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MAPPINGS BETWEEN MODULE LATTICES

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Abstract. We examine the properties of certain mappings between the lattice of ideals of a commutative ring R and the lattice of submodules of an R-module M, in particular considering when these mappings are lattice homomorphisms. We prove that the mapping λ from the lattice of ideals of R to the lattice of submodules of M defined by $\lambda(B) = BM$ for every ideal B of R is a (lattice) isomorphism if and only if M is a finitely generated faithful multiplication module. Moreover, for certain but not all rings R, there is an isomorphism from the lattice of ideals of R to the lattice of submodules of an R-module M if and only if the mapping λ is an isomorphism.

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1. Introduction

This paper is concerned with mappings, in particular homomorphisms, between lattices. Let L and L' be lattices. As usual, given a and b in L, the least upper bound and greatest lower bound of a and b are denoted by $a \vee b$ and $a \wedge b$, respectively. Given mappings φ and θ from the lattice L to the lattice L' we define $\varphi \leq \theta$ provided $\varphi(a) \leq \theta(a)$ for all $a \in L$. Clearly, $\varphi \leq \theta$ and $\theta \leq \varphi$ together imply $\varphi = \theta$. We begin with a very simple result.

Lemma 1.1. *Let L, L_1 and L_2 be lattices, let φ, φ_1 and φ_2 be mappings from L to L_1 and let θ, θ_1 and θ_2 be mappings from L_1 to L_2 such that $\varphi_1 \leq \varphi_2$, $\theta_1 \leq \theta_2$ and $\theta(a) \leq \theta(b)$ for all $a, b \in L_1$ with $a \leq b$. Then $\theta\varphi_1 \leq \theta\varphi_2$ and $\theta_1\varphi \leq \theta_2\varphi$.*

Proof. Let $a \in L$. Then $\varphi_1(a) \leq \varphi_2(a)$ and hence

$$\theta\varphi_1(a) = \theta(\varphi_1(a)) \leq \theta(\varphi_2(a)) = \theta\varphi_2(a).$$

On the other hand,

$$\theta_1\varphi(a) = \theta_1(\varphi(a)) \leq \theta_2(\varphi(a)) = \theta_2\varphi(a).$$

The result follows. $\square$

A mapping φ from a lattice L to a lattice L' is a *homomorphism* provided

$$\varphi(a \vee b) = \varphi(a) \ \vee \ \varphi(b) \text{ and } \varphi(a \wedge b) = \varphi(a) \ \wedge \ \varphi(b),$$

for all $a, b \in L$. A bijective (respectively, injective, surjective) homomorphism is called an *isomorphism* (respectively, *monomorphism, epimorphism*).

The next result is absolutely standard and easy to prove.

Lemma 1.2. *The following statements are equivalent for a bijection φ from a lattice L to a lattice L'.*

(i) *φ is an isomorphism.*
(ii) *$\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$ for all $a, b \in L$.*
(iii) *$\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$ for all $a, b \in L$.*

Moreover, in this case the inverse mapping $\varphi^{-1} : L' \to L$ is also an isomorphism.

Throughout this note all rings will be commutative with identity and all modules will be unital. Let R be a ring and M be any R-module. The collection of submodules of M form a lattice which we shall denote by $\mathcal{L}({}_RM)$ with respect to the following definitions:

$$L \vee N = L + N \text{ and } L \wedge N = L \cap N,$$

for all submodules L and N of M. Note that $\mathcal{L}({}_RM)$ is a lattice with least element the zero submodule, greatest element M and, for any given submodules L and N of M,

$$L \leq N \text{ in } \mathcal{L}({}_RM) \ \Leftrightarrow \ L \subseteq N \text{ in } M.$$

In particular, we shall denote the lattice $\mathcal{L}({}_RR)$ of ideals of R by $\mathcal{L}(R)$. We shall be interested in mappings between $\mathcal{L}(R)$ and $\mathcal{L}({}_RM)$.

Let R be a ring and M an R-module. Let L and N be submodules of M. Then $(L :_R N)$ will denote the set of elements $r \in R$ such that $rN \subseteq L$. Note that $(L :_R N)$ is an ideal of R. In particular, $(0 :_R M)$ is the *annihilator* of M in R and we shall denote it simply by $\text{ann}_R(M)$. As usual, M is called *faithful* in case $\text{ann}_R(M) = 0$.

