International Electronic Journal of Algebra
Volume 18 (2015) 92-106

ON UNIFORMITY IN LATTICES OF CLASSES OF MODULES DEFINED BY CLOSURE PROPERTIES

Hugo Alberto Rincón-Mejía and José Patricio Sánchez-Hernández

Received: 5 May 2015; Revised: 24 June 2015
Communicated by Abdullah Harmancı

Abstract. In this work we consider some big lattices of classes of modules defined by closure properties such as being closed under taking submodules, quotients, injective hulls, projective covers, products and directs sums. We obtain some results and characterizations of rings when we assume that those big lattices of classes of modules are atomic or uniform.

Mathematics Subject Classification (2010): 16D80, 16S99, 06B75, 06D15, 06F99
Keywords: Big lattices, uniform lattices, socle of a lattice, left local ring, semi-artinian ring, max ring

1. Introduction

Of great interest has been the study of several lattices associated to a ring. In particular, the study of big lattices of classes of modules defined by closure properties. The lattice aspects of those lattices have consequence in the features of the ring. In this work we consider some big lattices of classes of modules defined by closure properties such as being closed under taking submodules, quotients, injective hulls, projective covers, products and directs sums.

We use the notation $\mathbb{L}_{\leq}$, $\mathbb{L}_{/}$, $\mathbb{L}_{\oplus}$, $\mathbb{L}_{\prod}$, $\mathbb{L}_{ext}$, $\mathbb{L}_{E}$ and $\mathbb{L}_{P}$ described as follows. We denoted by $\mathbb{L}_{\leq}$ the class of classes of modules closed under taking submodules, by $\mathbb{L}_{/}$ the class of classes of modules closed under taking quotients, by $\mathbb{L}_{\oplus}$ the class of classes of modules closed under taking direct sums, by $\mathbb{L}_{\prod}$ the class of classes of modules closed under taking products, by $\mathbb{L}_{ext}$ the class of classes of modules closed under takings extensions, by $\mathbb{L}_{P}$ the class of classes of modules closed under taking projective covers and by $\mathbb{L}_{E}$ the class of classes of modules closed under taking injective hulls. In general, if ρ is a set of closure properties, we denote by $\mathbb{L}_{\rho}$ the class of classes of modules closed with respect to the closure properties in ρ. If ρ denotes a subset of $\{\leq, /, \oplus, \prod, ext, E, P\}$, we should notice that $\mathbb{L}_{\rho}$ becomes a big lattice ordered by class inclusion with meet given by class intersection. There are many lattices of module classes of this kind which are interesting to study by themselves. In this paper we will study when $\mathbb{L}_\rho$ is whether atomic or uniform.

In all of the following, R will denote an associative ring with identity. We say that an R-module M is compressible if for all nonzero submodule N there exists a monomorphism $\alpha : M \to N$. In a similar way, we say that an R-module M is cocompressible if for every nonzero quotient N there exists an epimorphism $\alpha : N \to M$. A ring R is called a V-ring if every simple module is injective. We denote by R-simp a complete set of representatives of isomorphism classes of simple modules. Let M, N be R-modules. Recall that N is a subquotient of M if N embeds in a quotient of M, equivalently if N is a quotient of a submodule of M. Thus each nonzero M has a simple subquotient. We refer the reader to [1] and [4], where the notation for lattices of classes of modules defined by closure properties are introduced, and for notation, terminology and for concepts on lattices, torsion theory and for information about lattices of modules classes, respectively.

2. Preliminaries

Definition 2.1. If ρ is a set of closure properties and if $\mathcal{C}$ is a class of R-modules, we denote by $\xi_\rho(\mathcal{C})$ the least class of modules containing $\mathcal{C}$ and being closed under the properties in ρ.

Definition 2.2. Let $\mathbb{L}$ be a lattice. We say that $\mathbb{L}$ is *bounded* if there exist $\underline{0}, \underline{1} \in \mathbb{L}$ such that for every $\mathcal{D} \in \mathbb{L}$, $\underline{0} \leq \mathcal{D}$ and $\mathcal{D} \leq \underline{1}$.

Definition 2.3. In a lattice $\mathbb{L}$ with least element $\underline{0}$, we say that an element $a' \in \mathbb{L}$ is a *pseudocomplement* of $a \in \mathbb{L}$ if a' is maximal such that $a \wedge a' = \underline{0}$. $\mathbb{L}$ is a *pseudocomplemented lattice* if all of its elements has a pseudocomplement.

Definition 2.4. Let $\mathbb{L}$ be a bounded lattice. We say that $0 \neq \mathcal{C} \in \mathbb{L}$ is an *atom* if for all $0 \neq \mathcal{D} \in \mathbb{L}$ such that $\mathcal{D} \leq \mathcal{C}$ we have that $\mathcal{D} = \mathcal{C}$.

It is clear that if $\mathcal{C}$ is an atom of $\mathbb{L}_\rho$ and $0 \neq M \in \mathcal{C}$, then $\xi_\rho(M) = \mathcal{C}$.

Definition 2.5. Let $\mathbb{L}$ be a bounded lattice. We define the *socle* of $\mathbb{L}$ as the join of all atoms of $\mathbb{L}$ and we denote it by $soc(\mathbb{L})$.

We describe socles for several big lattices of classes of modules.

Theorem 2.6. *The following statements hold for a ring R.*

(a) $soc(\mathbb{L}_{\leq}) = \{M \in R\text{-}Mod \mid M \text{ is compressible}\}$.
(b) $soc(\mathbb{L}_{/}) = \{M \in R\text{-}Mod \mid M \text{ is cocompressible}\}$.
(c) $soc(\mathbb{L}_{\oplus}) = \underline{0}$.

(d) $soc(\mathbb{L}_{\prod}) = \underline{0}$.
(e) $soc(\mathbb{L}_E) = \{E \in R\text{-}Mod \mid E \textit{ is injective}\}$.
(f) $soc(\mathbb{L}_P) = \{P \in R\text{-}Mod \mid P \textit{ is projective or has not a projective cover}\}$.
(g) $soc(\mathbb{L}_{\leq,/}) = \{S \in R\text{-}Mod \mid S \textit{ is simple}\} \cup \{0\}$.

Proof. (a) The atoms in $\mathbb{L}_{\leq}$ are indeed $\xi_{\leq}(M)$ with M being a compressible. In fact, if $\mathcal{C}$ is an atom and if $0 \neq M \in \mathcal{C}$, then $\xi_{\leq}(M) = \mathcal{C}$. If $0 \neq N \leq M$, then $\xi_{\leq}(N) = \mathcal{C}$ and then $M \in \xi_{\leq}(N)$. So M embeds in N, thus M is compressible. Hence, every $M \in soc(\mathbb{L}_{\leq})$ is compressible. On the other hand, it is clear that every compressible module generates an atom in $\mathbb{L}_{\leq}$. Hence, $soc(\mathbb{L}_{\leq}) = \{M \in R\text{-}Mod \mid M \text{ is compressible}\}$.

