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NIL CLEAN INDEX OF RINGS

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Abstract. Motivated by the concept of clean index of rings of Lee and Zhou we introduce the concept of nil clean index of rings. For any element a of a ring R with unity, we define $\eta(a) = \{e \in R \mid e^2 = e \text{ and } a - e \in \text{nil}(R)\}$, where $\text{nil}(R)$ is the set of all nilpotent elements of R. Then nil clean index of R is defined by $\sup\{\, |\eta(a)| \,:\, a \in R\}$ and it is denoted by $\text{Nin}(R)$. In this article, we characterize rings of nil clean indices 1, 2 and 3 and prove some interesting results pertaining them.

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1. Introduction

Rings R are associative rings with unity unless otherwise indicated, and modules (and bimodules) are unitary. The Jacobson radical, group of units, set of idempotents and set of nilpotent elements of a ring R are denoted by $\text{J}(R)$, $\text{U}(R)$, $\text{idem}(R)$ and $\text{nil}(R)$ respectively. Cyclic group of order m will be denoted by C_m. Notion of clean rings was first introduced by Nicholson [5], which was later extended to nil clean rings by Diesel [2]. Chen [1] characterized uniquely clean and uniquely nil clean rings completely. Further Lee and Zhou [3,4] introduced clean index of rings, which motivated us to introduced and study nil clean index of rings. For an element $a \in R$, if $a - e \in \text{nil}(R)$ for some $e^2 = e \in R$, then $a = e + (a - e)$ is called a nil clean expression of a in R and a is called a nil clean element. The ring R is called nil clean if each of its elements is nil clean. A ring R is uniquely nil clean if every element of R has a unique nil clean expression in R. For any element a of R, we denote $\eta(a) = \{e \in R \mid e^2 = e \text{ and } a - e \in \text{nil}(R)\}$ and nil clean index of R is defined by $\sup\{|\eta(a)| : a \in R\}$ and it is denoted by $\text{Nin}(R)$, where $|\eta(a)|$ denotes the cardinality of the set $\eta(a)$. Thus, R is uniquely nil clean if and only if R is a nil clean ring of nil clean index 1.

2. Elementary Properties

Some basic properties related to nil clean index are presented here as a preparation for the article.

Lemma 2.1. *Let R be a ring, and let $e, a, b \in R$. The following hold:*

(1) *If $e \in R$ is a central idempotent or a central nilpotent, then $|\eta(e)| = 1$, so $Nin(R) \geq 1$.*
(2) *$e \in \eta(a)$ iff $1-e \in \eta(1-a)$, and so $|\eta(a)| = |\eta(1-a)|$.*
(3) *If $f : R \to R$ is a homomorphism, then $e \in \eta(a)$ implies $f(e) \in \eta(f(a))$, and for converse part f must be monomorphism.*
(4) *If a ring R has at most n idempotents or at most n nilpotent elements, then $Nin(R) \leq n$.*

Proof. (1) Let e be a central idempotent, so we have $e = e + 0$, a nil clean expression of e. If possible let $e = a + n$ be another nil clean expression of e in R, where $a \in \text{idem}(R)$, $n \in \text{nil}(R)$ and $n^k = 0$ for some positive integer k. Then $(e-a)^{2k-1} = 0$ implies

$$e^{2k-1} - \binom{2k-1}{1} e^{2k-2}a + \cdots + \binom{2k-1}{2k-2}(-1)^{2k-2}ea^{2k-2} + (-1)^{2k-1}a^{2k-1} = 0,$$

$$(e + (-1)^{2k-1}a) - \left\{ \binom{2k-1}{1} - \binom{2k-1}{2} + \cdots + (-1)^{(2k-3)}\binom{2k-1}{2k-2} \right\} ea = 0.$$

Using elementary result of binomial coefficients, we get $(e-a)-(1+(-1)^{2k-3})ea = 0$. Hence $e = a$, i.e, $|\eta(e)| = 1$.

(2) $e \in \eta(a) \Leftrightarrow a - e$ is nilpotent $\Leftrightarrow e - a$ is nilpotent $\Leftrightarrow (1-a) - (1-e)$ is nilpotent $\Leftrightarrow 1 - e \in \eta(1-a)$, so we get $|\eta(a)| = |\eta(1-a)|$.