Define a mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}({}_RM)$ by

$$\lambda(B) = BM,$$

for all ideals B of R and define a mapping $\mu : \mathcal{L}({}_RM) \to \mathcal{L}(R)$ by

$$\mu(N) = (N \ :_R \ M),$$

for every submodule N of M. Note that $\lambda(B) \leq \lambda(C)$ for all ideals B, C of R with $B \leq C$ and $\mu(L) \leq \mu(N)$ for all submodules L, N of M with $L \leq N$. Note further that for each ideal B of R,

$$B \subseteq (BM :_R M) = \mu\lambda(B),$$

and thus

$$1 \leq \mu\lambda.$$

On the other hand, for each submodule N of M,

$$\lambda\mu(N) = (N :_R M)M \subseteq N,$$

gives that

$$\lambda\mu \leq 1.$$

From Lemma 1.1 it follows that $\lambda = \lambda 1 \leq \lambda(\mu\lambda) = (\lambda\mu)\lambda \leq 1\lambda = \lambda$ and thus

$$\lambda = \lambda\mu\lambda.$$

Similarly, Lemma 1.1 gives that $\mu = 1\mu \leq (\mu\lambda)\mu = \mu(\lambda\mu) \leq \mu 1 = \mu$, so that

$$\mu = \mu\lambda\mu.$$

Lemma 1.3. *With the above notation, the following statements are equivalent.*

(i) λ *is a surjection.*

(ii) $\lambda\mu = 1$.

(iii) $N = (N :_R M)M$ *for every submodule* N *of* M.

(iv) μ *is an injection.*

Proof. (i) $\Rightarrow$ (ii) Because $\lambda\mu\lambda = \lambda$. (ii) $\Leftrightarrow$ (iii) Clear. (ii) $\Rightarrow$ (iv) Clear. (iv) $\Rightarrow$ (ii) Because $\mu\lambda\mu = \mu$. (ii) $\Rightarrow$ (i) Clear. $\square$

The proof of the next result is similar to the proof of Lemma 1.3

Lemma 1.4. *With the above notation, the following statements are equivalent.*

(i) λ *is an injection.*

(ii) $\mu\lambda = 1$.

(iii) $B = (BM :_R M)$ *for every ideal* B *of* R.

(iv) μ *is a surjection.*

Moreover, in this case M *is faithful.*

Corollary 1.5. *With the above notation, the mapping* λ *is a bijection if and only if* μ *is a bijection. In this case* λ *and* μ *are inverses of each other.*

Proof. By Lemmas 1.3 and 1.4. $\square$

Again let R be a ring and let M be an R-module. Let $A = \mathrm{ann}_R(M)$. By defining

$$(r + A)m = rm \quad (r \in R,\, m \in M),$$

M becomes a faithful (R/A)-module with the property that a subset X of M is an R-submodule of M if and only if X is an (R/A)-submodule of M. Thus the lattice

$\mathcal{L}({}_RM)$ is identical to the lattice $\mathcal{L}({}_{R/A}M)$. We define a mapping $\overline{\lambda} : \mathcal{L}(R/A) \to \mathcal{L}({}_RM)$ by

$$\overline{\lambda}(B/A) = BM,$$

for every ideal B of R containing A. In addition, we define a mapping $\overline{\mu} : \mathcal{L}({}_RM) \to \mathcal{L}(R/A)$ by

$$\overline{\mu}(N) = (N :_R M)/A,$$

for every submodule N of M, noting that, of course, $A \subseteq (N :_R M)$ for every submodule N of M. Lemmas 1.3 and 1.4 have analogues for $\overline{\lambda}$ and $\overline{\mu}$. Now we define $\pi : \mathcal{L}(R) \to \mathcal{L}(R/A)$ by