(b) The proof is similar to the proof of (a).

(c) If $\mathcal{C} \in \mathbb{L}_{\oplus}$ is an atom, then $\xi_{\oplus}(M) = \mathcal{C}$, for each nonzero module M belonging to $\mathcal{C}$. There exists a set X such that $|M| < |M^{(X)}|$. Then $\xi_{\oplus}(M) = \mathcal{C} = \xi_{\oplus}(M^{(X)})$. Thus $M \in \xi_{\oplus}(M^{(X)})$ and so $M \cong (M^{(X)})^{(Y)}$ for some non void set Y. But $|M| < |M^{(X)}| \leq |(M^{(X)})^{(Y)}|$, a contradiction. Hence, $\mathbb{L}_{\oplus}$ has no atoms.

(d) It is similar to the proof of (c).

(e) If $\mathcal{C}$ is an atom of $\mathbb{L}_E$, then $\mathcal{C} = \xi_E(E) = \{0, E\}$ for some E injective. Thus the atoms are in a one to one correspondence with isomorphism classes of injective modules. Hence $soc(\mathbb{L}_E) = \{E \in R\text{-}Mod \mid E \text{ is injective}\}$.

(f) Let $\mathcal{C}$ be an atom of $\mathbb{L}_P$. If $0 \neq M \in \mathcal{C}$, then $\xi_P(M) = \mathcal{C}$. We have two cases: M has a projective cover or M does not have it. If the former case happens, then we have that $\{0, P(M)\} = \xi_P(P(M)) = \xi_P(M)$. Thus $M = P(M)$. On the other hand, if M does not have a projective cover, then $\mathcal{C} = \xi_P(M) = \{0, M\}$. Hence, the atoms are determined by projective modules or modules which do not have projective cover.

(g) If $\mathcal{C}$ is an atom of $\mathbb{L}_{\leq,/}$, then $\mathcal{C} = \{0, S\}$ for some simple module S, because, as we have already noted, every nonzero module has a simple module as a subquotient. On the other hand, every simple module generates an atom. Thus $soc(\mathbb{L}_{\leq,/}) = \{S \in R\text{-}Mod \mid S \text{ is simple}\}$. □

Definition 2.7. Let $\mathbb{L}$ be a complete lattice. We say that $\mathcal{C} \in \mathbb{L}$ is *essential* if for all $\underline{0} \neq \mathcal{D} \in \mathbb{L}$ we have that $\mathcal{D} \wedge \mathcal{C} \neq \underline{0}$. We denote $\mathfrak{E}(\mathbb{L}) = \bigwedge\{\mathcal{E} \in \mathbb{L} \mid \mathcal{E} \text{ is essential in } \mathbb{L}\}$.

If $\mathcal{E} \in \mathbb{L}$ is essential, it is clear that for all atom $\mathcal{C}$ in $\mathbb{L}$, we have that $\mathcal{C} \leq \mathcal{E}$. Thus $soc(\mathbb{L}) \leq \mathcal{E}$ for every essential element $\mathcal{E}$ in $\mathbb{L}$.

A class of R-modules is called *stable* when it is closed under taking injective hulls.

Proposition 2.8. *The following statements hold for a ring R.*

(a) $soc(\mathbb{L}_E) = \mathfrak{E}(\mathbb{L}_E)$.

(b) $soc(\mathbb{L}_P) = \mathfrak{E}(\mathbb{L}_P)$.

(c) $soc(\mathbb{L}_\oplus) = \mathfrak{E}(\mathbb{L}_\oplus)$.

(d) $soc(\mathbb{L}_{\prod}) = \mathfrak{E}(\mathbb{L}_{\prod})$.

Proof. (a) It suffices to show that $soc(\mathbb{L}_E)$ is essential. Let $\underline{0} \neq \mathcal{C} \in \mathbb{L}_E$. Since $\mathcal{C}$ is stable, then there exists an injective module $0 \neq E$ in $\mathcal{C}$. Thus $\mathcal{C} \wedge soc(\mathbb{L}_E) \neq \underline{0}$. Hence, $soc(\mathbb{L}_E)$ is essential and hence, $soc(\mathbb{L}_E) = \mathfrak{E}(\mathbb{L}_E)$.

(b) The proof is similar to the proof of (a).

(c) Let $0 \neq M \in R\text{-}Mod$ and define $\mathcal{C}_M = \{0 \neq L \in R\text{-}Mod | L$ is not a direct summand of $M\}$. We claim that $\mathcal{C}_M \in \mathbb{L}_\oplus$ and $\mathcal{C}_M$ is essential. Let $\{L_i\}_{i \in I}$ be a family of modules in $\mathcal{C}_M$. If $\oplus_{i \in I} L_i$ were a direct summand of M, then L_i would be a direct summand of M for each $i \in I$, a contradiction. Thus $\oplus_{i \in I} L_i \in \mathcal{C}_M$ and thus $\mathcal{C}_M$ is closed under direct sums. Now we show that $\mathcal{C}_M$ is essential in $\mathbb{L}_\oplus$. Let $\mathcal{D} \in \mathbb{L}_\oplus$ and $0 \neq N \in \mathcal{D}$. If $N \in \mathcal{C}_M$, then we have finished.

Now suppose that $N \notin \mathcal{C}_M$. There exists a set X such that $|M| < |N^{(X)}|$. Thus $N^{(X)}$ is not a direct summand of M. So $0 \neq N^{(X)} \in \mathcal{D} \wedge \mathcal{C}_M$. Therefore $\mathcal{C}_M$ is essential in $\mathbb{L}_\oplus$.

Since $M \notin \mathcal{C}_M$, $M \notin \mathfrak{E}(\mathbb{L}_\oplus)$. Thus $soc(\mathbb{L}_\oplus) = \underline{0} = \mathfrak{E}(\mathbb{L}_\oplus)$.

(d) The proof is similar to the proof of (c). □

Definition 2.9. Let R be a ring and let σ and ρ be such that $\sigma \subseteq \rho \subseteq \{\leq, /, \oplus, \prod, ext, E, P\}$. We say that $\mathcal{C} \in \mathbb{L}_\sigma$ is $\sigma - \rho$ *essential* if for all $\underline{0} \neq \mathcal{D} \in \mathbb{L}_\sigma$ such that $\mathcal{D} \leq \xi_\rho(\mathcal{C})$ then we have $\mathcal{C} \wedge \mathcal{D} \neq \underline{0}$. $\mathbb{L}_\sigma$ is called $\sigma - \rho$ *uniform* if each nonzero $\mathcal{C} \in \mathbb{L}_\sigma$ is $\sigma - \rho$ essential.

