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Research Article

An Evaluation of Mathematics Teachers' Conceptual Understanding of Irrational Numbers

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Abstract

The objective of this study was to examine mathematics teachers' conceptual understanding of irrational numbers. Data were gathered from semi-structured interviews with eight primary school mathematics teachers in Central Anatolia, Turkey. The interviews carried on with the teachers lasted about 45 minutes. To be able to record the interviews, permissions were taken from the teachers. After the interviews had been completed, they were decoded and begun to be analyzed. The data were analyzed via content analysis method. At the stage during which the interview form was prepared, at first the studies in the literature and the books prepared by the Ministry of National Education (MNE) for the secondary and high schools were examined.The results show that teachers have difficulty defining and recognizing irrational numbers, placing them on the number line, and doing operations with them. The teachers gave intuitive answers instead of using formal mathematics knowledge. Teachers' knowledge about irrational numbers is insufficient and they have misconceptions. The teachers' definitions, knowing of irrational numbers, showing the exact place of irrational numbers on

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number line and the difficulties they faced while doing operations on irrational set are taken into consideration so as to recommend providing studies involving a large number of participants.

Keywords: *Mathematics teachers, irrational numbers, rational numbers, conceptual understanding*

Introduction

In daily life people use numbers to count, order, and measure (Sirotic, 2004). Numbers and their perception is the fundamental theme of mathematics teaching. From pre-school through high school, mathematics teaching is an intense, ongoing, and difficult process (Güler, Kar & Işık, 2012). For students, successfully learning mathematics means it must be emphasized as one of the basic fields of learning (Baki, 2008) in both elementary and high school. However, it is not easy to comprehend how mathematical concepts are formed (Sirotic, 2004). Students need more than basic counting skills, especially to understand advanced mathematics. They need to develop concepts of order, number patterns, number operations, and relations between numbers. A basic perception of numbers is required to understand values and order concepts (Güler, Kar & Işık, 2012).

In a report on school mathematics standards, National Council of Teachers of Mathematics (NCTM, 2000) indicated that for high school students to understand a number system, they should be able to compare its characteristics to other number systems. It is very important to have sufficient knowledge about real numbers to understand advanced mathematical concepts (Güven, Çekmez & Karataş, 2011). On the other hand, the strict hierarchy among number sets and the abstract structure of irrational numbers are difficulty for students (Fischbein, Jehiam & Cohen, 1995; Güven, Çekmez & Karataş, 2011; Kara & Delice, 2012; Sirotic, 2004; Sirotic & Zazkis, 2007). Knowledge about real numbers is related to grasping rational and irrational numbers, so understanding irrational numbers is vital for understanding rational ones (Güven, Çekmez & Karataş, 2011). However, irrational numbers are not emphasized in school mathematics (Fischbein, Jehiam & Cohen, 1995). In addition, there are limited studies about the understanding of irrational numbers (Sirotic, 2004; Sirotic & Zazkis, 2007), even though it is clear that students have misconceptions about irrational numbers (Adıgüzel, 2013; Arcavi, Bruckheimer & Ben-Zvi, 1987; Ercire, 2014; Fischbein, Jehiam & Cohen, 1995; Güven, Çekmez & Karataş, 2011; Kara & Delice, 2012; Peled & Hershkovitz, 1999; Sirotic, 2004; Sirotic & Zazkis, 2007; Temel & Eroğlu, 2014; Zazkis, 2005).

The Studies on Understanding Irrational Numbers

Studies on understanding irrational numbers have addressed the following topics: determining which elements are rational or irrational (Adıgüzel, 2013; Ercire, 2014; Fischbein, Jehiam & Cohen, 1995; Güven, Çekmez & Karataş, 2011; Kara & Delice, 2012; Peled & Hershkovitz, 1999; Sirotic, 2004; Temel & Eroğlu, 2014), defining rational and irrational numbers (Adıgüzel, 2013; Ercire, 2014; Fischbein, Jehiam & Cohen, 1995; Güven, Çekmez & Karataş, 2011; Kara & Delice, 2012; Sirotic, 2004), representing and placing irrational numbers on the number line (Peled & Hershkovitz, 1999; Sirotic, 2004; Sirotic & Zazkis, 2007), doing operations with irrational numbers (Adıgüzel, 2013; Ercire, 2014; Güven, Çekmez & Karataş, 2011), and misconceptions about irrational numbers (Arcavi, Bruckheimer & Ben-Zvi, 1987; Fischbein, Jehiam & Cohen, 1995). Fischbein, Jehiam, and Cohen (1995) found that high school students and prospective mathematics teachers had difficulties distinguishing whether numbers were rational, irrational, or real. In particular, they had difficulties with $-\frac{2}{7}$ and $0.12122\ldots$, especially when irrational numbers were introduced such as $\sqrt{2}, \sqrt{3},$ and π. On the contrary, Peled and Hershkovitz (1999) concluded that prospective mathematics teachers knew the definition of irrational numbers but had difficulty representing them. Also, Ercire, Narlı and Aksoy (2016) have notified the thought that a number can be both rational and irrational as a result of the the idea of the 8th and 9th grades students thinking all the irrational numbers can be real.

One study focused on representations of irrational numbers and their place on the number line (Sirotic & Zazkis, 2007). The results revealed that prospective teachers had difficulties with this. Only those who used the Pythagorean Theorem were able to find the exact place of irrational numbers on the number line. They had inconsistent ideas about irrational numbers from their formal and algorithmic knowledge. Similarly, Peled and Hershkovitz (1999) showed that prospective teachers had difficulties placing the following numbers on the number line: $\sqrt{4}, \frac{1}{100}, 0.12, 0.25, \frac{1}{9}, \pi, \sqrt{5}, 0.3333\ldots$, and especially the latter three. These problems were mainly because of infinite and repeated numbers after the decimal and because they had difficulty converting between decimals and irrational numbers such as $\frac{1}{3} = 0.3333\ldots$

Güven, Çekmez, and Karataş (2011) adopted an holistic approach that included 80 prospective teachers. They focused on their ability to define rational and irrational numbers, place them on the number line, and do operations with both. Most participants were not able to define them correctly and confused irrational numbers with complex numbers. First grade teachers misunderstood π as a rational number while fourth grade teachers knew it was irrational. Prospective teachers thought that irrational numbers were the same as root numbers without looking at whether the decimals were repeated. They generally gave correct answers about placing numbers on the number line using intuition. With respect to operations, prospective teachers incorrectly thought that the sum of two irrational numbers and their product were always irrational. Moreover, in his master study Ercire (2014) indicates that the 8th grade students of secondary school and the 9th grade students of high school have the wrong learnings by considering that the results of addition and multiplication of irrational numbers are to be irrational. Besides, it is mentioned that the students' misconceptions have increased when two irrational numbers are added, however; their answers displaying the result of operation would be rational have risen when two irrational numbers are multiplied considering the cases of the multiplication of the rooted expressions. Likewise, Adıgüzel (2013) in her master thesis researches that knowledge of the 8th grade students of secondary school and the prospective mathematics teachers about irrational numbers and misconceptions have increased. In the study, she mentions that most of the prospective teachers have the misconception that irrational numbers are closed according to the addition. When multiplication is considered, it is seen that most of the prospective teachers (thinking the result of multiplication of the same rooted expressions) are aware of the fact that the closeness feature is not ensured.

