = O(3,R)$ or $G = SO(3,R)$. Denote by $G \ltimes Tr(3,R)$ the group of all transformations of E^3 generated by groups G and $Tr(3,R)$.

Proposition 4.1. *Let $G = O(3,R)$ or $G = SO(3,R)$. Assume that $\gamma(s)$ and $\eta(s)$ be two S-figures in E^3 and $s_0 \in S$. Then $\gamma(s) \overset{G\ltimes Tr(3,R)}{\sim} \eta(s), \forall s \in S$ if and only if $(\gamma(s) - \gamma(s_0)) \overset{G}{\sim} (\eta(s) - \eta(s_0)), \forall s \in S \setminus \{s_0\}$. In the case, $\gamma(s) \overset{G\ltimes Tr(n,R)}{\sim} \eta(s), \forall s \in S$, there exist $F \in O(3,R)$ and $a \in E^3$ such that $\eta(s) = F\gamma(s) + a, \forall s \in S$, where $a = \eta(s_0) - F\gamma(s_0)$.*

Proof. $\Rightarrow$ Assume that $\gamma(s) \overset{G\ltimes Tr(3,R)}{\sim} \eta(s), \forall s \in S$. Then there exists $F \in G$ and $a \in E^3$ such that $\eta(s) = F\gamma(s) + a, \forall s \in S$. In particularly, for $s = s_0$, we have $\eta(s_0) = F\gamma(s_0) + a$. This equality implies $a = \eta(s_0) - F\gamma(s_0)$. This equality and equalities $\eta(s) = F\gamma(s) + a, \forall s \in S$, imply equalities $\eta(s) = F\gamma(s) + \eta(s_0) - F\gamma(s_0), \forall s \in S$. These equalities imply equalities $\eta(s) - \eta(s_0) = F\gamma(s) - F\gamma(s_0) = F(\gamma(s) - \gamma(s_0)), \forall s \in S$. These equalities imply $(\gamma(s) - \gamma(s_0)) \overset{G}{\sim} (\eta(s) - \eta(s_0), \forall s \in S \setminus \{s_0\}$.

$\Leftarrow$ Assume that $(\gamma(s) - \gamma(s_0)) \overset{G}{\sim} (\eta(s) - \eta(s_0), \forall s \in S \setminus \{s_0\}$. Then there exists $F \in G$ such that $\eta(s) - \eta(s_0) = F(\gamma(s) - \gamma(s_0)), \forall s \in S \setminus \{s_0\}$. Put $a = \eta(s_0) - F\gamma(s_0)$. This equality and equalities $\eta(s) - \eta(s_0) = F(\gamma(s) - \gamma(s_0)), \forall s \in S \setminus \{s_0\}$ imply equalities $\eta(s) = F\gamma(s) + a, \forall s \in S$. Hence $\gamma(s) \overset{G\ltimes Tr(3,R)}{\sim} \eta(s), \forall s \in S$. □

Let $\gamma(s)$ and $\eta(s)$ be S-figures in E^3 and $s_0 \in S$. By Proposition 4.1, $\gamma(s) \overset{MO(3,R)}{\sim} \eta(s), \forall s \in S$ if and only if S-figures $\gamma(s) - \gamma(s_0)$ and $\eta(s) - \eta(s_0)$ are $O(3,R)$-equivalent on the set $S \setminus \{s_0\}$. Assume that S-figures $\gamma(s) - \gamma(s_0)$ and $\eta(s) - \eta(s_0)$ are $O(3,R)$-equivalent on the set $S \setminus \{s_0\}$. Then $rank(\gamma(s) - \gamma(s_0)) = rank(\eta(s) - \eta(s_0))$.

Let $\gamma(s)$ be an S-figure in E^3 and $s_0 \in S$. For the S-figure $(\gamma(s) - \gamma(s_0))$ following cases are possible: $rank(\gamma(s) - \gamma(s_0)) = 0$ or $rank(\gamma(s) - \gamma(s_0)) = m$, where $m \in N_3$.

Assume that $rank(\gamma(s) - \gamma(s_0)) = 0$ and $rank(\eta(s) - \eta(s_0)) = 0$. Then $\gamma(s) = \gamma(s_0), \forall s \in S$ and $\eta(s) = \eta(s_0), \forall s \in S$. In this case, it is obvious that $\gamma(s) \overset{MO(3,R)}{\sim} \eta(s)$. Moreover they are $MO(3,R)$-equivalent to the S-figure $\omega(s)$, where $\omega(s) = 0, \forall s \in S$.