Recall that a bounded lattice $\mathbb{L}$ is atomic if for all nonzero $\mathcal{C} \in \mathbb{L}$ there exists an atom $\mathcal{D} \in \mathbb{L}$ such that $\mathcal{D} \leq \mathcal{C}$.

Theorem 2.10. *Let $\mathbb{L}_\sigma$ be a $\sigma - \rho$ uniform lattice. If $\mathbb{L}_\sigma$ is atomic, then $\mathbb{L}_\rho$ is atomic.*

Proof. Suppose that $\mathbb{L}_\sigma$ is atomic and let $\underline{0} \neq \mathcal{C} \in \mathbb{L}_\rho$. Then $\mathcal{C} \in \mathbb{L}_\sigma$ and there exists an atom $\mathcal{D} \in \mathbb{L}_\sigma$ such thath $\mathcal{D} \leq \mathcal{C}$. So $\xi_\rho(\mathcal{D}) \leq \mathcal{C}$. We claim that $\xi_\rho(\mathcal{D})$ is an atom in $\mathbb{L}_\rho$. Indeed, let $\underline{0} \neq \mathcal{A} \in \mathbb{L}_\rho$ such that $\mathcal{A} \leq \xi_\rho(\mathcal{D})$. Therefore, $\mathcal{A} \in \mathbb{L}_\sigma$. Since $\mathbb{L}_\sigma$ is $\sigma - \rho$ uniform, $\mathcal{D} \wedge \mathcal{A} \neq \underline{0}$ and since $\mathcal{D}$ is an atom in $\mathbb{L}_\sigma$, we have $\mathcal{D} = \mathcal{D} \wedge \mathcal{A} \leq \mathcal{A}$. Thus $\xi_\rho(\mathcal{D}) \leq \mathcal{A}$ and $\xi_\rho(\mathcal{D}) = \mathcal{A}$. Then $\xi_\rho(\mathcal{D})$ is an atom in $\mathbb{L}_\rho$. Hence $\mathbb{L}_\rho$ is atomic. □

3. Atoms and uniformity

Recall that a bounded lattice $\mathbb{L}$ is called *uniform* if the meet of any nonzero elements of $\mathbb{L}$ is nonzero.

Theorem 3.1. *If $\mathbb{L}_\sigma$ is $\sigma - \rho$ uniform, then $\mathbb{L}_\sigma$ is uniform if and only if $\mathbb{L}_\rho$ is uniform.*

Proof. Suppose that $\mathbb{L}_\sigma$ is uniform and let $\underline{0} \neq \mathcal{C}, \mathcal{D} \in \mathbb{L}_\rho$. Then $\mathcal{C}, \mathcal{D} \in \mathbb{L}_\sigma$. By hypothesis, we have that $\mathcal{C} \wedge \mathcal{D} \neq \underline{0}$. Hence, $\mathbb{L}_\rho$ is uniform.

Conversely, let $\mathbb{L}_\rho$ be uniform and take $\underline{0} \neq \mathcal{C}, \mathcal{D} \in \mathbb{L}_\sigma$. Thus $\xi_\rho(\mathcal{C}) \wedge \xi_\rho(\mathcal{D}) \neq \underline{0}$. Let $\mathcal{A} = \xi_\rho(\mathcal{C}) \wedge \xi_\rho(\mathcal{D})$. Since $\mathcal{A} \in \mathbb{L}_\sigma$, $\mathcal{A} \leq \xi_\rho(\mathcal{C})$ and $\mathbb{L}_\sigma$ is $\sigma - \rho$ uniform, we have that $\mathcal{A} \wedge \mathcal{C} \neq \underline{0}$. Then since $\mathcal{A} \wedge \mathcal{C} \leq \mathcal{A} \leq \xi_\rho(\mathcal{D})$ and since $\mathcal{D}$ is $\sigma - \rho$ essential, we have that $\mathcal{A} \wedge \mathcal{C} \wedge \mathcal{D} \neq 0$. So $\mathcal{C} \wedge \mathcal{D} \neq 0$. Hence, $\mathbb{L}_\sigma$ is uniform. $\square$

Lemma 3.2. *$\mathbb{L}_\leq$ is $\{\leq\} - \{\leq, \oplus\}$ uniform.*

Proof. Let $0 \neq \mathcal{C} \in \mathbb{L}_\leq$ and $0 \neq N \in \xi_{\leq,\oplus}(\mathcal{C})$. We claim that $\mathcal{C} \wedge \xi_\leq(N) \neq \underline{0}$. Since $N \in \xi_{\leq,\oplus}(\mathcal{C})$, there exists a monomorphism $N \overset{\alpha}{\rightarrowtail} \oplus_{i\in I}\{M_i\}$ where $M_i \in \mathcal{C}$ $\forall i \in I$, so that, for every $0 \neq x \in N$, $\alpha(x)$ can be written as $m_{i_0} + \cdots + m_{i_k}$. Let us choose $0 \neq x \in N$ such that k be least. This choice yields $(0 : m_{i_s}) = (0 : m_{i_r})$ for all $0 \leq r, s \leq k$. So $(0 : x) = (0 : m_{i_0})$.

Therefore

$$Rx \cong \frac{R}{(0 : x)} = \frac{R}{(0 : m_{i_0})} \cong Rm_{i_0}.$$

So $Rx \overset{\cong}{\rightarrowtail} Rm_{i_0} \hookrightarrow M_{i_0}$ and $Rx \leq N$. Then $\mathcal{C} \wedge \xi_\leq(N) \neq \underline{0}$. Hence $\mathbb{L}_\leq$ is $\{\leq\} - \{\leq, \oplus\}$ uniform. $\square$

A ring R with an additive endomorphism D satisfying $D(ab) = D(a)b + aD(b)$ is called a *differential ring* and we say that R is a ring with derivation D.

Let k be a field with derivation D and let $k[y, D]$ denote the ring of differential polynomials in the indeterminate y with coefficients in k, i.e., the additive group of $k[y, D]$ is the additive group of the ring of polynomials in the indeterminate y with coefficients in k, and multiplication in $k[y, D]$ is defined by $ya = ay + D(a)$ for all $a \in K$, and its consequences.

Let k be a field of characteristic 0 and D be a derivation of k. A result due to Kolchin ([6, Theorem, p.771]) asserts the existence of a field $k \subseteq U$ and a derivation $\overline{D}$ of U extending D such that the equation

$$p(x, \overline{D}(x), \cdots, \overline{D}^{(n)}(x)) = 0 \qquad n \text{ arbitrary},$$

has a solution $\xi \in U$ for all $p(X) \in U[X_1, ..., X_{n+1}] - U$. Furthermore, every homogeneous linear differential equation in $\overline{D}$ over U has a nontrivial solution in U. Such a field U is called a universal extension of k or a universal differential field, see [6, Theorem, p. 771].