The most important conclusion was that prospective teachers thought of irrational and decimal numbers together (Arcavi, Bruckheimer & Ben-Zvi, 1987). Even though most knew when the irrational numbers had emerged historically, they did not know how they had been developed. They had the misunderstanding that decimal numbers were found before irrational numbers. According to the study, this situation resulted from not doing geometric evaluations of irrational numbers and only relating them to decimal numbers.

Irrational Numbers in Teaching Programs in Turkey

According to the teaching program, updated in 2013 by Turkey's Ministry of National Education (MNE), irrational numbers are a part of the "Numbers and Operations" unit, taught for the first time in the eighth grade. Eighth grade students are expected to be able to identify real numbers and relationships between rational and irrational numbers (MNE, 2013a). Let us determine what set of numbers the numbers below belong to, specifically, this is within the topic "Square Root Expressions." This topic states that, "the square roots of perfect squares should not be called rational numbers (they should not be written as the portion of two whole numbers).... π is introduced as an irrational number." The same issue is addressed in the teacher's book (MNE, 2015a), which also includes irrational numbers' historic relationship with geometry, guessing their approximate values, displaying them on the number line, their different representations, and their relationship with the other number sets. There are activities, examples, and problems.

Aşağıda verilen sayıların hangi sayı kümelerinin elemanı olduğunu belirleyelim.

$4 \ , \ -3 \ , \ 0 \ , \ \frac{1}{3} \ , \ 0{,}12 \ , \ \sqrt{25} \ , \ 2{,}012023034 \ldots \ , \ \sqrt{19}$

$\sqrt{7} = 2{,}645751311 \ldots \approx -2{,}6$ $\qquad -\sqrt{7} = -2{,}645751311 \ldots \approx -2{,}6$

$\sqrt{10} = 3{,}162277660 \ldots \approx 3{,}2$ $\qquad \sqrt{11} = 3{,}316624790 \ldots \approx 3{,}3$

$\sqrt{6} = 2{,}449489742 \ldots \approx 2{,}4$ $\qquad -\sqrt{6} = -2{,}449489742 \ldots \approx -2{,}4$

$2\sqrt{3} = 2 \cdot (1{,}732050807\ldots) \approx 3{,}464101615 \ldots \approx 3{,}5$

A number corresponds to each point taken on the number line. The corresponding number is a rational or irrational number.

$\sqrt{22} = \sqrt{2} \cdot \sqrt{11} = (1{,}414213562 \ldots) \cdot (3{,}316624790 \ldots) = 4{,}690415759 \ldots \approx 4{,}7$

The approximate values of these irrational numbers can be shown on the number line as follows.

Figure 1. The textbook explanation for the approximate values of irrational numbers and the example problem of placing them on the number line (MNE, 2015a)

The teaching program addresses irrational numbers for the second time in the ninth grade. In the unit "Equations and Inequalities," irrational and real number sets are explained in the topic "Real Numbers" (MNE, 2013b). For example, "$\sqrt{2}$ is proved not to be a rational number and it is placed on number line" and "the features of addition and multiplication with real numbers are examined." The ninth grade textbook (MNE, 2015b) includes the history of irrational numbers, their origins, their definitions, the proof of $\sqrt{2}$ as irrational, their place on the number line, their relationship with other number sets, and operations. At the eighth grade level, irrational numbers are defined as those that cannot be expressed as ratio of integers. At the ninth grade level, the definition is more detailed: expressed as decimals, irrational numbers are infinite and do not repeat. It is explained that π is not precisely equal to ratio estimates such as $\frac{22}{7}, \frac{25}{8}$, or $\frac{355}{113}$. The book also includes activities addressing whether operations done on irrational numbers are closed (see examples in the following section).

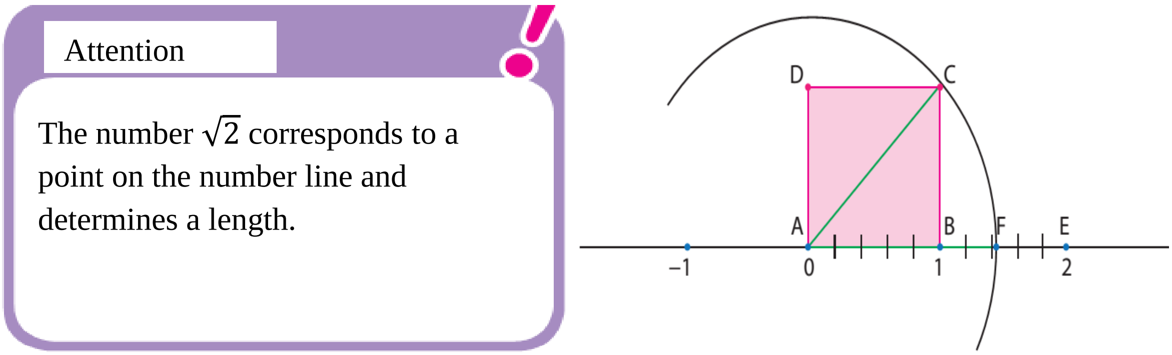

Figure 2. The textbook explanation for the exact place of irrational numbers on the number line (MNE, 2015b)

Prospective primary and high school teachers encounter irrational numbers several times during their undergraduate education. In the first year, they learn to define natural numbers according to the axioms of Peano within "General Mathematics" and then to form all other number sets (Güven, Çekmez & Karataş, 2011). In the primary and high school mathematics teacher's programs, irrational numbers are dealt with in detail in "Abstract Mathematics" and "Analysis I."