(3) is straightforward and (4) is clear from the definition of nil clean index. $\square$

Lemma 2.2. *If S is a subring of a ring R, where S and R may or may not share the same identity, then $Nin(S) \leq Nin(R)$.*

Proof. Since S is a subring of R, so all the idempotents and nilpotent elements of S are also idempotents and nilpotent elements of R. If $e \in \eta_S(a)$ i.e., $e^2 = e$ in S and $a-e \in \text{nil}(S)$, where $a \in S$, then $e^2 = e$ in R and $a-e \in \text{nil}(R)$, i.e., $e \in \eta_R(a)$. Therefore $\eta_S(a) \subseteq \eta_R(a)$ for all $a \in S$, implies $|\eta_S(a)| \leq |\eta_R(a)|$ for all $a \in S$ or $\sup_{a\in S} |\eta_S(a)| \leq \sup_{a\in S} |\eta_R(a)| \leq \sup_{a\in R} |\eta_R(a)|$. So we get $\text{Nin}(S) \leq \text{Nin}(R)$. $\square$

Lemma 2.3. *Let $R = S \times T$ be the direct product of two rings S and T. Then $Nin(R) = Nin(S)Nin(T)$.*

Proof. Since S and T are subrings of R, so $\mathrm{Nin}(S) \leq \mathrm{Nin}(R)$ and $\mathrm{Nin}(T) \leq \mathrm{Nin}(R)$. If $\mathrm{Nin}(S) = \infty$ or $\mathrm{Nin}(T) = \infty$, then $\mathrm{Nin}(R) = \infty$ and hence, $\mathrm{Nin}(R) = \mathrm{Nin}(S)\mathrm{Nin}(T)$ holds. So let $\mathrm{Nin}(S) = n < \infty$, $\mathrm{Nin}(T) = m < \infty$. Then $n,\ m \geq 1$ and there exist elements $s \in S$ and $t \in T$, such that $|\eta_S(s)| = n,\ |\eta_T(t)| = m$. If $s = e_i + n_i,\ i = 1, 2, \ldots, n$ and $t = f_j + m_j,\ j = 1, 2, \ldots, m,$ where $e_i's,\ f_j's$ are idempotents and $n_i's,\ m_j's$ are nilpotent elements of S and T respectively, then there exists an element $(s,t) \in R$, such that $(s,t) = (e_i, f_j) + (n_i, m_j)$, which are mn nil clean expression of $(s,t) \in R$. Hence $\mathrm{Nin}(R) \geq mn$.

If possible let $\mathrm{Nin}(R) > nm$, say $nm+1$, then there exists an element $(a,b) \in R$, such that it has at least $nm+1$ nil clean expression in R. That is $(a,b) = (g_i, h_i) + (c_i, d_i)$ where $i = 1, 2, \ldots, mn+1$, $(g_i,h_i)^2 = (g_i,h_i)$ and $(c_i,d_i) \in \mathrm{nil}(R)$. Then $a = g_i + c_i$ and $b = h_i + d_i$ are nil clean expressions for a and b respectively. Let $K = \{(g_i,h_i) \,|\, i = 1, 2, 3, \ldots, mn,\ mn+1\}$. Then $|K| = nm + 1$ implies $|\{g_i\}|.|\{h_i\}| = nm+1,$ and this implies $|\{g_i\}| > n$ or $|\{h_i\}| > m$, which gives $\mathrm{Nin}(S) > n$ or $\mathrm{Nin}(T) > m$, which is absurd. $\square$

Lemma 2.4. *Let I be an ideal of R with $I \subseteq nil(R)$ and let $n \geq 1$ be an integer. Then the following hold:*

(1) *If idempotents lift modulo I, then $Nin(R/I) = NinR$.*
(2) *If $Nin(R) \leq n$, then every idempotent of R/I can be lifted to at most n idempotents of R.*

Proof. (1) Let $a \in R$, then any idempotent $x + I \in \eta(a+I)$ is lifted to an idempotent e_x of R. Now from $(a+I)-(x+I) \in \mathrm{nil}(R/I)$ we get $(a+I)-(e_x+I) \in \mathrm{nil}(R/I),$ which means there exists some positive integer $k,$ such that $(a-e_x)^k + I = I$ which gives $a - e_x \in \mathrm{nil}(R)$ i.e., $e_x \in \eta(a)$. So the mapping $\eta(a) \to \eta(a+I)$ is onto, i.e., $|\eta(a)| \geq |\eta(a+I)|$ for all $a \in R$.

Conversely if $e \in \eta(a),$ then $a - e \in \mathrm{nil}(R)$, so there exists some positive integer k, such that $(a-e)^k = 0 \in I$. This implies $(a-e)^k + I = I$ and so $\{(a-I)-(e+I)\} \in \mathrm{nil}(R/I)$ which gives $e + I \in \eta(a+I)$. Therefore the mapping $\eta(a+I) \to \eta(a)$ is onto. i.e., $|\eta(a+I)| \geq |\eta(a)|,$ for all $a \in R$. Hence $|\eta(a)| = |\eta(a+I)|,$ for all $a \in R,$ which implies $\sup_{a\in R} |\eta(a)| = \sup_{(a+I)\in R/I} |\eta(a+I)|,$ consequently $\mathrm{Nin}(R) = \mathrm{Nin}(R/I)$.