Let k be a universal differential field with derivation D. We denote $R = K[y, D]$, see [3, p. 76]. We have the next theorem for R [3, Theorem 1.4].

Theorem 3.3. *The ring R has the following properties.*

(1) *R is a principal right and left ideal domain.*
(2) *R is a simple ring (and $soc(R) = 0$).*
(3) *R is a right V-ring.*
(4) *R is not a field.*
(5) *R has, up to isomorphism, a unique simple right R-module.*

Example 3.4. *$\mathbb{L}_{\leq}$ is not always $\{\leq\} - \{\leq, \prod\}$ uniform. Let R^* denote the opposite ring of R. Then by Theorem 3.3, R^* is a left local left V-ring and $soc(R^*) = 0$. Let S be the simple module. Since R^* is a left V-ring, then we have that S is an injective module. As R^*-simp has only one member, which is an injective module, then we have that R^* embeds in a product S^X. Thus $R^* \in \xi_{\leq,\prod}(S)$. Since $soc(R^*) = 0$, we have that $\xi_{\leq}(R^*) \wedge \xi_{\leq}(S) = \underline{0}$. Hence $\mathbb{L}_{\leq}$ is not $\{\leq\} - \{\leq, \prod\}$ uniform.*

Lemma 3.5. *$\mathbb{L}_{\leq,/}$ is $\{\leq, /\} - \{\leq, /, \oplus\}$ uniform.*

Proof. Let $\underline{0} \neq \mathcal{C} \in \mathbb{L}_{\leq,/}$ and $0 \neq N \in \xi_{\leq,/,\oplus}(\mathcal{C})$. We claim that $\mathcal{C} \wedge \xi_{\leq,/}(N) \neq \underline{0}$. By hypothesis there exists an epimorphism α such as in the following diagram:

$$\begin{array}{ccc} \oplus_{i \in I} M_i & \xrightarrow{\alpha} & M \\ & & \uparrow \\ & & N \end{array}$$

where $M_i \in \mathcal{C}$, $\forall i \in I$.

Choose $0 \neq x \in N$ such that $x = \alpha(m_{i_0} + \cdots + m_{i_k})$ with k least. Then $(0 : m_{i_s}) = (0 : m_{i_r})$ for all $0 \leq r, s \leq k$. Thus $(0 : x) = (0 : m_{i_0})$.

Therefore

$$Rx \cong \frac{R}{(0 : x)} \cong \frac{R}{(0 : m_{i_0})} \cong Rm_{i_0}.$$

So $Rx \overset{\cong}{\rightarrowtail} Rm_{i_0} \hookrightarrow M_{i_0}$ and $Rx \leq N$. Then $\mathcal{C} \wedge \xi_{\leq,/}(N) \neq \underline{0}$. Hence $\mathbb{L}_{\leq,/}$ is $\{\leq, /\} - \{\leq, /, \oplus\}$ uniform. □

Let $\mathcal{C}, \mathcal{D}$ be two classes of R-modules. We define

$$(\mathcal{C} : \mathcal{D}) = \left\{ L \in R\text{-}Mod \;\middle|\; \begin{array}{l} \text{there exist } M \in \mathcal{C},\ N \in \mathcal{D} \text{ and an exact} \\ \text{sequence } 0 \to M \to L \to N \to 0 \end{array} \right\}.$$

A class of R-modules closed under isomorphism is called with zero if it contains the zero module.

Remark 3.6. *Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ be three classes with zero of R-modules. Then $((\mathcal{C} : \mathcal{D}) : \mathcal{E}) = (\mathcal{C} : (\mathcal{D} : \mathcal{E}))$, see* [2, *Proposition* 2.2].

We can define recursively: $\mathcal{C}^{:0} = \{0\}$ and $\mathcal{C}^{:(n+1)} = (\mathcal{C} : \mathcal{C}^{:n})$.

Remark 3.7. *Let $\mathcal{C}$ be a class with zero of R-modules. Then $\xi_{ext}(\mathcal{C}) = \bigcup_{n\in\mathbb{N}} \mathcal{C}^{:n}$.*

Proof. First let us take $\bigcup_{n\in\mathbb{N}} \mathcal{C}^{:n} \in \mathbb{L}_{ext}$. Suppose that

$$0 \to N \to M \to L \to 0$$

is exact with $N, L \in \bigcup_{n\in\mathbb{N}} \mathcal{C}^{:n}$. Then $N \in \mathcal{C}^{:n}$ and $L \in \mathcal{C}^{:m}$ for some $n, m \in \mathbb{N}$. This means that $M \in (\mathcal{C}^{n} : \mathcal{C}^{:m}) = \mathcal{C}^{:(n+m)}$ by Remark 3.6. Thus $M \in \bigcup_{n\in\mathbb{N}} \mathcal{C}^{:n}$.

Now, if $\mathcal{C} \subseteq \mathcal{D} \in \mathbb{L}_{ext}$, then $(\mathcal{C} : \mathcal{C}) \subseteq (\mathcal{D} : \mathcal{D}) = \mathcal{D}$, and, inductively, $\mathcal{C}^{:n} \subseteq \mathcal{D}$ for all $n \in \mathbb{N}$. Hence $\bigcup_{n\in\mathbb{N}} \mathcal{C}^{:n} \subseteq \mathcal{D}$. □

It is easy to see that if $\mathcal{C}$ is an hereditary class, then $\xi_{ext}(\mathcal{C})$ is also an hereditary class. Similarly, if $\mathcal{C}$ is a cohereditary class, then $\xi_{ext}(\mathcal{C})$ is also a cohereditary class. Hence $\mathcal{C} \in \mathbb{L}_{\leq,/}$ implies that $\xi_{ext}(\mathcal{C}) \in \mathbb{L}_{\leq,/,ext}$.

Lemma 3.8. *$\mathbb{L}_{\leq}$ is $\{\leq\} - \{\leq, ext\}$ uniform.*

Proof. Let $\underline{0} \neq \mathcal{C} \in \mathbb{L}_{\leq}$. It suffices to show that for every nonzero $M \in \xi_{\leq,ext}(\mathcal{C})$ we have that $\mathcal{C} \wedge \xi_{\leq}(M) \neq \underline{0}$. It follows from Remark 3.7 that $\xi_{\leq,ext}(\mathcal{C}) = \xi_{ext}(\mathcal{C}) = \bigcup_{n\in\mathbb{N}} \mathcal{C}^{:n}$.

If $0 \neq M \in \bigcup_{n\in\mathbb{N}} \mathcal{C}^{:n}$, then $M \in \mathcal{C}^{:n}$, for some $n \in \mathbb{N}$. Let us choose the least n with this property.

If $n = 1$, then $M \in \mathcal{C}$.

If $n > 1$, then $M \in (\mathcal{C} : \mathcal{C}^{:(n-1)})$, and thus there exists an exact sequence

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

with $L \in \mathcal{C}$ and $N \in \mathcal{C}^{:(n-1)}$.