Consider the case $rank(\gamma(s) - \gamma(s_0)) = rank(\eta(s) - \eta(s_0)) = 3$. In this case, there exists a subset $\{s_1, s_2, s_3\}$ of S such that vectors $\gamma(s_1) - \gamma(s_0), \gamma(s_2) - \gamma(s_0), \gamma(s_3) - \gamma(s_0)$ are linearly independent.

The following theorem follows from Proposition 4.1 and Theorem 3.1.

Theorem 4.1. *Let $\gamma(s)$ and $\eta(s)$ be S-figures in E^3 and $s_0 \in S$. Assume that $rank(\gamma(s) - \gamma(s_0)) = 3$ and a subset $\{s_1, s_2, s_3\}$ of $S \setminus Z(\gamma(s) - \gamma(s_0))$ such that vectors $\gamma(s_1) - \gamma(s_0), \gamma(s_2) - \gamma(s_0), \gamma(s_3) - \gamma(s_0)$ are linearly independent.*

Assume that $\gamma(s) \overset{MO(3,R)}{\sim} \eta(s), \forall s \in S$.Then following equalities hold for all $s \in S \setminus Z(\gamma(s) - \gamma(s_0))$, and for all $i \in N_3$:

$$\begin{cases} Z(\gamma(s) - \gamma(s_0)) = Z(\eta(s) - \eta(s_0)) \\ rank(\gamma(s) - \gamma(s_0)) = rank(\eta(s) - \eta(s_0)) \\ \langle \gamma(s_i) - \gamma(s_0), \gamma(s) - \gamma(s_0) \rangle = \langle \eta(s_i) - \eta(s_0), \eta(s) - \eta(s_0) \rangle. \end{cases} \tag{4.1}$$

Conversely, assume that the equalities (4.1) hold. Then the matrix $(\Lambda(\eta(s_1) - \eta(s_0), \eta(s_2) - \eta(s_0), \eta(s_3) - \eta(s_0)))^{-1}$ exists, the unique $F \in O(3,R)$ exists and the unique $a \in E^3$ exists such that $\eta(s) = F\gamma(s) + a, \forall s \in S$. In this case, F has following form: $F = ((\Lambda(\eta(s_1) - \eta(s_0), \eta(s_2) - \eta(s_0), \eta(s_3) - \eta(s_0)))^{-1} \cdot \Lambda(\gamma(s_1) - \gamma(s_0), \gamma(s_2) - \gamma(s_0), \gamma(s_3) - \gamma(s_0))$ and $a = \eta(s_0) - F\gamma(s_0)$.

Remark 4.1. By Proposition 4.1, Theorem 4.1 implies easy an analogue of Theorem 4.1 for the group $MSO(3,R)$. This result is omitted.

5. Complete systems of invariants of a parametric figure in the space-time $E^3 \times R$ for the groups $Gal_1(3,R)$ and $Gal_1^+(3,R)$

Let $E^3 \times R = \{(x,t)|x \in E^3, t \in R\}$ be space-time. Let S be a set such that it has at least two elements and R be the field of real numbers. Put $S \times R = \{(s,r)|s \in S, r \in R\}$.

Definition 5.1. A mapping $\gamma : S \times R \to E^3$ will be called $S \times R - figure$ in E^3. It is denoted as follows: $\gamma(s,t)$, where $s \in S, t \in R$.

Denote by $\Phi(S \times R, E^3)$ the set of all mappings of the $S \times R$ to E^3.

Definition 5.2. The subset $\{(\gamma(s,t),t)|t \in R, s \in S)\}$ of $E^3 \times R$, where $\gamma(s,t)$ is a $S \times R$-figure in E^3 will be called a motion of the set S in the space-time $E^3 \times R$.

Theorem 3.1 implies following theorem:

Theorem 5.1. *Let $\gamma(s,t)$ and $\eta(s,t)$ be $S \times R$-figures in E^3 such that $rank(\gamma(s,t)) = 3$. Assume that elements $(s_1,t_1),(s_2,t_2),(s_3,t_3)$ of the set $(S \times R) \setminus Z(\gamma(s,t))$ such that vectors $\gamma(s_1,t_1),\gamma(s_2,t_2),\gamma(s_3,t_3)$ are linearly independent in E^3.*

Assume that $\gamma(s,t) \overset{O(3,R)}{\sim} \eta(s,t), \forall(s,t) \in S \times R$.Then following equalities hold for all $(s,t) \in S \times R \setminus Z(\gamma(s,t))$ and for all $i \in N_3$:

$$\begin{cases} Z(\gamma(s,t)) = Z(\eta(s,t)) \\ rank(\gamma(s,t)) = rank(\eta(s,t)) \\ \langle \gamma(s_i,t_i), \gamma(s,t)\rangle = \langle \eta(s_i,t_i), \eta(s,t)\rangle. \end{cases} \tag{5.1}$$