(2) Let $a \in R$ such that $a^2 - a \in I$. If $a - e \in I \subseteq \mathrm{nil}(R)$, for some $e^2 = e \in R$, then $e \in \eta(a)$. But $|\eta(a)| \leq \mathrm{Nin}(R) \leq n$. So there are at most n such elements. $\square$

Lemma 2.5. *Let* $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, *where* A *and* B *are rings,* ${}_AM_B$ *is a bimodule. Let Nin*$(A) = n$ *and Nin*$(B) = m$. *Then*

1. *Nin*$(R) \geq |M|$.
2. *If* $(M, +) \cong C_{p^k}$, *where* p *is a prime and* $k \geq 1$, *then Nin*$(R) \geq n + [\frac{n}{2})(|M| - 1)$, *where* $[\frac{n}{2})$ *denotes the least integer greater than or equal to* $\frac{n}{2}$.
3. *Either Nin*$(R) \geq nm + |M| - 1$ *or Nin*$(R) \geq 2nm$.

Proof. (1) Let $\alpha = \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}$. Then $\left\{ \begin{pmatrix} 1_A & w \\ 0 & 0 \end{pmatrix} | w \in M \right\} \subseteq \eta(\alpha)$ as $\begin{pmatrix} 1_A & w \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix}$ is nilpotent. So $\mathrm{Nin}(R) \geq |\eta(\alpha)| \geq |M|$.

(2) Let $q = p^k$ and $a = e_i + n_i, \ i = 1, 2, \ldots n$ be n distinct nil clean expressions of a in A. For any $e = e^2 \in A$, $(M, +) = eM \oplus (1 - e)M$. Since $(M, +) \cong C_{p^k}$, so $(M, +)$ is indecomposable and hence $M = eM$ or $M = (1 - e)M$. Assume $(1 - e_1)M = \cdots = (1 - e_s)M = M$ and $e_{s+1}M = \cdots = e_nM = M$.

If $s \geq (n - s)$ (i.e., $s \geq [\frac{n}{2})$), then for $\alpha = \begin{pmatrix} 1_A - a & 0 \\ 0 & 0 \end{pmatrix}$ we have

$$\eta(\alpha) \supseteq \left\{ \begin{pmatrix} 1_A - e_i & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1_A - e_j & w \\ 0 & 0 \end{pmatrix} : 1 \leq i \leq n, 1 \leq j \leq s, 0 \neq w \in M \right\}$$

So $|\eta(\alpha)| \geq n + s(q - 1)$.

If $s \leq (n - s)$ (i.e., $n - s \geq [\frac{n}{2})$), then for $\beta = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$

$$\eta(\beta) \supseteq \left\{ \begin{pmatrix} e_i & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} e_j & w \\ 0 & 0 \end{pmatrix} : 1 \leq i \leq n, s + 1 \leq j \leq n, 0 \neq w \in M \right\}$$

So $|\eta(\beta)| \geq n + (n - s)(q - 1)$. Hence $\mathrm{Nin}(R) \geq n + [\frac{n}{2})(q - 1)$.

(3) Let $a = e_i + n_i, \ i = 1, 2, \ldots n$ and $b = f_j + m_j, \ j = 1, 2, \ldots m$ be distinct nil clean expressions of a and b in A and B respectively.

Case I: $e_{i_0}M(1 - f_{j_0}) + (1 - e_{i_0})Mf_{j_0} = 0$ for some i_0 and j_0. Then $e_{i_0}w = wf_{i_0}$ for all $w \in M$. Thus for $\alpha = \begin{pmatrix} 1_A - a & 0 \\ 0 & b \end{pmatrix}$

$$\eta(\alpha) \supseteq \left\{ \begin{pmatrix} 1_A - e_i & 0 \\ 0 & f_j \end{pmatrix}, \begin{pmatrix} 1_A - e_{i_0} & w \\ 0 & f_{j_0} \end{pmatrix} ; 1 \leq i \leq n, 1 \leq j \leq m; 0 \neq w \in M \right\}$$

So $|\eta(\alpha)| \geq mn + |M| - 1$.