$$\pi(C) = (C + A)/A,$$

for every ideal C of R. It is clear that

$$\lambda = \overline{\lambda}\pi \text{ and } \overline{\mu} = \pi\mu.$$

We shall prove that a domain R is Prüfer if and only if the above mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}({}_RM)$ is a (lattice) homomorphism for every (cyclic) R-module M (Theorem 2.3). Given a general ring R, if M is a faithful multiplication module then the mapping λ is a homomorphism (Theorem 2.12). Moreover, λ is an isomorphism if and only if M is a finitely generated faithful multiplication module and in this case the inverse of λ is μ which is also an isomorphism (Theorem 4.3). Furthermore, in case R is a semilocal Noetherian domain or a Dedekind domain then the lattices $\mathcal{L}(R)$ and $\mathcal{L}({}_RM)$ are isomorphic if and only if the above mapping λ is an isomorphism (Corollary 5.4 and Theorem 5.5).

2. The Mapping λ

Let R be a ring and let M be an R-module. Let the mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}({}_RM)$ be as before, so that $\lambda(B) = BM$ for every ideal B of R. It is clear that

$$\lambda(B \vee C) = \lambda(B) \vee \lambda(C),$$

for all ideals B and C of R. Thus λ is a homomorphism if and only if

$$\lambda(B \wedge C) = \lambda(B) \wedge \lambda(C),$$

for all ideals B and C of R. It will be convenient for us to call the module M a λ-*module* in case the above mapping λ is a homomorphism.

Lemma 2.1. *The following statements are equivalent for an R-module M.*

(i) *M is a λ-module.*

(ii) *$(B_1 \cap \cdots \cap B_n)M = (B_1M) \cap \cdots \cap (B_nM)$ for every positive integer n and ideals $B_i\,(1 \leq i \leq n)$ of R.*

(iii) *$(B \cap C)M = BM \cap CM$ for all finitely generated ideals B and C of R.*

Proof. (i) $\Rightarrow$ (ii) By induction on n,

$$(B_1\cap\cdots\cap B_n)M = \lambda(B_1\wedge\cdots\wedge B_n) = \lambda(B_1)\wedge\cdots\wedge\lambda(B_n) = (B_1M)\cap\cdots\cap(B_nM).$$

(ii) $\Rightarrow$ (iii) Clear.

(iii) $\Rightarrow$ (i) Suppose that (iii) holds. Let G and H be any ideals of R. Then

$$\lambda(G\wedge H) = (G\cap H)M \subseteq GM\cap HM = \lambda(G)\wedge\lambda(H),$$

so that $\lambda(G\wedge H)\leq\lambda(G)\wedge\lambda(H)$. Let $m\in GM\cap HM$. There exist a positive integer n and elements $g_i\in G, h_i\in H (1\leq i\leq n)$ with

$$m\in(g_1M+\cdots+g_nM)\cap(h_1M+\cdots+h_nM) = G'M\cap H'M,$$

where $G' = Rg_1+\cdots+Rg_n\subseteq G$ and $H' = Rh_1+\cdots+Rh_n\subseteq H$. Now by hypothesis, $G'M\cap H'M = (G'\cap H')M\subseteq(G\cap H)M$. Thus $m\in(G\cap H)M$. We have proved that $GM\cap HM\subseteq(G\cap H)M$ so that $\lambda(G\wedge H)\leq\lambda(G)\wedge\lambda(H)$. It follows that $\lambda(G\wedge H) = \lambda(G)\wedge\lambda(H)$ for all ideals G and H of R. Thus λ is a homomorphism and M is a λ-module. $\square$

Let R be a domain with field of fractions F. Let I be any non-zero ideal of R. Then I^* is the R -submodule of F consisting of all elements $f\in F$ such that $fI\subseteq R$. Note that I^*I is an ideal of R. The ideal I is called *invertible* provided $I^*I = R$ (see, for example, [5, p. 67]). It is well known that I is an invertible ideal of R if and only if there exists an R-submodule X of F such that $IX = R$. The domain R is called *Prüfer* in case every non-zero finitely generated ideal of R is invertible. For a very comprehensive account of the properties of Prüfer domains see [5].