Thus $0 \neq L \in \mathcal{C} \wedge \xi_{\leq}(M)$. Hence $\mathbb{L}_{\leq}$ is $\{\leq\} - \{\leq, ext\}$ uniform. □

Lemma 3.9. *$\mathbb{L}_{/}$ is $\{/\} - \{/, ext\}$ uniform.*

Proof. The proof is similar to that of Lemma 3.8. □

Lemma 3.10. $\mathbb{L}_{\leq,/}$ *is* $\{\leq,/\}-\{\leq,/,ext\}$ *uniform.*

Proof. Let $\underline{0}\neq\mathcal{C}\in\mathbb{L}_{\leq,/}$. It is easy to see that $\xi_{\leq,/,ext}(\mathcal{C})=\xi_{ext}(\mathcal{C})=\bigcup_{n\in\mathbb{N}}\mathcal{C}^{:n}$. Suppose that $0\neq N$ is such that $N\in\xi_{\leq,/,ext}(\mathcal{C})$. We will show that $\xi_{\leq,/}(N)\wedge\mathcal{C}\neq\underline{0}$. As $N\in\bigcup_{n\in\mathbb{N}}\mathcal{C}^{:n}$, let us take the least n such that $N\in\mathcal{C}^{:n}$.

If $n=1$, then $N\in\mathcal{C}$ and we are done.

If $n>1$, there exists an exact sequence

$$0\to L\xrightarrow{f}N\xrightarrow{g}M\to 0$$

with $L\in\mathcal{C}$ and $M\in\mathcal{C}^{:(n-1)}$.

Then L is a nonzero subquotient of N in $\mathcal{C}$. Thus $0\neq L\in\mathcal{C}\wedge\xi_{\leq,/}(N)$. Hence $\mathbb{L}_{\leq,/}$ is $\{\leq,/\}-\{\leq,/,ext\}$ uniform. □

Remark 3.11. ([2, *Lemma* 1.7; 6, *Sec.* 2]) $\mathbb{L}_{\leq,/}$ *is a pseudocomplemented big lattice. The pseudocomplement for* $\mathcal{C}\in\mathbb{L}_{\leq,/}$ *is given by*

$$\mathcal{C}^{\perp_{\{\leq,/\}}}=\{M\in R\text{-}Mod\mid M \text{ does not have a nonzero subquotient in } \mathcal{C}\}.$$

Moreover, $\mathcal{C}^{\perp_{\{\leq,/\}}}$ *belongs to* $\mathbb{L}_{\leq,/,\oplus,ext}$.

Lemma 3.12. $\mathbb{L}_{\leq,/}$ *is* $\{\leq,/\}-\{\leq,/,\oplus,ext\}$ *uniform.*

Proof. Let $\underline{0}\neq\mathcal{C}\in\mathbb{L}_{\leq,/}$ and $0\neq N\in\xi_{\leq,/,\oplus,ext}(\mathcal{C})$. We claim that $\mathcal{C}\wedge\xi_{\leq,/}(N)\neq\underline{0}$. As a consequence of Remark 3.11, $(\mathcal{C}^{\perp_{\{\leq,/\}}})^{\perp_{\{\leq,/\}}}$ belongs to $\mathbb{L}_{\leq,/,\oplus,ext}$ and also it contains $\mathcal{C}$. Then it is clear that $\xi_{\leq,/,\oplus,ext}(\mathcal{C})\leq(\mathcal{C}^{\perp_{\{\leq,/\}}})^{\perp_{\{\leq,/\}}}$.

It is easy to see that $(\mathcal{C}^{\perp_{\{\leq,/\}}})^{\perp_{\{\leq,/\}}}$ consists precisely of the modules such that each one of its nonzero subquotient has a nonzero subquotient in $\mathcal{C}$.

In particular, since N is a nonzero subquotient of itself, then N has a nonzero subquotient belonging to $\mathcal{C}$. Then $\mathcal{C}\wedge\xi_{\leq,/}(N)\neq\underline{0}$. Hence $\mathbb{L}_{\leq,/}$ is $\{\leq,/\}-\{\leq,/,\oplus,ext\}$ uniform. □

The following result is a direct consequence of Theorem 2.6.

Theorem 3.13. *Let R be a ring. Then the following statements hold.*

(1) $\mathbb{L}_{\leq}$ *is atomic if and only if each nonzero* $M\in R\text{-}Mod$ *has a compressible submodule.*
(2) $\mathbb{L}_{/}$ *is atomic if and only if each nonzero* $M\in R\text{-}Mod$ *has a cocompressible quotient.*
(3) $\mathbb{L}_{\oplus}$ *and* $\mathbb{L}_{\prod}$ *are never atomic.*
(4) $\mathbb{L}_{E}$ *and* $\mathbb{L}_{P}$ *are always atomic.*

Lemma 3.14. *For each ring R,* $\mathbb{L}_{\leq,/}$ *is atomic.*

Proof. It follows from the fact that every nonzero module has a nonzero simple subquotient. □

Lemma 3.15. *For each ring R, $\mathbb{L}_{\leq,/,\oplus}$ is atomic.*

Proof. By Lemma 3.5, we have that $\mathbb{L}_{\leq,/}$ is $\{\leq,/\}-\{\leq,/,\oplus\}$ uniform. Then by Theorem 2.10, $\mathbb{L}_{\leq,/,\oplus}$ is atomic. □

Lemma 3.16. *For each ring R, both $\mathbb{L}_{\leq,/,ext}$ and $\mathbb{L}_{\leq,/,\oplus,ext}$ are atomic.*

Proof. The proof is similar to that of Lemma 3.15. □

Theorem 3.17. *The following statements are equivalent for a ring R.*

(i) *$\mathbb{L}_{\leq}$ is uniform.*
(ii) *R is left local and left semi-artinian.*

Proof. Suppose that $\mathbb{L}_{\leq}$ is uniform. Let S, S' be simple modules. Thus $\{0,\ S\}$, $\{0,\ S'\} \in \mathbb{L}_{\leq}$. By hypothesis, $\{0, S\} \wedge \{0, S'\} \neq \underline{0}$. So $\{0, S\} = \{0, S'\}$ and hence $S \cong S'$. Then R is a left local ring.

Let $0 \neq M \in R\text{-}Mod$. Then $\xi_{\leq}(M) \wedge \{0, S\} \neq \underline{0}$. Therefore $S \in \xi_{\leq}(M)$. Thus S embeds in M. Hence R is left semi-artinian.