Conversely, assume that the equalities (5.1) hold. Then the matrix $(\Lambda(\eta(s_1,t_1),\eta(s_2,t_2),\eta(s_3,t_3)))^{-1}$ exists and the unique $F \in O(3,R)$ exists such that $\eta(s,t) = F\gamma(s,t), \forall(s,t) \in S \times R$. In this case, F has following form: $F = ((\Lambda(\eta(s_1,t_1),\eta(s_2,t_2),\eta(s_3,t_3)))^{-1} \cdot \Lambda(\gamma(s_1,t_1),\gamma(s_2,t_2),\gamma(s_3),t_3)$.

Let $\gamma(s,t)$ be an $S \times R$-figure in E^3. For fixed $s_0 \in S$, the set $\{\gamma(s_0,t), t \in R\}$ will be called an $R - path$ in E^3.

Definition 5.3. Two $S \times R$-figures $\gamma(s,t)$ and $\eta(s,t)$ in E^3 are called $Gal_1(3,R)$-equivalent if there exists $g \in O(3,R)$, $a \in E^3, b \in E^3$ such that $\eta(s,t) = g\gamma(s,t) + ta + b, \forall s \in S, \forall t \in R$. In this case, we write $\gamma(s,t) \overset{Gal_1(3,R)}{\sim} \eta(s,t)$.

Definition 5.4. Two $S \times R$-figures $\gamma(s,t)$ and $\eta(s,t)$ in E^3 are called $Gal_1^+(3,R)$-equivalent if there exists $g \in SO(3,R)$, $a \in E^3, b \in E^3$ such that $\eta(s,t) = g\gamma(s,t) + ta + b, \forall s \in S, \forall t \in R$. In this case, we write $\gamma(s,t) \overset{Gal_1^+(3,R)}{\sim} \eta(s,t)$.

Definition 5.5. Two motions $(\gamma(s,t),t)$ and $(\eta(s,t),t)$ of S-figures $\gamma(s,t)$ and $\eta(s,t)$ in E^3 are called $Gal_1(3,R)$-equivalent if there exists $g \in O(3,R)$, $a,b \in E^3$ such that $(\eta(s,t),t) = (g\gamma(s,t) + ta + b, t), \forall s \in S, \forall t \in R$. In this case, we write $(\gamma(s,t),t) \overset{Gal_1(3,R)}{\sim} (\eta(s,t),t)$.

Definition 5.6. . Two motions $(\gamma(s,t),t)$ and $(\eta(s,t),t)$ of S-figures $\gamma(s,t)$ and $\eta(s,t)$ in E^3 are called $Gal_1^+(3,R)$-equivalent if there exists $g \in SO(3,R)$, $a,b \in E^3$ such that $(\eta(s,t),t) = (g\gamma(s,t) + ta + b, t), \forall s \in S, \forall t \in R$. In this case, we write $(\gamma(s,t),t) \overset{Gal_1^+(3,R)}{\sim} (\eta(s,t),t)$.

Definitions 5.3 and 5.5 imply that two $S \times R$-figures $\gamma(s,t)$ and $\eta(s,t)$ in E^3 are $Gal_1(3,R)$-equivalent if and only if two motions $(\gamma(s,t),t)$ and $(\eta(s,t),t)$ of S-figures $\gamma(s,t)$ and $\eta(s,t)$ in E^3 are $Gal_1(3,R)$-equivalent.

Definitions 5.4 and 5.6 imply that two $S \times R$-figures $\gamma(s,t)$ and $\eta(s,t)$ in E^3 are $Gal_1^+(3,R)$-equivalent if and only if two motions $(\gamma(s,t),t)$ and $(\eta(s,t),t)$ of $S \times R$-figures $\gamma(s,t)$ and $\eta(s,t)$ in E^3 are $Gal_1^+(3,R)$-equivalent.