Aim and Research Questions

Previous research has showed that prospective teachers have difficulties and misconceptions with irrational numbers, which will directly affect students' learning. Thus, it is vital to find explore teachers' knowledge of irrational numbers, which is the goal of the present study. There are few studies on this topic, so the present study will contribute to identifying and correcting any misconceptions. The study was organized around the following questions.

- What are the mathematics teachers' cases of defining and knowing the irrational and rational numbers?
- What are the mathematics teachers' knowledge status and strategies (they use) about demonstrating the exact place of irrational numbers on the number line?
- What strategies do they use to place rational and irrational numbers on the number line?
- What are the mathematics teachers' knowledge whose reflections on practice about the closeness feature of the addition and multiplication with irrational and rational?

Research Methodology

Study and Participants

The present study was carried out during fall semester of the 2015–2016 academic year in Central Anatolia. All the elementary schools in the city center were informed about the study, where there were 124 elementary mathematics teachers. Eight volunteered for the study, three women and five men. To select participants, the maximum variation sampling method was used (Büyüköztürk et al., 2011). The group had a range of teaching experience. There were two teachers were in each of these four categories: 1–5, 5–10, 10–15 and 15 or more years' experience. They were made anonymous with the codes T1, T2, T3, etc.

Data Collection and Analysis

Data were collected through semi-structured interviews. To prepare the interview form, previous literature (Fischbein, Jehiam & Cohen, 1995; Güven, Çekmez & Karataş, 2011; Peled & Hershkovitz, 1999; Sirotic, 2004; Sirotic & Zazkis, 2007) and textbooks prepared were examined. The form was revised from the one used by Güven, Çekmez, and Karataş (2011), which focuses on three issues: "Defining rational and irrational numbers and classifying numbers as rational and irrational", "Finding the place of irrational numbers on the number line and writing other numbers between the two irrational and rational numbers" and "operations made with rational and irrational numbers." The form was reorganized and improved through consultations with three experts in mathematics education and five prospective mathematics teachers. Table 1 shows the form's categories and questions.

Table 1

Questions and Categories on the Interview Form

Categories	Questions
Defining and classifying rational and irrational numbers	Define the rational and irrational numbers. $A = \left\{-5, \frac{22}{7}, \frac{1}{3}, 3\sqrt{3}, 1.92713\ldots, 3+\sqrt{2}, 1.2555\ldots, \sqrt[3]{3}, \sqrt{36}, \frac{3}{\pi}, 3.14\right\}$ Write which numbers are rational and which are irrational in the set, with reasons for each.
Place irrational numbers on the number line, placing other numbers between two rational and irrational numbers	Show the exact place of $\sqrt{2}$ number on the number line. Can we always find an irrational number between two irrational numbers? Why? • Can you write an irrational number between $\sqrt{2}$ and $\sqrt{3}$? Can we always find a rational number between two irrational numbers? Why? • Can you write a rational number between $\sqrt{2}$ and $\sqrt{3}$ Can we always find another rational number between two rational numbers? Why? • Can you write a rational number between $\frac{1}{3}$ and $\frac{1}{2}$? Can we always find an irrational number between two rational numbers? Why?

Categories	Questions
Operations with rational and irrational numbers	• Can you write an irrational number between $\frac{1}{3}$ and $\frac{1}{2}$? Is the adding of two irrational numbers always an irrational number? Why? • If the answer is no, we have seen that the adding operation is not closed on the irrational numbers set. Thus, addition is not definable with the irrational numbers set. So how can we do this operation on the irrational numbers set? Explain. Is the multiplication of two irrational numbers always an irrational number? Why? • If the answer is no, we have seen that irrational numbers are not closed under multiplication set. Thus, multiplication is not definable for irrational numbers. So how can we do this operation on the irrational numbers set? Explain. Is the sum of two rational numbers always a rational number? Why? Is the multiplication of two rational numbers always a rational number? Why?

Data was collected over one month. One-on-one interviews were conducted on days that the teachers were at their schools. The 45-minute interviews were done in a quiet environment where teachers could think aloud. They were recorded with the permission of the teachers. Interviews were processed with a qualitative content analysis method. First, answers were coded and checked by two mathematics educators. Coding following methods in previous studies (Fischbein, Jehiam & Cohen, 1995; Güven, Çekmez & Karataş, 2011; Peled & Hershkovitz, 1999; Sirotic, 2004; Sirotic & Zazkis, 2007). Coded answers were summarized as charts and supported with direct quotations from the interviews.

Findings

Defining and Recognizing Rational and Irrational Numbers

Table 2 summarizes teachers' ability to define and recognize rational and irrational numbers. The table shows that they used inadequate language to describe the numbers. Only two teachers (T5 and T8) were able to use formal mathematical definitions and none were able to formally define irrational numbers. Their definitions were usually deficient because they did not mention that the numbers between the numerator and denominator are prime numbers. In defining rational numbers, they did not indicate which set the numerator and denominator belonged to. Irrational numbers were inadequately defined as "not rational." They also omitted the decimal representation of irrational numbers. These problems were common, even with the two teachers who used formal definitions. They also had a misconception that repeated decimals like $0.\overline{3}$ were irrational. The following are the definitions given by teachers T1, T5, and T6.

Table 2

Definitions Used by Each Teacher (T1, T2, etc.) for Rational and Irrational Numbers

Definition	Rational Numbers	Irrational Numbers
Deficient	T1, T2, T3, T4, T6, T7	T1, T2, T3, T4, T5, T6, T7, T8
Formal (Mathematical)	T5, T8	-

> **T1:** *Rational numbers can be written as* $a, b \in \mathbb{Z}$ *and* $\frac{a}{b}$ *when* $b \neq 0$. *We can define irrational numbers as numbers that can't be written in this form or numbers that can continue indefinitely after the decimal point.*
>
> **T5:** *Rational numbers that can be written as* $\left\{\frac{a}{b};\ a, b \in \mathbb{Z} \text{ and } b \neq 0\right\}$ *that meet the following condition:* $(a, b) = 1$. *We can define irrational numbers as numbers that can't be written this way.*
>
> **T6:** *Irrational numbers are the numbers that can't be rational ones. Irrational numbers written as* $\frac{p}{q}$ *are also written as* $(p, q) = 1$.