Case II: $e_iM(1-f_j) + (1-e_i)Mf_j \neq 0$ for all i and j. Take $0 \neq w_{ij} \in e_iM(1-f_j) + (1-e_j)Mf_j$ for each pair (i,j). Then for $\alpha = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$

$$\eta(\alpha) \supseteq \left\{ \begin{pmatrix} e_i & 0 \\ 0 & f_j \end{pmatrix}, \begin{pmatrix} e_i & w_{ij} \\ 0 & f_j \end{pmatrix}; 1 \leq i \leq n, 1 \leq j \leq m; 0 \neq w_{ij} \in M \right\}$$

So $|\eta(\alpha)| \geq 2mn$.

Combining Case I and II we have, either $\text{Nin}(R) \geq nm + |M| - 1$ or $\text{Nin}(R) \geq 2nm$. □

Lemma 2.6. *Let $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, where A and B are rings, ${}_AM_B$ is a bimodule with $(M,+) \cong C_{2^r}$. Then $Nin(R) = 2^r Nin(A) Nin(B)$.*

Proof. Let $k = \text{Nin}(A)$ and $l = \text{Nin}(B)$. Let $a = e_i + n_i,\ i = 1, 2, \ldots k$ and $b = f_j + m_j,\ j = 1, 2, \ldots l$ be distinct nil clean expressions of a and b in A and B respectively. Write $M = \{0,\ x,\ 2x, \ldots, (2^r-1)x\}$, for any $e = e^2 \in A$, either $M = eM$ or $M = (1_A - e)M$; so $ex \in \{0, x\}$. Suppose $e_1x \neq e_2x$, say $e_1x = 0$ and $e_2x = x$. Then

$$ax = n_1x = x + n_2x = (1+n_2)x.$$

Because $ax \in M,\ ax = ix$ for some $2 \leq i \leq 2^k$. Then $n_1x = ix \Rightarrow 0 = i^px$ (Since $n^p = 0$ for some $p \in \mathbb{N}$), which gives i is even, so let $i = 2j$. Now $(1+n_2)x = (2j)x \Rightarrow (1+n_2)^rx = (2j)^rx = j^k(2^k)x = 0 \Rightarrow x = 0$ (as $(n+1) \in \text{U}(A)$) a contradiction as $x \neq 0$. So $e_1x = e_2x = \cdots = e_nx$. Similarly $xf_1 = xf_2 = \cdots = xf_l$.

Case I: $e_ix = 0$ and $xf_j = 0$. For $\alpha = \begin{pmatrix} 1_A - a & 0 \\ 0 & b \end{pmatrix}$ we have

$$\begin{pmatrix} 1_A - a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 1_A - e_i & w \\ 0 & f_j \end{pmatrix} + \begin{pmatrix} -n_i & -w \\ 0 & m_j \end{pmatrix}, \qquad \begin{array}{r} i = 1, 2, \ldots, k \\ j = 1, 2, \ldots, l, \forall w \in M. \end{array}$$

Therefore, in this case, $\text{Nin}(R) \geq |\eta(\alpha)| \geq 2^rkl$.

Case II: $e_ix = x,\ xf_j = x$. Then

$$\beta = \begin{pmatrix} 1_A - a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 1_A - e_i & w \\ 0 & f_j \end{pmatrix} + \begin{pmatrix} -n_i & -w \\ 0 & m_j \end{pmatrix}, \qquad \begin{array}{r} i = 1, 2, \ldots k, \\ j = 1, 2, \ldots, l, \forall\ w \in M. \end{array}$$

Therefore, in this case, $\text{Nin}(R) \geq |\eta(\alpha)| \geq 2^rkl$.

Case III: $e_ix = x,\ xf_j = 0$. Then

$$\gamma = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} e_i & w \\ 0 & f_j \end{pmatrix} + \begin{pmatrix} n_i & -w \\ 0 & m_j \end{pmatrix}, \qquad i = 1, 2, \dots, k$$
$$j = 1, 2, \dots, l, \forall\ w \in M.$$

Therefore, in this case, $\text{Nin}(R) \geq |\eta(\alpha)| \geq 2^r kl$.

Case IV: $e_i x = 0,\ x f_j = x$. Then

$$\delta = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} e_i & w \\ 0 & f_j \end{pmatrix} + \begin{pmatrix} n_i & -w \\ 0 & m_j \end{pmatrix}, \qquad i = 1, 2, \dots, k$$
$$j = 1, 2, \dots, l,\ \forall\ w \in M.$$

Therefore, in this case, $\text{Nin}(R) \geq |\eta(\alpha)| \geq 2^r kl$.