Lemma 2.2. *Let R be a domain, let B and C be invertible ideals of R and let M be the R-module $B+C$. Then M is a λ-module only if M is also an invertible ideal of R.*

Proof. We are given that

$$(B\cap C)(B+C) = \lambda(B\wedge C) = \lambda(B)\wedge\lambda(C) = B(B+C)\cap C(B+C).$$

Note that

$$(B\cap C)(B+C)\subseteq BC,$$

and

$$BC\subseteq B(B+C)\cap C(B+C).$$

Thus $BC = (B\cap C)(B+C)$. It follows that the ideal $B+C$ is invertible. $\square$

The next result is presumably known but we do not have a reference for it.

Theorem 2.3. *The following statements are equivalent for a (commutative) domain R.*

(i) *R is Prüfer.*
(ii) *Every R-module is a λ-module.*
(iii) *Every homomorphic image of a λ-module is a λ-module.*
(iv) *Every cyclic R-module is a λ-module.*
(v) *Every ideal of R is a λ-module.*
(vi) *Every 2-generated ideal of R is a λ-module.*

Proof. (i) $\Rightarrow$ (ii) Let M be any R-module. Let B and C be non-zero finitely generated ideals of R. Let $D = B + C$. Then D is a finitely generated ideal of R and hence $D^*D = R$. Let $G = D^*B$ and $H = D^*C$ so that G and H are ideals of R such that $G + H = D^*B + D^*C = D^*(B + C) = R$. Note that

$$BH = BD^*C = C(D^*B) = CG.$$

Now

$$B \cap C = (B \cap C)(G + H) \subseteq BH + CG = BH = CG \subseteq B \cap C,$$

so that $B \cap C = BH = CG$. Next

$$\begin{aligned} BM \cap CM &= (G + H)(BM \cap CM) \subseteq GCM + HBM \\ &= (B \cap C)M \subseteq BM \cap CM, \end{aligned}$$

and hence $BM \cap CM = (B \cap C)M$. By Lemma 2.1, M is a λ-module.

(ii) $\Rightarrow$ (iii) Clear.

(iii) $\Rightarrow$ (iv) Because the R-module R is clearly a λ-module.

(ii) $\Rightarrow$ (v) $\Rightarrow$ (vi) Clear.

(iv) $\Rightarrow$ (i) Let B, C and E be ideals of R. By (iv) and Lemma 2.1 we know that $B(R/E) \cap C(R/E) = (B \cap C)(R/E)$ and hence that $(B+E) \cap (C+E) = (B \cap C)+E$. It follows that $B \cap (C + E) \subseteq (B \cap C) + E$ and hence that $B \cap (C + E) \subseteq (B \cap C) + (B \cap E)$. Therefore $B \cap (C + E) = (B \cap C) + (B \cap E)$ for all ideals B, C and E of R. By [5, Theorem 25.2], R is a Prüfer domain.

(vi) $\Rightarrow$ (i) By Lemma 2.2, every non-zero 2-generated ideal of R is invertible because every non-zero principal ideal is clearly invertible. By [5, Theorem 22.1], R is a Prüfer domain. □

Note that if R is a domain which is not Prüfer then Theorem 2.3 shows that, despite the fact that the R-module R is a λ-module, there exists an ideal of R which is not a λ-module and there exists a homomorphic image of R which is not a λ-module. Thus in general the class of λ-modules is not closed under taking submodules and homomorphic images.

There is another class of rings for which every module is a λ-module. If R is a ring then an R-module M will be called a *chain module* provided the lattice $\mathcal{L}(_RM)$

is a chain, that is for any submodules N and L of M either $N \subseteq L$ or $L \subseteq N$. The ring R will be called a *chain ring* if the R-module R is a chain module.

Proposition 2.4. *Let R be any chain ring. Then every R-module is a λ-module.*

Proof. Let M be any R-module. Let B and C be ideals of R. Without loss of generality we can suppose that $B \subseteq C$ because R is a chain ring. Then $BM \subseteq CM$ and we have:

$$(B \cap C)M = BM = BM \cap CM.$$

By Lemma 2.1 M is a λ-module. □

To find other examples of λ-modules we first prove a simple lemma.

Lemma 2.5. *Let R be any ring. Then*

(a) *Every direct summand of a λ-module is a λ-module.*

(b) *Every direct sum of λ-modules is also a λ-module.*

Proof. (a) Let K be a direct summand of a λ-module M. Let B and C be any ideals of R. Then

$$\begin{aligned} BK \cap CK &= (K \cap BM) \cap (K \cap CM) = K \cap (BM \cap CM) \\ &= K \cap (B \cap C)M = (B \cap C)K. \end{aligned}$$

By Lemma 2.1 K is a λ-module.