Theorem 5.2. *Assume that $\gamma(s,t)$ and $\eta(s,t)$ be two $S \times R$-figures in E^3 and $s_0 \in S$ such that R-paths $\gamma(s_0,t)$ and $\eta(s_0,t)$ have following forms $\gamma(s_0,t) = c_1 + tc_2$ and $\eta(s_0,t) = d_1 + td_2$ for some vectors c_1, c_2, d_1, d_2 in E^3. Then*

$\gamma(s,t) \overset{Gal_1(3,R)}{\sim} \eta(s,t), \forall (s,t) \in S \times R$ if and only if $(\gamma(s,t) - \gamma(s_0,t)) \overset{O(3,R)}{\sim} (\eta(s,t) - \eta(s_0,t)), \forall (s,t) \in S \times R$. In the case $\gamma(s,t) \overset{Gal_1(3,R))}{\sim} \eta(s,t), \forall (s,t) \in S \times R$, there exist $F \in O(3,R)$ and vectors a_1, a_2 of E^3 such that $\eta(s,t) = F\gamma(s,t) + a_1 + ta_2, \forall (s,t) \in S \times R$, where $a_1 + ta_2 = \eta(s_0,t) - F\gamma(s_0,t) = (d_1 - Fc_1) + t(d_2 - Fc_2), \forall t \in R$.

Proof. $\Rightarrow$ Assume that $\gamma(s,t) \overset{Gal_1(3,R))}{\sim} \eta(s,t), \forall (s,t) \in S \times R$. Then there exists $F \in O(3,R)$ and vectors a_1, a_2 of E^3 such that $\eta(s,t) = F\gamma(s,t) + a_1 + ta_2, \forall (s,t) \in S \times R$. In particularly, for $s = s_0$, we have $\eta(s_0,t) = F\gamma(s_0,t) + a_1 + ta_2$. This equality implies $a_1 + ta_2 = \eta(s_0,t) - F\gamma(s_0,t)$. This equality and equalities $\eta(s,t) = F\gamma(s,t) + a_1 + ta_2, \forall (s,t) \in S \times R$ imply equalities $\eta(s,t) = F\gamma(s,t) + \eta(s_0,t) - F\gamma(s_0,t), \forall (s,t) \in S \times R$. These equalities imply equalities $\eta(s,t) - \eta(s_0,t) = F\gamma(s,t) - F\gamma(s_0,t) = F(\gamma(s,t) - \gamma(s_0,t)), \forall (s,t) \in S \times R$. These equalities imply $(\gamma(s,t) - \gamma(s_0)) \overset{O(3,R)}{\sim} (\eta(s,t) - \eta(s_0)), \forall (s,t) \in S \times R$.

$\Leftarrow$ Conversely, assume that $(\gamma(s,t) - \gamma(s_0,t)) \overset{O(3,R)}{\sim} (\eta(s,t) - \eta(s_0,t)), \forall (s,t) \in S \times R$. Then there exists $F \in O(3,R)$ such that $\eta(s,t) - \eta(s_0,t) = F(\gamma(s,t) - \gamma(s_0,t)), \forall (s,t) \in S \times R$. This equality implies following equality $\eta(s,t) = F\gamma(s,t) + \eta(s_0,t) - F\gamma(s_0,t), \forall (s,t) \in S \times R$. This equality and the above equalities $\gamma(s_0,t) = c_1 + tc_2$ and $\eta(s_0,t) = d_1 + td_2$ imply following equalities $\eta(s,t) = F\gamma(s,t) + d_1 + td_2 + F(c_1 + tc_2) = F\gamma(s,t) + (d_1 + Fc_1) + t(d_2 + Fc_2)$. Since $(d_1 + Fc_1)$ and $(d_1 + Fc_1)$ are vectors of E^3 the equality $\eta(s,t) = F\gamma(s,t) + (d_1 + Fc_1) + t(d_2 + Fc_2)$ means that $\gamma(s,t) \overset{Gal_1(3,R))}{\sim} \eta(s,t), \forall (s,t) \in S \times R$. □

Let $\gamma(s,t)$ and $\eta(s,t)$ be $S \times R$-figures in E^3 and $(s_0,t) \in S \times R$. By Theorem 5.2, $\gamma(s,t) \overset{Gal_1(3,R)}{\sim} \eta(s,t), \forall (s,t) \in S \times R$ if and only if $S \times R$-figures $\gamma(s,t) - \gamma(s_0,t)$ and $\eta(s,t) - \eta(s_0,t)$ are $O(3,R)$-equivalent on the set $S \times R$. Assume that $S \times R$-figures $\gamma(s,t) - \gamma(s_0,t)$ and $\eta(s,t) - \eta(s_0,t)$ are $O(3,R)$-equivalent on the set $S \times R$. Then $rank(\gamma(s,t) - \gamma(s_0,t)) = rank(\eta(s,t) - \eta(s_0,t))$.

Let $\gamma(s,t)$ be a $S \times R$-figure in E^3 and $(s_0,t) \in S \times R$. For the $S \times R$-figure $(\gamma(s,t) - \gamma(s_0,t))$ following cases are possible: $rank(\gamma(s,t) - \gamma(s_0,t)) = 0$ or $rank(\gamma(s,t) - \gamma(s_0,t)) = m$, where $m \in N_3$.