On the other hand for $\alpha = \begin{pmatrix} c & z \\ 0 & d \end{pmatrix} \in R$ we have

$$\eta(\alpha) = \left\{ \begin{pmatrix} e & w \\ 0 & f \end{pmatrix} \in R,\ e \in \eta(c),\ f \in \eta(d),\ w = ew + we \right\}.$$

Therefore, $|\eta(\alpha)| \leq |M||\eta(c)||\eta(d)| \leq 2^r kl$ and hence $\text{Nin}(R) \leq 2^r kl$. Thus, $\text{Nin}(R) = 2^r kl = 2^r \text{Nin}(A)\text{Nin}(B)$. $\square$

Lemma 2.7. *Let A and B be rings and ${}_AM_B$ a nontrivial bimodule. If $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ is a formal triangular matrix ring, then $Nin(A) < Nin(R)$ and $Nin(B) < Nin(R)$.*

Proof. Let $k = \text{Nin}(A)$ and let $a = e_i + n_i$ $(i = 1,\ 2, \dots, k)$ be k distinct nil clean expressions of a in A. If $e_1 M = 0$. Then

$$\begin{pmatrix} 1_A - a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1_A - e_i & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -n_i & 0 \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1_A - e_1 & x \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -n_1 & -x \\ 0 & 0 \end{pmatrix} \qquad \forall\ 0 \neq x \in M.$$

There are at least $k+1$ distinct nil clean expressions of $\begin{pmatrix} 1_A - a & 0 \\ 0 & 0 \end{pmatrix}$ in R.

If $e_1 M \neq 0$, then $e_1 x \neq 0$ for some $x \in M$. So we have

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e_i & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} n_i & 0 \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} e_1 & e_1 x \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} n_1 & -e_1 x \\ 0 & 0 \end{pmatrix} \qquad \forall\ 0 \neq x \in M.$$

There are at least $k+1$ distinct nil clean expressions of $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ in R.
So in any case $\mathrm{Nin}(R) \geq k+1 > k = \mathrm{Nin}(A)$. Similarly, $\mathrm{Nin}(R) > \mathrm{Nin}(B)$. □

Lemma 2.8. *Let R be a ring with unity, then $In(R) \geq Nin(R)$, where $In(R)$ is the clean index of R.*

Proof. Definition of $\mathrm{In}(R)$ is similar to that of $\mathrm{Nin}(R)$ where nilpotent is replaced by unit, for details one can see [3]. Let $\mathrm{Nin}(R) = k$, then there is at least an element $a \in R$, such that it has k nil clean expressions in R, i.e., $a = e_i + n_i$, $i = 1,\ 2, \cdots, k$, where $e_i \in \mathrm{idem}(R)$ and $n_i \in \mathrm{nil}(R)$. From this we get, $a - 1 = e_i + (n_i - 1)$ are k clean expression for $(a-1) \in R$, and therefore $\mathrm{In}(R) \geq k$, hence $\mathrm{In}(R) \geq \mathrm{Nin}(R)$. □

3. Rings of Nil Clean Index 1

Lemma 3.1. *$Nin(R) = 1$, if and only if R is abelian and for any $0 \neq e^2 = e \in R$, $e \neq n + m$ for any $n, m \in nil(R)$.*

Proof. Let $e^2 = e \in R$, then for any $r \in R$, we have $e + 0 = [e + er(1-e)] + [-er(1-e)]$, where $\{e + er(1-e)\}^2 = e + er(1-e)$ and $\{-er(1-e)\}^2 = er(1-e)er(1-e) = 0$ i.e., $-er(1-e) \in \mathrm{nil}(R)$. Since $\mathrm{Nin}(R) = 1$, so $e = e + er(1-e)$ which gives $er = ere$, Similarly $re = ere$, hence $er = re$ i.e., R is abelian. Again, if $e = n + m$ for some $n, m \in \mathrm{nil}(R)$, then $e + (-m) = 0 + n$, since $\mathrm{Nin}(R) = 1$, this is not possible.

Conversely, suppose R is abelian and no nonzero idempotent can be written as a sum of two nilpotent elements. We know that $\mathrm{Nin}(S) \geq 1$ for any ring S. Suppose if possible $a \in R$ has two nil clean expressions

$$a = e_1 + n_1 = e_2 + n_2, \text{ where } e_1, e_2 \in \mathrm{idem}(R) \text{ and } n_1, n_2 \in \mathrm{nil}(R). \quad (1)$$

If $e_1 = e_2$, we have nothing to prove. So let $e_1 \neq e_2$. Now multiplying equation (1) by $(1 - e_1)$ we get,

$$e_1(1-e_1) + n_1(1-e_1) = e_2(1-e_1) + n_2(1-e_2)$$
$$e_2(1-e_1) = n_1(1-e_1) - n_2(1-e_2). \quad (2)$$