Conversely, suppose that R is left local and left semi-artinian. It suffices to show that $\xi_{\leq}(N) \wedge \xi_{\leq}(M) \neq \underline{0}$ for any nonzero $N, M \in R\text{-}Mod$. But this follows from the fact that as R is left local and left semi-artinian, then a copy of the simple module $_RS$ embeds both in M and in N. Thus $S \in \xi_{\leq}(N) \wedge \xi_{\leq}(M)$. □

Theorem 3.18. *The following statements are equivalent for a ring R.*

(i) *$\mathbb{L}_{\leq,ext}$ is uniform.*
(ii) *R is left local and left semi-artinian.*

Proof. By Lemma 3.8, we have that $\mathbb{L}_{\leq}$ is $\{\leq,\}-\{\leq, ext\}$ uniform. Then by Theorem 3.1, $\mathbb{L}_{\leq,ext}$ is uniform if and only if $\mathbb{L}_{\leq}$ is uniform. But from Theorem 3.17 we have that $\mathbb{L}_{\leq}$ is uniform if and only if R is left local and left semi-artinian. Hence $\mathbb{L}_{\leq,ext}$ is uniform if and only if R is left local and left semi-artinian. □

Theorem 3.19. *The following statements are equivalent for a ring R.*

(i) *$\mathbb{L}_{\leq,\oplus}$ is uniform.*
(ii) *R is left local and left semi-artinian.*

Proof. The proof is similar to that of Theorem 3.18. □

Theorem 3.20. *The following statements are equivalent for a ring R.*

(i) *$\mathbb{L}_{/}$ is uniform.*

(ii) *R is left max and left local.*

Proof. Suppose that $\mathbb{L}_{/}$ is uniform. Let S, S' be simple modules. Thus $\{0, S\}$,$\{0, S'\} \in \mathbb{L}_{/}$. Then by hypothesis, $\{0, S\} \wedge \{0, S'\} \neq \underline{0}$. Thus $\{0, S\} = \{0, S'\}$ and $S \cong S'$. Hence R is left local.

Let $0 \neq M \in R\text{-}Mod$. Then $\xi_{/}(M) \wedge \{0, S\} \neq \underline{0}$. Therefore $S \in \xi_{/}(M)$. Thus there exists an epimorphism $M \twoheadrightarrow S$. Then there exists $N \leq M$ such that $M/N \cong S$. This implies that N is a maximal submodule of M. Hence R is left max ring.

Conversely, suppose that R is left local and left max. Let $0 \neq N, M \in R\text{-}Mod$. Since R is left max, there exist maximal submodules $M_1 \leq M$ and $N_1 \leq N$ respectively. Therefore $M/M_1 \cong S \cong N/N_1$, because R is left local. Thus $0 \neq S \in \xi_{/}(M) \wedge \xi_{/}(N)$. Hence, $\mathbb{L}_{/}$ is uniform. □

Theorem 3.21. *The following statements are equivalent for a ring R.*

(i) $\mathbb{L}_{/,ext}$ *is uniform.*

(ii) *R is left max and left local.*

Proof. It follows from Theorem 3.20 and Lemma 3.9. □

Theorem 3.22. *The following statements are equivalent for a ring R.*

(i) $\mathbb{L}_{\oplus}$ *is uniform.*

(ii) *R is semisimple and left local.*

Proof. Suppose that $\mathbb{L}_{\oplus}$ is uniform. Let $0 \neq M, P \in R\text{-}Mod$ with P being a projective module. By hypothesis, $\xi_{\oplus}(M) \wedge \xi_{\oplus}(P) \neq \underline{0}$. So there exist sets X, Y such that $M^{(X)} \cong P^{(Y)}$. Since P is projective, then $P^{(Y)}$ and $M^{(X)}$ are projective. Hence each module M is projective. Thus R is a semisimple ring.

Let S, S' be two simple modules. Then $\xi_{\oplus}(S) \wedge \xi_{\oplus}(S') \neq \underline{0}$. So there exist sets X, Y such that $S^{(X)} \cong S'^{(Y)}$. Therefore $S \cong S'$. Thus R is left local.

Conversely, suppose that R is semisimple and left local. Let $0 \neq M, N \in R\text{-}Mod$ and let S denote the simple module. Since R is semisimple, then $M \cong S^{(X)}$ and $N \cong S^{(Y)}$ for some sets X, Y. Let Z be a infinite set such that $|X|, |Y| \leq |Z|$. Then $M^{(Z)} \cong (S^{(X)})^{(Z)} \cong S^{(X \times Z)} \cong S^{(Z)} \cong S^{(Y \times Z)} \cong (S^{(Y)})^{(Z)} = N^{(Z)}$. Therefore $0 \neq M^{(Z)} \in \xi_{\oplus}(M) \wedge \xi_{\oplus}(N)$. Hence $\mathbb{L}_{\oplus}$ is uniform. □

Theorem 3.23. *The following statements are equivalent for a ring R.*

(i) $\mathbb{L}_{\prod}$ *is uniform.*

(ii) *R is semisimple and left local.*

Proof. Suppose that $\mathbb{L}_{\prod}$ is uniform. Let $0 \neq M, E \in R\text{-}Mod$ with E as an injective module. Then by hypothesis, $\xi_{\prod}(M) \wedge \xi_{\prod}(E) \neq \underline{0}$. So there exist sets X, Y such that $M^X \cong E^Y$. Since E is injective, then E^Y is also injective. Thus M^X is injective. Therefore M is injective. Hence R is semisimple.

Let S, S' be two simple modules. Then $\xi_{\prod}(S) \wedge \xi_{\prod}(S') \neq \underline{0}$. So there exist sets X, Y such that $S^X \cong S'^Y$. Therefore $S \cong S'$. Thus R is left local.

Conversely, suppose that R is semisimple and left local. Let $0 \neq M, N \in R\text{-}Mod$. Then, if S is the simple module, $M \cong S^{(X)}$, $N \cong S^{(Y)}$ for some sets X, Y. Let Z be a infinite set such that $|M|, |N| \leq |Z|$. We can assume that R is non trivial. Then $2 \leq |M|, |N|$ and thus $|2^Z| \leq |M^Z| \leq |Z^Z| \leq |(2^Z)^Z| = |2^{(Z \times Z)}| = |2^Z|$. Hence $|M|^{|Z|} = |2|^{|Z|}$. Similarly, $|N|^{|Z|} = |2|^{|Z|}$. Then $|M|^{|Z|} = |N|^{|Z|}$.

On the other hand, there exists a set A such that $M^Z \cong S^{(A)}$. Since M^Z is infinite, then $S^{(A)}$ is infinite. Therefore $|S^{(A)}| = max\{|S|, |A|\}$, because $|M^Z| = |S^{(A)}|$. Since $|S| \leq |S^{(X)}| = |M| < |M^Z| = |S^{(A)}| = max\{|S|, |A|\}$, we have that $|S| < max\{|S|, |A|\}$. Thus $|A| = max\{|S|, |A|\} = |M^Z|$.