Table 3 presents numbers teachers defined as rational or irrational. They had misconceptions and hesitations about some numbers. They may have had difficulty because their representations of π and other irrational numbers were ineffective, for example, they treated $\frac{3}{\pi}$ and 1.92713... as rational numbers. There was also a misconception arising from using $\pi = \frac{22}{7}$ in their courses; they incorrectly though that $\frac{22}{7}$ was irrational. Some teachers were able to decide whether $\sqrt{36}$ and $3+\sqrt{2}$ were rational or irrational. Five teachers' comments are presented to show this.

Table 3

The Situations Revealing the Teachers Knowing the Given Numbers in a Number Set as Rational or Irrational

The ones who have done correctly	The ones who have done incorrectly	The reasons leading to incorrect answers
T1, T3, T4, T7	T2, T5, T6, T8	$\frac{22}{7}, \frac{3}{\pi}, \sqrt{36}, 3+\sqrt{2}, 1{,}92713 \ldots$

T1: *Numbers that can be written as* $\frac{a}{b}$ *are rational; others are irrational. But I am not sure about* $3+\sqrt{2}$*. The sum of rational and irrational numbers ... [thinking] ... This number can be rational. I'm not exactly sure about that.*

T2: *Let's write them as two sets, rational and irrational.* $A = \left\{-5, \frac{1}{3}, 1.255\ldots, \sqrt{36}, 3.14\right\}$, $A \subset \mathbb{Q}$ *leads to* $B \subset \mathbb{Q}'$ *to which leads to* $B = \left\{\frac{22}{7}, 3\sqrt{3}, 1.192713\ldots, 3+\sqrt{2}, \sqrt[3]{3}, \frac{3}{\pi}\right\}$*. I think of* $\frac{22}{7}$ *as* π*, so I have included it with irrational numbers because they are equal in some books. However, I do not know why this is the case. I am not quite sure if this number is irrational or rational.*

T3: *Let's write them as separate sets... Irrational numbers:* $\left\{3\sqrt{3}, 1.192713\ldots, 3+\sqrt{2}, \sqrt[3]{3}, \frac{3}{\pi}\right\}$*. These cannot be remembered or one can't take their square root... Let's write the ones that are rational... Rational numbers:* $\left\{-5, \frac{22}{7}, \frac{1}{3}, 1.2555\ldots, \sqrt{36}, 3.14\right\}$ *Here* $\frac{22}{7}$ *is not* π*. Since the children always ask this, I need to do the division ... I compare it to* π *to show them that they are not the same.*

T6: *Now... Rational numbers are like this:* $\left\{-5, \frac{22}{7}, \frac{1}{3}, 1.192713\ldots, \sqrt{36}, \frac{3}{\pi}, 3.14, 1.2555\ldots\right\}$*. Because even though the digits after the decimal point of* $1.192713\ldots$ *are different, there is a number in its denominator and it can be written as* $\frac{a}{b}$*. Thus, I think it is rational. In the same way, the number* $1.2555\ldots$ *is also rational. There is not a contradictory example* *with a prime number between them...* $\frac{3}{\pi}$ *... [thinking]... I think this number also goes here, it is relatively prime number between the numerator and denominator. But* π *that is infinite [thinking]... I think it is rational... I do not see anything contradictory with the characteristics of rational numbers. In any case, it could be made a relatively prime number by abbreviating. Irrational numbers:* $\{3\sqrt{3}, 3 + \sqrt{2}, \sqrt[3]{3}\}$*. These numbers are not exactly clear. Their roots can't be taken.*

T8: *The rational numbers:* $\left\{-5, \frac{1}{3}, 1.2555\ldots, \sqrt{36}, 3.14\right\}$ *...* $\frac{22}{7}$ *this needs dividing... We need to check if this number repeats or not. [doing the division operation]... it is as if* π*... ...* π *is also a number that is not rational but irrational. When divided it is [looking at* $\frac{22}{7} = 3.14$ *the teacher stops]... Therefore, this number is* π *...* π *is also an irrational number. For* $1.2555\ldots$ *the number 5 repeats... Since we can write the repeated numbers as rational, they are rational. Because we can take a square root of* $\sqrt{36}$*, it is rational. ...*3.14 *is a rational decimal number. Irrational numbers:* $\left\{\frac{22}{7}, 3\sqrt{3}, 1.192713\ldots, 3 + \sqrt{2}, \sqrt[3]{3}, \frac{3}{\pi}\right\}$ *Since* $3\sqrt{3}$ *is the product of a rational and irrational number, it is irrational.* $1.192713\ldots$*, since I do not seen a repeated decimal, it is irrational. ...*$3 + \sqrt{2}$ *is irrational because the sum of a rational and irrational number is also irrational. We can't take the square root of* $\sqrt[3]{3}$*, so it is irrational.* π *is irrational because the quotient of a rational number is also irrational. Sums, differences, products, and quotients of a rational and irrational number are always irrational.*

Teachers' Strategies for Placing Irrational Numbers on the Number Line and in Sequences

Table 4 summarizes teachers' strategies for placing numbers on the number line. Teachers used geometric approaches, approximate value or decimal representation, and estimated representation. Only teachers who used the geometric approach and the Pythagorean Theorem could show the exact place of $\sqrt{2}$. Teachers T4 and T7 correctly placed $\sqrt{2}$ on the number line using geometric length. Teachers who used the approximate or decimal value were unsuccessful. They state that only the approximate values $\sqrt{2}$ can be displayed but its exact place cannot be found. Similarly, teachers who used an estimated representation believed that the exact place of $\sqrt{2}$ could not be found and gave answers such as "somewhere between 1 and 2." Their comments follow.

Table 4

The strategies that teachers used to place $\sqrt{2}$ on the number line

Teachers	Themes	Teachers' Narratives
T4, T7	Geometric approach	T4: We can determine its exact place with the help of the Pythagorean Theorem and a circle with a $\sqrt{2}$ radius.
T5, T8	Approximate value/decimal representation approach	T5: When $\sqrt{2} \cong 1.4$, we can determine its approximate place on the number line.
T1, T2, T3, T6	Estimated representation	T2: It is a number between $1 < \sqrt{2} < 2$ but we can't determine its exact place on the number line.