Since R is Abelian, $e_2(1-e_1) \in \mathrm{idem}(R)$ and $n_1(1-e_1)$, $n_2(1-e_2)$ are nilpotent elements. So (2) gives a contradiction if $e_2(1-e_1) \neq 0$. On other hand if $e_2(1-e_1) = 0$, then (1) implies $e_1(1-e_2) = n_1 - n_2$ which is again a contradiction. This implies $|\eta(a)| \leq 1$ for all $a \in R$, hence $\mathrm{Nin}(R) = 1$. □

Theorem 3.2. *Nin*$(R) = 1$ *if and only if* R *is an abelian ring.*

Proof. ($\Rightarrow$) This is done in Lemma 3.1.

($\Leftarrow$) Let R be an abelian ring and e a non zero idempotent of R. We claim that e can not be written as sum of two nilpotent elements. Suppose $e = a+b$ where $a^n = 0$, $b^m = 0$, and $n < m$. Then $(e-a)^m = 0$ and by using binomial theorem we get

$$e^m - \binom{m}{1}ae^{(m-1)} + \binom{m}{2}a^2e^{(m-2)} - \cdots + (-1)^{(n-1)} \binom{m}{n-1}a^{(n-1)}e^{(m-n+1)} = 0$$

which gives

$$e[1 - \binom{m}{1}a + \binom{m}{2}a^2 - \cdots + (-1)^{(n-1)}\binom{m}{n-1}a^{(n-1)} + (-1)^n\binom{m}{n}a^n + (-1)^{(n+1)}\binom{m}{n+1}a^{(n+1)} + \cdots + (-1)^m a^m] = 0$$

and this gives $e(1-a)^m = 0$. Therefore we get, $e = 0$ (since $1 - a \in U(R)$). Similarly, if $n > m$, then $(e-b)^n = 0$ and so $e = 0$, a contradiction. Hence, no nonzero idempotent can be written as sum of two nilpotent elements and therefore $\mathrm{Nin}(R) = 1$. □

Above theorem gives the following observations:

(1) A ring R with $\mathrm{Nin}(R) = 1$ is always Dedekind finite, but the converse is not true by Example 4.3.

(2) Rings with trivial idempotents have nil clean index one and consequently the local rings are of nil clean index one. If $\mathrm{Nin}(R) = 1$, then it is easy to see that idempotents of $R[[x]]$ are idempotents of R, and for any $\alpha = \alpha_0 + \alpha_1 x + \cdots \in R[[x]]$, it is easy to see that $\eta_{R[[x]]}(\alpha) \subseteq \eta_R(\alpha_0)$, this gives $\mathrm{Nin}(R[x]) = \mathrm{Nin}(R[[x]]) = 1$. But if $\mathrm{Nin}(R) > 1$, then there is some noncentral idempotent $e \in R$, such that $er \neq re$ for some $r \in R$. So either $er(1-e) \neq 0$ or $(1-e)re \neq 0$. Let $er(1-e) \neq 0$, then we have $a = e+er(1-e) = [e+er(1-e)x^i]+[er(1-e)(1-x^i)]$ where i is a positive integer, are infinitely many nil clean expression of a in $R[x]$ which implies $\mathrm{Nin}(R[x]) = \infty$. Now we have the following theorem.

Theorem 3.3. *Let* R *be a ring, Nin*$(R[[x]])$ *is finite iff Nin*$(R) = 1$.

Proof. If $\mathrm{Nin}(R) = 1$, then it is easy to see that idempotents of $R[[x]]$ are idempotents of R, and for any $\alpha = \alpha_0 + \alpha_1 x + \cdots \in R[[x]]$, it is easy to see that $\eta_{R[[x]]}(\alpha) \subseteq \eta_R(\alpha_0)$, this gives $\mathrm{Nin}(R[x]) = \mathrm{Nin}(R[[x]]) = 1$. But if $\mathrm{Nin}(R) > 1$ then, there is some noncentral idempotent $e \in R$, such that $er \neq re$ for some $r \in R$. So either $er(1-e) \neq 0$ or $(1-e)re \neq 0$. Let $er(1-e) \neq 0$, then we have $a = e+er(1-e) = [e+er(1-e)x^i]+[er(1-e)(1-x^i)]$ where i is a positive integer, are infinitely many nil clean expression of a in $R[x]$ which implies $\mathrm{Nin}(R[x]) = \infty$. Hence the theorem follows. □

Corollary 3.4. *$Nin(R[[x]])$ is 1 or infinite.*

4. Rings of Nil Clean Indices 2 and 3

In this section, we characterize the rings of nil clean indices 2 and 3. From the discussion above we see that such rings should be non abelian. For rings A and B and for a bimodule ${}_AM_B$, we denote by $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ the formal triangular matrix ring.