(b) Let L_i $(i \in I)$ be any collection of λ-modules and let $L = \oplus_{i \in I} L_i$. Given any ideals B and C of R we have:

$$\begin{aligned} BL \cap CL &= (\oplus_{i \in I} BL_i) \cap (\oplus_{i \in I} CL_i) = \oplus_{i \in I}(BL_i \cap CL_i) \\ &= \oplus_{i \in I}(B \cap C)L_i = (B \cap C)L. \end{aligned}$$

By Lemma 2.1 L is a λ-module. □

Corollary 2.6. *Let R be any ring and let an R-module $M = M_1 \oplus M_2$ be the direct sum of a projective submodule M_1 and a semisimple submodule M_2. Then the module M is a λ-module.*

Proof. Clearly every simple module and the R-module R are λ-modules. Apply Lemma 2.5. □

For any ring R, every semisimple R-module is a λ-module but the mapping λ is not a monomorphism (if R is not von Neumann regular) and seldom an epimorphism, as the following result shows.

Proposition 2.7. *Let R be any ring and let M be a semisimple R-module. Then the above mapping λ is a homomorphism. Moreover,*

(a) *λ is a monomorphism only if R is von Neumann regular, and*

(b) *λ is an epimorphism only if M is cyclic.*

Proof. The first part follows by Corollary 2.6.

Let $M = \oplus_{i\in I} U_i$ for some non-empty collection of simple R-modules $U_i\,(i \in I)$. For each $i \in I$ let $P_i = \mathrm{ann}_R(U_i)$. First suppose that λ is a monomorphism. Let $0 \neq a \in R$. Then $\lambda(Ra) = \oplus_{j\in J} U_j$, where J is the set of elements $i \in I$ such that $a \notin P_i$. Clearly we have

$$\lambda(Ra^2) = \oplus_{j\in J} U_j = \lambda(Ra),$$

and hence $Ra = Ra^2$. This proves (a).

To prove (b), we suppose now that λ is an epimorphism. Suppose that I contains at least two elements. For each $i \in I$ let $B_i = \cap_{j\neq i} P_j$. By [4, Theorem 2.2] $R = P_i + B_i$ for all $i \in I$. It follows that $P_i \neq P_j$ for all distinct i, j in I. Moreover if $A = \cap_{i\in I} P_i$ then R/A is semisimple Artinian. Thus I is finite and M is a finite direct sum of non-isomorphic simple modules. In this case it is well known that M is cyclic. □

Let $\mathbb{Z}$ denote the ring of rational integers. Let M denote the $\mathbb{Z}$-module $\oplus_{p\in\mathcal{S}}(\mathbb{Z}/\mathbb{Z}p)$, for any infinite set $\mathcal{S}$ of primes in $\mathbb{Z}$. Then Proposition 2.7 shows that the mapping $\lambda : \mathcal{L}(\mathbb{Z}) \to \mathcal{L}(_{\mathbb{Z}}M)$ is a homomorphism that is neither a monomorphism nor an epimorphism. The question remains, for a given ring R, which R-modules M are λ-modules. This is certainly the case if $\lambda : \mathcal{L}(R) \to \mathcal{L}(_RM)$ is a bijection (see Lemma 1.2) but note the following simple fact.

Example 2.8. *Let R be any ring and let F be a free R-module of rank ≥ 2. Then the mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}(_RF)$ is a monomorphism but not an epimorphism.*

Proof. Given ideals B and C of R, $\lambda(B) = \lambda(C)$ implies that $BF = CF$ and hence $B = C$. It follows that λ is an injection. By Corollary 2.6 λ is a monomorphism. However, if F has a basis $f_i\,(i \in I)$ then, for each $j \in I$, $Rf_j \neq \lambda(B)$ for any ideal B of R. Thus λ is not an epimorphism. □

Next we consider when the mapping λ is a surjection for a given module M. The module M is called a *multiplication module* in case λ is a surjection, in which case M satisfies the equivalent conditions of Lemma 1.3. In other words, M is a multiplication module if and only if for each submodule N of M there exists an ideal B of R such that $N = BM$. Multiplication modules have been extensively studied (see, for example, [1] - [4], [8] - [11]). Note the following simple fact about multiplication modules that is included for completeness.