T7: *I think of it this way. When we draw a square whose sides have a length of 1, the length of the diagonal is $\sqrt{2}$. Now we can draw a circle with $\sqrt{2}$ as the radius... When we think of it this way it exactly coincides with $\sqrt{2}$. How can we carry this to the number line?[thinking]... We can do it as an axis... [displaying it on the coordinate axis]... Now when we draw a circle whose center is at $(0,0)$, intercept of the circle is the exact place of $\sqrt{2}$ on the number line.*

T8: *The exact place of $\sqrt{2}$ on the number line ... No, it can't be [placed]... We can show it approximately. Namely, it is smaller than a certain rational number and larger than another rational number. Now how can we accommodate the number ... $\sqrt{2}$ between them? [thinking] $(1.1)^2 = 1.21$ ..., $(1.2)^2 = 1.44$ ..., $(1.4)^2 = 1.96$..., $(1.5)^2 = 2.25$... In other words, I can say that $\sqrt{2}$ is between 1.4 and 1.5 That is, I think it is something like this: $\left(\frac{14}{10}, \sqrt{2}, \frac{15}{10}\right)$.*

T1: *Since $\sqrt{2}$ is irrational and we can't exactly determine what the number is, we only can approximate its place on the number line. Where is its place on the number line? [thinking]... It must be somewhere between 1 and 2. Let's draw a coordinate plane... Since $\sqrt{2}$ is somewhere between 1 and 2, the x-intercept is between 1 and 2 but it is closer to 1. [drawing]... In other words, we can approximate its place. It is because in the definition of an irrational number the decimals continue infinitely after the decimal point, so it does not represent a point on the number line. Therefore, I think its exact place can't be found. We can only approximate its place.*

All teachers believed that another irrational number could be written between two irrational numbers. The strategies they used while to place a irrational number between two other irrational numbers are presented in Table 5. They used the following approaches: rational thinking, decimal number, root, and intuitive representation. When the answers of the teachers are examined, the decimal number approach is the most valid, in which an irrational number is placed between $\sqrt{2}$ and $\sqrt{3}$. Those using the rational thinking and root approaches did not place them correctly and had difficulties making approximate decimals of irrational numbers. Teacher T3 used intuitive representation, who was correct, but not at the intended interval. Teacher T3 mentioned that there is always another irrational number between two irrational numbers and used the example $\sqrt{2} < \sqrt{3} < \sqrt{7}$. Teachers' comments follow.

Table 5

Strategies Teachers Used to Place an Irrational Number between Two Irrational Numbers

Teachers	Themes	Sample Answers
T1, T2, T4	Rational thinking approach	T1: $\sqrt{2} < \frac{\sqrt{3}-\sqrt{2}}{2} < \sqrt{3}$, T4: $\sqrt{2} < \frac{\sqrt{5}}{2} < \sqrt{3}$
T6, T7, T8	Decimal number approach	T8: $\left(\frac{15}{10}, \frac{17}{10}\right)$ is between $\sqrt{2} = \left(\frac{14}{10}, \frac{15}{10}\right)$ and $\sqrt{3} = \left(\frac{17}{10}, \frac{18}{10}\right)$; there are infinite irrational numbers.
T5	Root using approach	T5: $\sqrt{2} < \sqrt[3]{9} < \sqrt{3}$
T3	Intuitive representation	T3: $\sqrt{2} < \sqrt{3} < \sqrt{7}$

T2: *For example, we can place* $\sqrt{\frac{5}{2}}$ *between* $\sqrt{2}$ *and* $\sqrt{3}$ *In other words,* $\sqrt{2} < \sqrt{\frac{5}{2}} < \sqrt{3}$ *because if there is a rational number between two rational numbers, there can be an irrational number between two irrational numbers. I think there is a root form of expressing these rational numbers.*

T7: *It can be written as* $\sqrt{2} +$ *... any sum not like* $\sqrt{3}$ *becomes irrational... For instance, I can add something like 0.*00 *and write several numbers as we approach* $\sqrt{3}$ *and then I can decide whether it is larger than* $\sqrt{3}$ *by squaring and comparing them. The approximate value... can be written like this:* $\sqrt{2} < \sqrt{2} + 0{,}01 < \sqrt{3}$.

T5: *Between* $\sqrt{2}$ *and* $\sqrt{3}$*... [thinking]...* $\sqrt{2}$, $\sqrt[3]{9}$, $\sqrt{3}$*...When we take the cubes of all of them, it can be seen that the number I have written stays between these two numbers.*

T3: *For example,* $\sqrt{3}$ *is between* $\sqrt{2}$ *and* $\sqrt{7}$.

Teachers indicated that another rational number can always be written between two other rational numbers. The strategies that teachers used are presented in Table 6. When the teachers wanted to place a rational number between $\frac{1}{3}$ and $\frac{1}{2}$ only teacher T7 gave an incorrect answer. Teachers had the following approaches: generalization, decimal number, finding the middle number, and intuitive representation. Unlike with irrational numbers, teachers had no difficulties placing rational numbers between two rational numbers, as indicated from their explanations.

Table 6

Strategies Used to Place a Rational Number between Two Rational Numbers

Teachers	Themes	Sample Answers
T1,T8	Generalization approach	T8: We find the number that is exactly between two rational numbers and can repeat this operation infinitively.
T2,T3,T6	Decimal number approach	T6: $\frac{1}{3} = 0.\overline{3}$ and $\frac{1}{2} = 0.5$ for $\frac{1}{3} < 0.4 < \frac{1}{2}$.
T4, T5	Finding middle number approach	T4: When we add these two numbers and divide them they become $\frac{1}{3} < \frac{5}{12} < \frac{1}{2}$.
T7	Intuitive representation	T7: For instance, I write $\frac{1}{3}$ between $\frac{1}{2}$ and $\frac{1}{4}$ …

T1: *For instance, we can write a rational number between $\frac{1}{2}$ and $\frac{1}{3}$. We can write another rational number between them. In fact, we can always write other numbers between them since there are infinite numbers in this space.*

T3: *It definitely can be written. When we think of them as decimal numbers: $\frac{1}{3} =$ 0.33 ... and $\frac{1}{2} = 0.5$; $0.4 = \frac{2}{5}$ is between these two numbers. Therefore, we can write a rational number between two rational numbers.*

T5: *A rational number between $\frac{1}{3}$ and $\frac{1}{2}$. Since $\frac{1}{3} = \frac{4}{12}$ and $\frac{1}{2} = \frac{6}{12}$, $\frac{5}{12}$ is between them. That is, $\frac{1}{3} < \frac{5}{12} < \frac{1}{2}$.*

T7: *I think about it like this. [showing it on the number line] For example, I can write $\frac{1}{3}$ between $\frac{1}{2}$ and $\frac{1}{4}$. It is easier to see this way. When we think like this, we can also write $\frac{1}{4}$ between $\frac{1}{3}$ and $\frac{1}{5}$.*

The teachers stated that an irrational number always can be placed between two rational numbers. The strategies they used are presented in Table 7. The strategies are based on the root, decimal number, and intuitive representation. Answers using the root and decimal number approaches were correct. Answers based on the root approach place $\frac{1}{\sqrt{5}}$ between $\frac{1}{3}$ and $\frac{1}{2}$. Answers based on the decimal number approach used irrational numbers with infinite and non-repeating decimals to place them in order: $\frac{1}{3} < 0.358\ldots < \frac{1}{2}$. The intuitive approach did provide a correct answer. These three approaches are illustrated with comments by teachers T8, T7, and T4.