Theorem 4.1. *$Nin(R) = 2$ if and only if $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, where $Nin(A) = Nin(B) = 1$ and ${}_AM_B$ is a bimodule with $|M| = 2$.*

Proof. ($\Leftarrow$) For $\alpha_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix} \in R$, $\left\{\begin{pmatrix} 0 & \omega \\ 0 & 1_B \end{pmatrix}; \omega \in M\right\} \subseteq \eta(\alpha_0)$. So, $\mathrm{Nin}(R) \geq |\eta(\alpha_0)| \geq |M| = 2$. For any $\alpha = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in R$,

$$\eta(\alpha) = \left\{\begin{pmatrix} e & w \\ 0 & f \end{pmatrix}; \ e \in \eta(a), \ f \in \eta(b), \ w = ew + wf\right\}.$$

Because $|M| = 2$, $|\eta(a)| \leq 1$, $|\eta(b)| \leq 1$, it follows that $|\eta(\alpha)| \leq 2$. Hence, $\mathrm{Nin}(R) = 2$.

($\Rightarrow$) Suppose R is non abelian and let $e^2 = e \in R$ be a non central idempotent. If neither $eR(1-e)$ nor $(1-e)Re$ is zero, then take $0 \neq x \in eR(1-e)$ and $0 \neq y \in (1-e)Re$. Then $e = e + 0 = (e + x) - x = (e + y) - y$ are three distinct nil clean expressions of e in R. So without loss of generality, we can assume that $eR(1-e) \neq 0$ but $(1-e)Re = 0$. The Peirce decomposition of R gives

$$R = \begin{pmatrix} eRe & eR(1-e) \\ 0 & (1-e)R(1-e) \end{pmatrix}.$$

As above $2 = \mathrm{Nin}(R) \geq |eR(1-e)|$; so $|eR(1-e)| = 2$. Write $eR(1-e) = \{0, x\}$. Suppose $a = e_1 + n_1 = e_2 + n_2$ are distinct nil clean expressions of a in eRe. If $e_1x = x$, then

$$\begin{aligned}
\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} n_1 & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} e_2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} n_2 & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} e_1 & x \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} n_1 & x \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

are three distinct nil clean expressions of $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in R$. If $e_1x = 0$, then

$$\begin{aligned}\begin{pmatrix} a & 0 \\ 0 & 1_B \end{pmatrix} &= \begin{pmatrix} e_1 & 0 \\ 0 & 1_B \end{pmatrix} + \begin{pmatrix} n_1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} e_2 & 0 \\ 0 & 1_B \end{pmatrix} + \begin{pmatrix} n_2 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} e_1 & x \\ 0 & 1_B \end{pmatrix} + \begin{pmatrix} n_1 & x \\ 0 & 1_B \end{pmatrix}\end{aligned}$$

are three distinct nil clean expressions of $\begin{pmatrix} a & 0 \\ 0 & 1_B \end{pmatrix}$ in R. This contradiction shows that $\mathrm{Nin}(eRe) = 1$. Similarly, $\mathrm{Nin}((1-e)R(1-e)) = 1$. □

The next proposition gives a sufficient condition for rings to have nil clean index 3.

Proposition 4.2. *If* $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, *where* $Nin(A) = Nin(B) = 1$ *and* ${}_AM_B$ *is a bimodule with* $|M| = 3$, *then* $Nin(R) = 3$.

Proof. This is similar to the proof of the implication "(⇐)" of Proposition 4.1. □

The condition of Proposition 4.2 is a sufficient condition, but not necessary, as shown by the following example.

Example 4.3. $R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}$ *is a ring of nil clean index* 3.

We see that, $nil(R) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$. *Using Lemma 2.1, we get* $Nin(R) \leq 4$. *Also,*

$$\eta\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right\},$$

thus $Nin(R) \geq 3$. *Similarly, by verifying for each element we see that* $Nin(R) = 3$. *But it is not of the form* $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$. □

Next we have the following proposition for the full matrix ring.

Proposition 4.4. *Let* $R = M_n(S)$, *where* S *is a ring with unity and let* $n \geq 2$ *be an integer. Then*

(1) $Nin(R) \geq 3$.
(2) $Nin(R) = 3$ *iff* $n = 2$ *and* $S \cong \mathbb{Z}_2$.