Assume that $rank(\gamma(s,t) - \gamma(s_0,t)) = 0$ and $rank(\eta(s,t) - \eta(s_0,t)) = 0$. Then $\gamma(s,t) = \gamma(s_0,t), \forall (s,t) \in S \times R$, and $\eta(s,t) = \eta(s_0,t), \forall (s,t) \in S \times R$. In this case, it is obvious that $\gamma(s,t) \overset{Gal_1(3,R)}{\sim} \eta(s,t)$. Moreover they are $Gal_1(3,R)$-equivalent to the $S \times R$-figure $\omega(s,t)$, where $\omega(s,t) = 0, \forall (s,t) \in S \times R$.

Below we consider the case $rank(\gamma(s,t) - \gamma(s_0,t)) = rank(\eta(s,t) - \eta(s_0,t)) = 3$. In this case, there exists a subset $\{(s_1,t_1),(s_2,t_2),(s_3,t_3)\}$ of S such that vectors $\gamma(s_1,t_1) - \gamma(s_0,t_1), \gamma(s_2,t_2) - \gamma(s_0,t_2), \gamma(s_3,t_3) - \gamma(s_0,t_3)$ are linearly independent.

The following theorem follows from Theorems 5.1, and 5.2.

Theorem 5.3. *Let $\gamma(s,t)$ and $\eta(s,t)$ be $S \times R$-figures in E^3 and $s_0 \in S$ such that R-paths $\gamma(s_0,t)$ and $\eta(s_0,t)$ have following forms $\gamma(s_0,t) = c_1 + tc_2$ and $\eta(s_0,t) = d_1 + td_2$ for some vectors c_1, c_2, d_1, d_2 in E^3. Assume that $rank(\gamma(s,t) - \gamma(s_0,t)) = 3$ and a subset $\{(s_1,t_1),(s_2,t_2),(s_3,t_3)\}$ of $(S \times R) \setminus Z(\gamma(s) - \gamma(s_0))$ such that vectors $\gamma(s_1,t_1) - \gamma(s_0,t_1), \gamma(s_2,t_2) - \gamma(s_0,t_2), \gamma(s_3,t_3) - \gamma(s_0,t_3)$ are linearly independent.*

Assume that $\gamma(s,t) \overset{Gal_1(3,R)}{\sim} \eta(s,t), \forall (s,t) \in S \times R$. Then, following equalities hold: for all $(s,t) \in (S \times R) \setminus Z(\gamma(s,t) - \gamma(s_0,t))$ and for all $i \in N_3$:

$$\begin{cases} Z(\gamma(s,t) - \gamma(s_0,t)) = Z(\eta(s,t) - \eta(s_0,t)) \\ rank(\gamma(s,t) - \gamma(s_0,t)) = rank(\eta(s,t) - \eta(s_0,t)) \\ \langle \gamma(s_i,t_i) - \gamma(s_0,t_i), \gamma(s,t) - \gamma(s_0,t) \rangle = \langle \eta(s_i,t_i) - \eta(s_0,t_i), \eta(s,t) - \eta(s_0,t) \rangle. \end{cases} \tag{5.2}$$

Conversely, assume that the equalities (5.2) hold. Then the matrix $(\Lambda(\eta(s_1,t_1) - \eta(s_0,t_1), \eta(s_2,t_2) - \eta(s_0,t_2), \eta(s_3,t_3) - \eta(s_0,t_3)))^{-1}$ exists, the unique $F \in O(3,R)$ exists and the unique function $a(t)$ exists such that $a(t) \in E^3, \forall t \in R$ and $\eta(s,t) = F\gamma(s,t) + a(t), \forall (s,t) \in S \times R$. In this case, F has following form: $F = ((\Lambda(\eta(s_1,t_1) - \eta(s_0,t_1), \eta(s_2,t_2) - \eta(s_0,t_2), \eta(s_3,t_3) - \eta(s_0,t_3)))^{-1} \cdot \Lambda(\gamma(s_1,t_1) - \gamma(s_0,t_1), \gamma(s_2,t_2) - \gamma(s_0,t_2), \gamma(s_3,t_3) - \gamma(s_0,t_3))$ and $a(t) = \eta(s_0,t) - F\gamma(s_0,t) = (d_1 - Fc_1) + t(d_2 - Fc_2)$.

Remark 5.1. Analogues of Theorems 5.2, and 5.3 for the group $Gal_1^+(3,R)$ also are true. They are omitted.

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Author's contributions

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