Table 7

Strategies Used to Place an Irrational Number between Two Rational Numbers

Teachers	Themes	Sample Answers
T5, T6, T8	Root approach	T5: Between $\frac{1}{3}$ and $\frac{1}{2}$ … To illustrate, $\frac{1}{2} = \frac{1}{\sqrt{4}}$ and we can place $\frac{1}{\sqrt{5}}$ between them.
T2, T3, T7	Decimal number approach	T3: When $\frac{1}{3} = 0.\overline{3}$ and $\frac{1}{2} = 0.5$, a number that is infinite and irregularly continuous like $\frac{1}{3} <$ $0.358 \ldots < \frac{1}{2}$ can be placed.
T1, T4	Intuitive representation	T1: In other words, since there are infinite numbers at this interval, we can always write other numbers at this interval.

T8: *Between* $\frac{1}{3}$ *and* $\frac{1}{2}$*... We look at the denominators and the numerators ... if we can find a number that is larger than 2 and smaller than 3, it will be when we write this number as the denominator as long as its numerator becomes 1. Our number that is larger than 2 because* $2^2 = 4$ *so* $\left(\sqrt{5}\right)^2 = 5$ *;* $3 > \sqrt{5} > 2$ *. Thus,* $\frac{1}{3} < \frac{1}{\sqrt{5}} < \frac{1}{2}$*. Because the inequality whose root we have taken is being kept as* $\frac{1}{9} < \frac{1}{5} < \frac{1}{4}$*. That is, numbers like these can be found.*

T7: *If we think that* $\frac{1}{2} = 0.5$ *and* $\frac{1}{3} = 0.3333$ *... then we can write a number that is infinite and continuous after the decimal. I can see it more easily when thinking of it as decimal.*

T4: *There must be... But, can we find it exactly?... It will take some thinking...*

The teachers indicated that a rational number can always be written between two irrational numbers, and Table 8 shows that strategies they used. They were based on approximate value, the root, and intuition. The approximate value approach correctly shows that $\sqrt{2} < 1.6 < \sqrt{3}$ and the root approach expresses it as $\sqrt{2} < \frac{3}{2} < \sqrt{3}$. Teachers using intuition estimated incorrect answers such as "a rational number that one that is located between 1 and 2." Illustrative answers are from teachers T5, T6, and T7.

Table 8

Strategies Used to Place a Rational Number between Two Irrational Numbers

Teachers	Themes	Sample Answers
T1, T3, T5, T8	Approximate value approach	T1: To illustrate, since $\sqrt{2} = 1.41$ and $\sqrt{3} = 1.7$ there is definitely a rational number at this interval, for instance, $\frac{3}{2}$.
T2, T7	Root approach	T2: I can write a rational number like $2 < x^2 < 3$ and if I take their roots…. 1.6 is between them.
T4, T6	Intuitive representation	T4: Finding the approximate number of these numbers, a rational number can be squeezed between them. Both are between 1 and 2… again a rational number between 1 and 2 can be at this interval.

T5: *There is always a rational number between two irrational numbers. For example, if I try to place a rational number between $\sqrt{2}$ and $\sqrt{3}$. They are roughly $\sqrt{2} = 1.4$... and $\sqrt{3} = 1.7$. I can also write $\frac{3}{2}$. That is, $\sqrt{2} < \frac{3}{2} < \sqrt{3}$.*

T7: *We can do that as well. In other words, I am looking for x in the expression $\sqrt{2} < x < \sqrt{3}$. If we take their roots it becomes $2 < x^2 < 3$. Now I am looking for a rational number that has a root between 2 and 3. This could be a number like 1.7.*

T6: *I don't know... [thinking]... Since I have not been able to determine their approximate values, the rational numbers I am going to write can't be between these two numbers... I do not remember anything. For instance, if I know the approximate values like $\sqrt{2} = 1.2$ and $\sqrt{3} = 1.8$... I can place a rational number in this interval. Because I know their approximate values, I can say that it is a rational number between 1 and 2.*

Teachers' Thoughts on Operations with Irrational and Rational Numbers

The last research problem examines teachers' thoughts on whether irrational and rational number sets are closed under addition and multiplication, as shown in Table 9. Three teachers (T1, T7, and T8) indicated that irrational number sets are not closed under addition while five teachers (T2, T3, T4, T5, T6) indicated that they are. They used examples such as $\sqrt{2} + \left(-\sqrt{2}\right) = 0$ to show that addition is not always closed. These two positions are illustrated with comments from teachers T1 and T3.

Table 9

Teachers' Answers about Adding Two Irrational Numbers

Teachers	Addition Result	Sample Answers
T1, T7, T8	Not always irrational. T1 and T7 do not know why sums are not always irrational but T8 has indicated this is a kind of exception.	T7: For example $\sqrt{2} - \sqrt{2} = 0$ or $(\sqrt{5} + 2) - \sqrt{5} = 2$. When it is like this, it's not always be irrational. That is, there are cases that it can be irrational…
T2, T3, T4, T5, T6	Always irrational.	T5: No exception comes to mind… To illustrate, I think since $\sqrt{2}$ and $\sqrt{3}$ are irrational, their sums are also irrational, like $\sqrt{2} + \sqrt{3}$. i.e., is it possible to get a rational number from the addition of two irrational numbers? [Thinking]… I don't think so…