Similarly, there exists a set B such that $N^Z \cong S^{(B)}$ and $|N^Z| = |B|$. Thus $|A| = |B|$. Then $M^Z \cong S^{(A)} \cong S^{(B)} \cong N^Z$. So $0 \neq M^Z \in \xi_{\prod}(M) \wedge \xi_{\prod}(N)$. Hence $\mathbb{L}_{\prod}$ is uniform. □

Proposition 3.24. *The following statements are equivalent for a ring R.*

(i) $\mathbb{L}_E$ *is uniform.*
(ii) R *is trivial.*

Proof. Suppose that R is not trivial. Let $0 \neq E$ be an injective module. Then there exists a set X such that $|E^X| > |E|$. Since E and E^X are injective, then $\{0, E\}, \{0, E^X\} \in \mathbb{L}_E$ and their meet is $\{0\}$. Hence $\mathbb{L}_E$ is not uniform.

The converse is immediate. □

Proposition 3.25. *The following statements are equivalent for a ring R.*

(i) $\mathbb{L}_P$ *is uniform.*
(ii) R *is trivial.*

Proof. Suppose that R is not trivial. Let $0 \neq P$ be a projective module. Then there exists a set X such that $|P^{(X)}| > |P|$. Since P and $P^{(X)}$ are projective, then $\{0, P\}, \{0, P^{(X)}\} \in \mathbb{L}_P$ and their meet is $\{0\}$. Thus $\mathbb{L}_P$ is not uniform.

The converse is immediate. □

Proposition 3.26. *The following statements are equivalent for a ring R.*

(i) $\mathbb{L}_{ext}$ *is uniform.*
(ii) R *is trivial.*

Proof. Suppose that R is not trivial. Let $0 \neq E$ be an injective module. Then $\{0, E^X\} \in \mathbb{L}_{ext}$ for any infinite set X. Let X, Y be infinite sets such that $|E^X| < |E^Y|$. Then $E^X \ncong E^Y$. Thus $\{0, E^X\} \wedge \{0, E^Y\} = \underline{0}$. Hence $\mathbb{L}_{ext}$ is not uniform.

The converse is immediate. □

Theorem 3.27. *The following statements are equivalent for a ring R.*

(i) $\mathbb{L}_{\leq,/}$ *is uniform.*
(ii) R *is left local.*

Proof. Suppose that $\mathbb{L}_{\leq,/}$ is uniform. Let $S, S' \in R\text{-}Mod$ be simple modules. Then $\{0, S\} \wedge \{0, S'\} \neq \underline{0}$, since both belong to $\mathbb{L}_{\leq,/}$, which is uniform. Thus $S \cong S'$. Hence R is left local.

Conversely, suppose that R is left local. Let $0 \neq M, N \in R\text{-}Mod$. Then there exist simple modules S and S' which are subquotients of M and N respectively. Thus $S \cong S'$, since R is left local. So $\underline{0} \neq \{0, S\} \leq \xi_{\leq,/}(M) \wedge \xi_{\leq,/}(N)$. Hence $\mathbb{L}_{\leq,/}$ is uniform. □

The following theorem establishes, that left local rings are the rings whose big lattice of Serre classes is uniform.

Theorem 3.28. *The following statements are equivalent for a ring R.*

(i) $\mathbb{L}_{\leq,/,ext}$ *is uniform.*
(ii) R *is left local.*

Proof. By Lemma 3.10, we have that $\mathbb{L}_{\leq,/}$ is $\{\leq, /\} - \{\leq, /, ext\}$ uniform. Then by Theorem 3.1, $\mathbb{L}_{\leq,ext}$ is uniform if and only if $\mathbb{L}_{\leq}$ is uniform. But from Theorem 3.27 we have that $\mathbb{L}_{\leq}$ is uniform if and only if R is left local. Thus $\mathbb{L}_{\leq,/,ext}$ is uniform if and only if R is left local. □

Theorem 3.29. *The following statements are equivalent for a ring R.*

(i) $\mathbb{L}_{\leq,/,\oplus}$ *is uniform.*
(ii) R *is left local.*

Proof. It follows from Theorem 3.27 and Lemma 3.5. □

The following theorem describes rings for which the lattice of hereditary torsion theories is uniform.

Theorem 3.30. *The following statements are equivalent for a ring R.*

(i) $\mathbb{L}_{\leq,/,\oplus,ext}$ *is uniform.*
(ii) R *is left local.*

Proof. It follows from Theorem 3.27 and Lemma 3.12. □

Theorem 3.31. *The following statements are equivalent for a ring R.*

(i) $\mathbb{L}_{\leq,E}$ *is uniform.*

(ii) R *is left semi-artinian and left local.*

Proof. Assume that $\mathbb{L}_{\leq,E}$ is uniform. Let $0 \neq M, S \in R\text{-}Mod$, where S is a simple module. Since $\mathbb{L}_{\leq,E}$ is uniform and $\xi_{\leq,E}(E) = \xi_{\leq}(E)$ for all injective module E, then there exists $0 \neq N \in R\text{-}Mod$ such that $N \in \xi_{\leq}(E(S)) \wedge \xi_{\leq}(E(M))$, i.e., $N \leq E(S)$ and $N \leq E(M)$. Then $0 \neq N \cap S$ and $S \leq N \leq E(M)$. Thus, $0 \neq M \cap S$. Therefore $S \leq M$. Hence R is left semi-artinian.

If S and S' are two simple modules, then, as in the above paragraph, we have that $S \leq S'$. Therefore $S \cong S'$. Hence R is left local.

Conversely, suppose that R is left semi-artinian left local ring. Let M, N be two nonzero modules. If S is the simple module, then S embeds both in N and M. Therefore $0 \neq S \in \xi_{\leq,E}(M) \wedge \xi_{\leq,E}(N)$. Hence $\mathbb{L}_{\leq,E}$ is uniform. □

Theorem 3.32. *The following statements are equivalent for a ring R.*

(i) $\mathbb{L}_{\oplus,/}$ *is uniform.*

(ii) R *is left local and left max.*

Proof. Assume that $\mathbb{L}_{\oplus,/}$ is uniform. Let S be a simple module. Then $\xi_{\oplus,/}(S) = \{S^{(X)} | X \text{ is a set }\}$. Clearly, $\xi_{\oplus,/}(S)$ is an atom. Therefore, $\mathbb{L}_{\oplus,/}$ has only one atom, because $\mathbb{L}_{\oplus,/}$is uniform. Hence R has only one type of simple, and thus R is left local.

Let $0 \neq M \in R\text{-}Mod$. Then $\xi_{\oplus,}(M) = \{N \mid \exists \alpha : M^{(X)} \twoheadrightarrow N\}$. Since $\xi_{\oplus,/}(S)$ is the unique atom, and since $\mathbb{L}_{\oplus,/}$ is uniform, we have that $\xi_{\oplus,/}(S) \leq \xi_{\oplus,/}(M)$. Therefore, there exists an epimorphism $M^{(X)} \twoheadrightarrow S$ for some set X, because $S \in \xi_{\oplus,/}(M)$. Then S is a quotient of M. Thus M has a maximal submodule. Hence R is left max.