Lemma 2.9. *Let R be a ring and let M be an R-module with annihilator A in R. Then the R-module M is a multiplication module if and only if the (R/A)-module M is a multiplication module.*

Proof. Let N be any submodule of M and let B be any ideal of R. Then $N = BM$ if and only if $N = ((B + A)/A)M$. The result follows. □

It is quite possible for the mapping λ to be a surjection but not an injection, as we shall see shortly. Note the following fact about multiplication modules which we shall require later.

Lemma 2.10. *(See* [4, Theorem 1.2]*.) Let R be any ring. Then an R-module M is a multiplication module if and only if for each maximal ideal P of R either*

(a) *for each m in M there exists p in P such that $(1 - p)m = 0$, or*

(b) *there exist $x \in M$ and $q \in P$ such that $(1 - q)M \subseteq Rx$.*

Corollary 2.11. *Let R be any ring. Then an R-module M is a finitely generated multiplication module if and only if for each maximal ideal P of R there exist $m \in M$, $p \in P$ such that $(1 - p)M \subseteq Rm$.*

Proof. Suppose that M is a finitely generated multiplication module. There exist a positive integer n and elements $m_i \in M\,(1 \le i \le n)$ such that $M = Rm_1 + \cdots + Rm_n$. Let P be any maximal ideal of R. By the lemma, either (a) or (b) in the lemma holds. Suppose that (a) holds. Then for each $1 \le i \le n$ there exists $p_i \in P$ such that $(1 - p_i)m_i = 0$. Let $p = 1 - [(1 - p_1) \dots (1 - p_n)] \in P$ and note that $(1 - p)m_i = 0$ for all $1 \le i \le n$. Thus $(1 - p)M = 0$. This proves the necessity.

Conversely, suppose that the module M has the stated property. Suppose that the ideal $\sum_{x \in M} (Rx :_R M)$ is proper and let Q be a maximal ideal of R such that

$$\sum_{x \in M} (Rx :_R M) \subseteq Q.$$

By hypothesis, there exist $u \in M$, $q \in Q$ such that $(1 - q)M \subseteq Ru$ and hence $1 - q \in (Ru :_R M) \subseteq Q$, a contradiction. Thus $R = \sum_{x \in M} (Rx :_R M)$ and there exist a positive integer k and elements $u_i \in M\,(1 \le i \le k)$ such that $R = (Ru_1 :_R M) + \cdots + (Ru_k :_R M)$. It follows that

$$M = RM = (Ru_1 :_R M)M + \cdots + (Ru_k :_R M)M \subseteq Ru_1 + \cdots + Ru_k \subseteq M.$$

Hence $M = Ru_1 + \cdots + Ru_k$ and M is finitely generated. By [4, Corollary 1.5] M is a multiplication module. □

The next result shows that multiplication modules are λ-modules in certain cases.

Theorem 2.12. *Let R be any ring. Then every faithful multiplication R-module is a λ-module.*

Proof. By Lemma 2.1 and [4, Theorem 1.6]. □

Corollary 2.13. *Let R be a ring and let M be any multiplication module with $A = ann_R(M)$. Then the mapping $\overline{\lambda} : \mathcal{L}(R/A) \to \mathcal{L}({}_RM)$ is a homomorphism.*

Proof. By Lemma 2.9 and Theorem 2.12. □

Corollary 2.14. *Let R be any ring and let an R-module M be a direct summand of a direct sum of faithful multiplication R-modules. Then M is a λ-module.*

Proof. By Lemma 2.5 and Theorem 2.12. □

For any ring R, the R-module $R \oplus R$ is a faithful λ-module which is not a multiplication module and therefore the converse of Theorem 2.12 is false. Note further that in general if R is a domain which is not Prüfer then there exists a cyclic R-module which is not a λ-module (see Theorem 2.3). Clearly cyclic modules are multiplication modules. This shows that the modules in Theorem 2.12 need to be faithful. Moreover in Theorem 2.12, although λ is an epimorphism it need not be an isomorphism as the following result shows.