> **T1:** *The addition of two irrational numbers is always irrational... [thinking]... For instance, when we add $\sqrt{2}$ and $\sqrt{3}$, $\sqrt{2} + \sqrt{3}$ is an irrational number. Well, if we add two other irrational numbers, can the result become irrational?... [thinking]... So, if we use $\sqrt{2}$ and $-\sqrt{2}$ as irrational numbers, the sum is $\sqrt{2} + \left(-\sqrt{2}\right) = 0$, and 0 is rational. So we have an example. Hence, the sum of two irrational numbers is not always irrational.*
>
> **M:** *We have seen that irrational numbers are not closed under addition. Therefore, addition is not definable with the irrational number set. So how can we do this operation on the irrational number set? Explain.*

> **T1:** *But the irrational number set is closed. However, it is not closed under addition ... [thinking]... Yes, if it does not have the characteristics of being closed, then the operation is not definable. I wonder what we do if we accept it as closed... But, the result is not always irrational... Thus, either $-\sqrt{2}$ isn't irrational or there is a contradiction ... I don't know... It should be closed but we are adding two irrational numbers... But, we do this when there is no closure, I have not thought about the reason... There is a conflict here but I do not know...*

> **T3:** *Is the addition of two irrational numbers always irrational?... [thinking]... Yes, I am thinking that of an example... To illustrate, $\sqrt{2} + \sqrt{2} = 2\sqrt{2}$. Similarly, if we try it with $\sqrt{3}$, it always becomes irrational. Thus, it is always irrational.*

The teachers also stated that the product of two irrational numbers was not always irrational. They presented examples such as $\sqrt{2} \cdot \sqrt{2} = 2$. Their answers with roots provided correct answers. However, when asked how this operation is carried out, they had difficulty answering. Their answers about multiplying irrational numbers are presented in Table 10, which were coded as: think of it as a real number, think of it as an exponential/root number, acceptable, and do not know. Only teachers T2 and T3 knew how this operation was done. Both explained that the product of two irrational numbers is defined by real numbers and this is a closed operation that can be defined. Responses from teachers T2, T1, and T6 are presented below.

Table 10

Teachers' Answers about Multiplying Two Irrational Numbers

Teachers	Reason	Sample Answers
T2, T3	Think of real numbers	T3: We are doing it on real numbers… When we multiply them. We don't think of them as irrational numbers, as more general real numbers.
T1, T5	Think as exponential/root numbers	T5: Multiplication can be done with exponential numbers. Of course, we do not teach it like this… When we teach the root numbers, we teach that if the degrees of the root are equal, we can multiply the numbers. However, if there is a need to explain it, $\sqrt{3} = 3^{\frac{1}{2}}$ and since the exponents are added when their roots are equal, we can do $3^{\frac{1}{2}}.3^{\frac{1}{2}} = 3^{\frac{1}{2}+\frac{1}{2}} = 3^1 = 3$. We can get rid of the root this way. When the exponential expression is a whole

Teachers	Reason	Sample Answers
		number, it can't be a root number.
T8	Acceptable	T8: I think closure is vital both for rational and irrational numbers. There is no closure property for irrational numbers in this case… I think it's not important to trying to show that irrational numbers are closed … Because it doesn't provide for any operations. [Thinking]… I think we are doing it with exceptions.
T4, T6, T7	I don't know	T7: [Thinking]… I don't know the answer to this question…. In other words, the addition and multiplication are not closed but I do not know how we do it.

T2: *[thinking]... That's right, it's not closed... I hadn't noticed that before... that is, I don't know how the operations are done but... Or it could be like this: if we think of all these numbers as real, there's no problem. Because every irrational number is a real number at the same time.*

T1: *[thinking]... We can do that by making use of the properties of the root numbers... For example,* $\sqrt{2} \cdot \sqrt{8} = \sqrt{2 \cdot 8} = \sqrt{16} = 4$. *It can be written as root expressions with a common root... Here we can infer the conclusion... that irrational numbers are not always closed under multiplication.*

T6: *[thinking]... Hence, an irrational number is the root of a rational number... I wonder if we think of it this way... Or do we do it by ignoring not-closed operations for irrational numbers... I do not know ... But I think we make use of the root expression... It is because when a square root is taken, our number can be taken out... So, I think we can do our operation... Namely, nothing comes to my mind except for this...*

Despite difficulties, teachers correctly indicated that rational numbers are closed under addition, as presented in Table 11. There are no exceptions and the result is always in the form $\frac{a}{b}$ with whole number numerators and denominators. Thus, teachers use rational numbers frequently and think about the operations on this set in more detail, as shown in statements by teachers T2 and T4.

Table 11

Teachers' Answers about the Sum of Two Rational Numbers

Teachers	Reasons	Sample Answers
T2, T3, T8	There aren't any exceptions	T3: We can’t find an example of a number that is not rational as a result of adding two rational numbers.
T4, T5, T6	The result is always in the form of $\frac{a}{b}$	T5: Since we equalize the denominator and add directly while doing adding rational numbers, the result is always in the form $\frac{a}{b}$ as rational numbers.
T7	The numerator and denominator are always whole numbers	T7: We work with whole numbers… Because the numerator and denominator are always whole numbers for the rational numbers. Thus, I find a rational number in the end. As all whole number are rational, the result is always rational.
T1	Rational numbers are closed	T1: It is because rational numbers are closed under addition.

T2: *Yes, the sum of two rational numbers is a rational number. There is no exception to this... For instance* $\frac{1}{2}+\frac{1}{3}=\frac{1}{6}$ *and it is valid for all rational numbers.*

T4: *Its sum is always rational because when two rational numbers are added together, the denominators are equalized and the numerators are added... A rational number in the form of* $\frac{a}{b}$ *is always the result.*

Similarly, teachers knew that the product of two rational numbers was always rational, as presented at Table 12. The result is always $\frac{a}{b}$ as rational numbers are closed under multiplication, and teachers knew how to verify this, as shown by comments by teachers T3 and T5.

Table 12

Teachers' Answers about the Multiplying Two Rational Numbers

Teachers	Reasons	Sample Answers
T2, T3, T8	There aren't any exceptions	T8: I think it is rational. I have not seen any exceptions. If we multiply by 0, the result is 0… That is a rational number.
T4, T5, T6, T7	The result is always in the form of $\frac{a}{b}$	T4: When two rational numbers are multiplied, the numerators and denominators are multiplied and again there is a rational number like $\frac{a}{b}$.
T1	Rational numbers are closed	T1: Now since the numerator and denominator of rational numbers are whole numbers, their products and sums are whole numbers. Namely, $\frac{a}{b}.\frac{c}{d}=\frac{a\cdot c}{b\cdot d}$. Hence, closure is verified and the result is rational.