Proof. For $a = E_{11}$, $E_{11} + \sum_{i=2}^{n} r_i E_{1i}$ and $E_{11} + \sum_{i=2}^{n} s_i E_{i1}$ are contained in $\eta_R(a)$ $\forall r_i, s_i \in S$ $(2 \leq i \leq n)$. So

$$\text{Nin}(R) \geq |\eta_R(a)| \geq 2|S|^{n-1} - 1.$$

(1) If $|S| \geq 3$ or $n \geq 3$, then $\text{Nin}(R) \geq \min\{2.3^{2-1} - 1,\ 2.3^{3-1} - 1\} = 5$. Also, $\text{Nin}(M_2(\mathbb{Z}_2)) = 3$. So $\text{Nin}(R) \geq 3$.
(2) If $\text{Nin}(R) = 3$, then $3 = \text{Nin}(R) \geq 2|S|^{n-1} - 1$ i.e., $2 \geq |S|^{n-1}$. So we must have $n = 2$ and $|S| = 2$. So $S \cong \mathbb{Z}_2$. Converse part is obviously true as $\text{Nin}(M_2(\mathbb{Z}_2)) = 3$. □

Theorem 4.5. *Let R be a ring. If $Nin(R) = 3$, then one of the following holds:*

(1) $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, *where A and B are rings with $Nin(A) = Nin(B) = 1$ and ${}_AM_B$ is a bimodule with $|M| = 3$.*

(2) $R = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$, *where A and B are rings with $Nin(A) = Nin(B) = 1$ and ${}_AM_B$, ${}_BN_A$ are bimodules with $|M| = |N| = 2$.*

Proof. Let $\text{Nin}(R) = 3$. Then R is non abelian. Let $e \in R$ be a noncentral idempotent. Set $A = eRe$, $B = (1-e)R(1-e)$, $M = eR(1-e)$, $N = (1-e)Re$. Since e is noncentral, so M and N are not both zero, so we have two cases:
Case I: $M \neq 0$, $N = 0$ or $M = 0$, $N \neq 0$. Without loss of generality let $M \neq 0$, $N = 0$. Then $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$. Clearly by Lemma 2.5, $2 \leq |M| \leq \text{Nin}(R) = 3$. Also, by Lemma 2.7, we have $\text{Nin}(A) < \text{Nin}(R)$ and $\text{Nin}(B) < \text{Nin}(R)$. By Lemma 2.6, if $|M| = 2$, then $3 = \text{Nin}(R) = 2\text{Nin}(A)\text{Nin}(B)$, which is a contradiction. So $|M| = 3$. Now by Lemma 2.5, we see that

$$\begin{aligned} 3 = \text{Nin}(R) \geq \text{Nin}(A)\text{Nin}(B) + |M| - 1 \quad &\text{or} \quad \text{Nin}(R) \geq 2\text{Nin}(A)\text{Nin}(B) \\ \Rightarrow \quad \text{Nin}(A)\text{Nin}(B) \leq 1 \quad &\text{or} \quad \text{Nin}(A)\text{Nin}(B) \leq \frac{3}{2} \\ \Rightarrow \quad \text{Nin}(A)\text{Nin}(B) = 1, & \end{aligned}$$

that is $\text{Nin}(A) = \text{Nin}(B) = 1$. So we get (1).
Case II: Let $N \neq 0$ and $M \neq 0$, so $|N| \geq 2$ and $|M| \geq 2$. Now

$$\eta(e) \supseteq \{e + w, e + z; w \in M, 0 \neq z \in N\}.$$

Thus

$$3 = \mathrm{Nin}(R) \geq |\eta(e)| \geq |M| + |N| - 1 \Rightarrow 4 \leq |M| + |N| \leq 4 \ \Rightarrow |M| = |N| = 2.$$

Again $C = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \subseteq R$, so $\mathrm{Nin}(C) \leq \mathrm{Nin}(R) = 3$. But

$$\mathrm{Nin}(C) = 2\mathrm{Nin}(A)\mathrm{Nin}(B) \leq 3 \Rightarrow \mathrm{Nin}(A) = \mathrm{Nin}(B) = 1, \text{ so this proves (2).} \qquad \square$$

Note: Ring homomorphisms in general do not preserve the nil clean index. For example, if we consider a ring R of nil clean index 2, then R cannot be abelian, so $\mathrm{Nin}(R[[x]])$ can not be finite. But R is a homomorphic image of $R[[x]]$. However in case of $\mathrm{Nin}(R) = 1$, we have the following result.

Theorem 4.6. *The homomorphic image of a ring R with Nin$(R) = 1$ is again a ring with Nin$(R) = 1$, provided idempotents of R can be lifted modulo the kernel of the homomorphism.*

Proof. Straightforward. $\square$

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