Suppose that R is left local and left max. Let $0 \neq M, N \in R\text{-}Mod$ and let S denote the unique type of simple modules. Since R is left max, then there exists a maximal submodule $M' \leq M$. Therefore $M/M' \cong S$. Thus there exists an epimorphism $M \twoheadrightarrow S$. Then $S \in \xi_{\oplus,/}(M)$ and, moreover, $\xi_{\oplus,/}(S) \leq \xi_{\oplus,/}(M)$.

Similarly, for N we have that $\xi_{\oplus,/}(S) \leq \xi_{\oplus,/}(N)$. Then $\underline{0} \neq \xi_{\oplus,/}(S) \leq \xi_{\oplus,/}(M) \wedge \xi_{\oplus,/}(N)$. Hence $\mathbb{L}_{\oplus,/}$ is uniform. □

Proposition 3.33. *The following statements are equivalent for a ring R.*

(i) $\mathbb{L}_{ext,E}$ *is uniform.*

(ii) R *is trivial.*

Proof. Suppose that R is not trivial. Let $0 \neq E$ be an injective module. Then $\{0, E^X\} \in \mathbb{L}_{ext,E}$ for any infinite set X. Let X, Y be infinite sets such that $|E^X| < |E^Y|$. Then $E^X \ncong E^Y$. Thus $\{0, E^X\} \wedge \{0, E^Y\} = \underline{0}$. Hence $\mathbb{L}_{ext,E}$ is not uniform.

The converse follows immediately. □

Proposition 3.34. *Let R be a ring. Then $\mathbb{L}_{ext,P}$ is uniform if and only if R is trivial.*

Proof. The proof is similar to that of Proposition 3.33. □

Remark 3.35. (Lemma of Bumby) *If A, B are injective modules such that A is isomorphic to a submodule of B and B is isomorphic to a submodule of A, then $A \cong B$, see* [5, *Proposition* 3.60].

Theorem 3.36. *The following statements are equivalent for a ring R.*

(i) *$\mathbb{L}_{E,\prod}$ is uniform.*

(ii) *R is left local and left semi-artinian.*

Proof. Assume that $\mathbb{L}_{E,\prod}$ is uniform. Notice that if E is injective, then $\xi_{E,\prod}(E) = \{E^X \mid X \text{ is a set}\}$. Let $0 \neq M \in R\text{-}Mod$ and S be a simple module. Then since $\mathbb{L}_{E,\prod}$, is uniform, there exist sets X, Y such that $E(S)^X \cong E(M)^Y$. Therefore $S \leq E(M)$. Thus $S \leq M$, since $E(M)$ is the injective hull of M. Hence R is left semi-artinian.

If we take $M = S'$, where S' is a simple module, then $S \leq S'$. So $S \cong S'$. Hence R is left local.

Suppose that R is left local and left semi-artinian ring. Let M, N be two nonzero modules and let S be the unique type of simple module. Put $E = E(S)$. Since S embeds in M and since R is left semi-artinian, we have that $E \leq E(M)$. On the other hand, $soc(M) = S^{(X)}$ for some set X. Thus $soc(M)$ is essential in M, because R is left semi-artinian. Then $E(M) \leq E(soc(M))$. Therefore, $E(M) = E(soc(M)) \leq E(S^{(X)}) \leq E(S^X) \leq E(S)^X$. Thus, $E(M) \leq E^X$.

If X is infinite, then $E^X \leq E(M)^X \leq (E^X)^X \cong E^{X \times X} \cong E^X$. Then, by the Lemma of Bumby, $E^X \cong E(M)^X$.

If X is finite, take $Y = \mathbb{N}$. Then $E(M) \leq E(S)^X \leq E(S)^Y = E^Y$ and proceed as in the case when X is infinite.

Thus in any case, we have that $E^X \cong E(M)^X$. Similarly for N, there exists a set Z such that $E^Z \cong E(N)^Z$.

Let Y be an infinite set such that $|X|, |Z| \leq |Y|$. Then, $(E(M)^X)^Y \cong (E^X)^Y \cong E^{X \times Y} \cong E^Y \cong E^{Z \times Y} \cong (E^Z)^Y \cong (E(N)^Z)^Y$. Thus $E(M)^Y \cong E(N)^Y$. Then, $\xi_{E,\prod}(M) \wedge \xi_{E,\prod}(S) \neq \underline{0}$. Hence $\mathbb{L}_{E,\prod}$ is uniform. □

Recall that the lattice of natural classes is precisely $\mathbb{L}_{\leq,E,\oplus}$, and it is also denoted $R-nat$. Dauns and Zhou proved that $R-nat$ is a complete boolean lattice [4 Theorem 5.1.5, p. 119].

Proposition 3.37. *The following statements are equivalent for a ring R.*

(i) *$R-nat$ is uniform.*

(ii) *R is trivial.*

Proof. It follows from the fact that $R-nat$ is a complete boolean lattice. □

Acknowledgment. The authors wish to express their gratitude to the referee for his valuable recommendations which improved substantially this work.

References

[1] A. Alvarado-García, C. Cejudo-Castilla, H. A. Rincón-Mejía and I. F. Vilchis-Montalvo, *On classes of modules closed under injective hulls and Artinian principal ideals rings*, Int. Electron. J. Algebra, 15 (2014), 1-12.

[2] A. Alvarado-García, H. A. Rincón-Mejía and J. Ríos-Montes, *On big lattices of classes of R-modules defined by closure properties*, Advances in ring theory, 19-36, Trends Math., Birkhäuser/Springer Basel AG, Basel, 2010.

[3] J. H. Cozzens, Homological Properties of the Ring of Differential Polynomials, Thesis (Ph.D.)-Rutgers The State University of New Jersey - New Brunswick, 1969.

[4] J. Dauns and Y. Zhou, Classes of Modules, Pure and Applied Mathematics (Boca Raton), 281, Chapman & Hall/CRC, Boca Raton, FL, 2006.

[5] C. Faith, Algebra: Rings, Modules and Categories, I, Die Grundlehren der mathematischen Wissenschaften, Band 190, Springer-Verlag, New York-Heidelberg, 1973.

[6] E. R. Kolchin, *Galois theory of differential fields*, Amer. J. Math., 75 (1953), 753-824.

Hugo Alberto Rincón-Mejía and **José Patricio Sánchez-Hernández**
Facultad de Ciencias
Universidad Nacional Autónoma de México
Circuito Exterior, C.U.
04510 México, D.F.
e-mails: hurincon@gmail.com (H. A. Rincón-Mejía)
patriciosanchezh@hotmail.com (J. P. Sánchez-Hernández)