T3: *Yes, the product of two rational numbers is a rational number. We can't find any exception.*

T5: *The product of two rational numbers is always rational. There is not really any exception... Because when multiplying, the numerator is multiplied with the numerator and the denominator is multiplied with denominator. The result is again in the form of a rational number* $\frac{a}{b}$.

Discussion and Conclusion

According to the interviews, mathematics teachers were not able to correctly define rational and irrational numbers. They focused on the representations of the rational numbers. Teachers who were able to use formal mathematical definitions clearly expressed the properties of the numerator and denominator as well as the representations of rational numbers. Shortcomings in definitions were identified for irrational numbers. Teachers often defined this set as "the numbers that are not rational." Since the rational numbers were not correctly defined, defining irrational numbers in this way is inadequate. In defining irrational numbers, they did not mention the decimal representation or showed incomplete knowledge, which are also some of the most important results from this research. For example, they incorrectly described the decimal representation of irrational numbers as "the numbers that are infinite after the decimal," so students will learn this incorrect information. Teachers did not indicate that numbers after the decimal are non-repeating and infinite, which is the correct way to identify an irrational number. These results confirm other studies with current and prospective teachers about inadequate or incorrect definitions of rational and irrational numbers (Arcavi, Bruckheimer & Ben-Zvi, 1987; Fischbein, Jehiam & Cohen, 1995; Güven, Çekmez & Karataş, 2011).

Unclear definitions of irrational and rational numbers were the underlying cause of the misconceptions when discussing numbers such as $\frac{22}{7}, \frac{3}{\pi}, \sqrt{36}, 3+\sqrt{2},$ and $1.92713...$. For example, teachers using the estimate of $\pi = \frac{22}{7}$ during classroom exercises created confusion, as π is irrational and $\frac{22}{7}$ is rational, which has been reported in other studies (Fischbein, Jehiam & Cohen, 1995; Güven, Çekmez & Karataş, 2011; Sirotic, 2004; Temel & Eroğlu, 2014). Güven, Çekmez and Karataş (2011) stated that students' thoughts on the representations of numbers affected their ideas about the nature of the numbers, which also seems to be true for the teachers. For example, teacher T6 knew the π number is irrational but said that $\frac{3}{\pi}$ was rational, because of too much focus on fractional representations of rational numbers. Another teacher said $1.92713...$ was rational due to a lack of knowledge about the decimal representation of irrational numbers. Teachers were also unclear about numbers expressed as roots, such as hesitations about $\sqrt{36}$ being irrational or rational. Similar results have been obtained from studies with primary school students (Temel & Eroğlu, 2014), high school students (Fischbein, Jehiam & Cohen, 1995; Kara & Delice, 2012) and prospective teachers (Güven, Çekmez & Karataş, 2011). Hence teachers should be careful with these concepts at all grades, as participants in various studies repeated the same mistakes. Güven, Çekmez and Karataş (2011) stated that diversifying examples about different representations of irrational numbers would help students recognize irrational numbers better, which would probably be useful for teachers as well.

In other studies, prospective teachers had difficulties determining the place of irrational numbers on the number line and did so intuitively (Peled & Hershkovitz, 1999; Sirotic, 2004; Sirotic & Zazkis, 2007). Teachers also had difficulties determining the exact place of $\sqrt{2}$ on the number line. Other studies have shown that teachers were not able to relate irrational numbers to the concept of geometric length and relied on intuition. Sirotic and Zazkis (2007) remarked that irrational numbers are quite difficult but in school programs it is mentioned that each point on the real number axis has an opposite. Turkish eighth and ninth grade textbooks include activities about locating irrational numbers on the number line, but this study shows that teachers' explanations are incorrect. To locate irrational numbers correctly, the geometric approach is recommended, that is, using the Pythagorean Theorem (Güven, Çekmez & Karataş, 2011; Peled & Hershkovitz, 1999; Sirotic, 2004; Sirotic & Zazkis, 2007).

In this study, teachers had difficulty placing rational and irrational numbers in sequence. Only the teachers who used the decimal number approach were able to do so correctly; other strategies produced incorrect answers. Teachers were relatively comfortable placing irrational numbers between two rational numbers, especially those using root and decimal number approaches. To place a rational number between two irrational numbers, the approximate value and root approaches were effective. Institutive answers were sometimes correct but explanations were weak. Teachers were more comfortable placing rational numbers between two rational numbers and said they have used this in the classroom. They have even used this approach to incorrectly explain how to locate irrational numbers on the number line. Sirotic and Zazkis (2007) indicated that prospective teachers focus on decimal representations to do operations with irrational numbers, which does not contribute to learning the concept of irrational numbers. Teachers used decimal representations in many problems with irrational numbers. Güven, Çekmez and Karataş (2011) concluded that despite the fact that prospective teachers have high correct answer rates at placing the numbers on the number line, they use intuitive expressions.

Güler, Kar and Işık (2012) showed that prospective teachers had difficulty explaining why irrational numbers are not closed under multiplication. The present study showed a similar pattern. Five teachers (T2–6) stated that the sum of irrational numbers was "always irrational." These teachers suggested there is no exception to this, so they incorrectly thought that irrational numbers were closed under addition. This can be a great obstacle for their students and doing textbook exercises. These teachers may transfer their misconceptions to their students. Teachers who said "it is not always irrational" were unsure what in what cases would be acceptable.

Teachers said that the product of two irrational numbers was not always irrational, so they have a better understanding of multiplication that addition with irrational numbers. Their experience with root expressions is probably the reason for this. Even so, they have difficulties explaining how to multiply irrational numbers. Only two teachers (T2 and T3) said they could do this operation as it is defined in the real numbers set under which multiplication is closed. Teachers had conceptual shortcomings about operations on irrational numbers, but were much clearer on rational numbers. Teachers were aware that both sums and products of two rational numbers are rational and that rational numbers are closed under both addition and multiplication. This is probably a result of more extensive study of rational numbers and using them in classroom.

This study was limited to semi-structured interviews with eight mathematics teachers. Additional studies with more participants are recommended. This study showed that instead of using formal mathematical knowledge about irrational numbers, teachers gave intuitive answers. It shows they have difficulties defining and recognizing irrational numbers, representing their exact place on the number line, and adding and multiplying with them. These misconceptions are important to address because they may transfer their difficulties and fallacies to their students. This will also increase the value of textbook exercises. It is hoped this study will be useful to mathematics teachers and students who are working with irrational numbers.

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