Proposition 2.15. *Let R be a ring and let I be a proper ideal of R which is generated by idempotent elements such that $ann_R(I) = 0$. Then the R-module I is a faithful multiplication module and the mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}({}_RI)$ is an epimorphism but not a monomorphism.*

Proof. Let J be any ideal of R with $J \subseteq I$. Let $a \in J$. Then $a \in Re_1 + \cdots + Re_n$ for some positive integer n and idempotent elements $e_i (1 \leq i \leq n)$ of I. It is well known that there exists an idempotent $e \in I$ such that $Re_1 + \cdots + Re_n = Re$. Then $a = be$ for some $b \in R$ and hence $a = ae \in JI$. It follows that $J = JI$. Hence the R-module I is a faithful multiplication module. By Theorem 2.12 λ is an epimorphism. However, $\lambda(R) = RI = I^2 = \lambda(I)$, so that λ is not an injection. □

To illustrate Proposition 2.15 we have the following example.

Example 2.16. *Let a ring $R = \prod_{i \in I} F_i$ be the direct product of any infinite collection of fields F_i $(i \in I)$. Then R is a commutative von Neumann regular ring whose socle $S = \oplus_{i \in I} F_i$. Moreover for any proper ideal B of R with $S \subseteq B$ the R-module B is a faithful multiplication module such that the mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}({}_RB)$ is an epimorphism but not a monomorphism.*

Proof. It is clear that R is a commutative von Neumann regular ring. Let $f = \{f_i\}$ be any non-zero element of R where the ith component f_i is an element of F_i for each $i \in I$. For each $j \in I$ let $e_{(j)}$ denote the element in R whose jth component is 1 and all of whose other components are 0. There exists $k \in I$ such that $f_k \neq 0$. Then $e_{(k)}f$ is a non-zero element of U_k where U_k is the ideal of R consisting of all elements $\{u_i\}$ in R such that $u_i = 0$ for all $i \neq k$. Clearly U_i is a simple submodule of ${}_RR$ for each $i \in I$ and $\oplus_{i \in I} U_i$ is an essential submodule of ${}_RR$ so that $S = \oplus_{i \in I} U_i = \oplus_{i \in I} F_i$ is the socle of R. Let B be any proper ideal of R with $S \subseteq B$. Then B is generated by idempotent elements and $\mathrm{ann}_R(B) = 0$. Apply Proposition 2.15. □

Note that in Proposition 2.15 the ideal I is not finitely generated.

3. The Mapping μ

Given a ring R, in this section we shall consider the mapping μ from the lattice $\mathcal{L}({}_RM)$ of submodules of an R-module M to the lattice $\mathcal{L}(R)$ of ideals of R defined by $\mu(N) = (N :_R M) = \mathrm{ann}_R(M/N)$. We shall say that M is a *μ-module* in case the mapping μ is a homomorphism. First note the following simple fact.

Lemma 3.1. *Let R be a ring and let M be an R-module. Then M is a μ-module if and only if $(N + L :_R M) = (N :_R M) + (L :_R M)$ for all submodules N and L of M.*

Proof. Let N and L be any submodules of M. Clearly

$$\mu(N \wedge L) = (N \cap L :_R M) = (N :_R M) \cap (L :_R M) = \mu(N) \wedge \mu(L).$$

Next note that

$$\mu(N \vee L) = \mu(N) \vee \mu(L) \Leftrightarrow (N + L :_R M) = (N :_R M) + (L :_R M).$$

The result follows. □

Corollary 3.2. *Let M be a μ-module for a ring R such that $M = N + L$ for some submodules N and L. Then there exists $a \in R$ such that $aM \subseteq N$ and $(1 - a)M \subseteq L$.*

Proof. Note that Lemma 3.1 gives:

$$R = (M :_R M) = (N + L :_R M) = (N :_R M) + (L :_R M).$$

Clearly the result follows. □

In contrast to Theorem 2.3, no (non-zero) ring R has the property that the mapping $\mu : \mathcal{L}({}_RM) \to \mathcal{L}(R)$ is a homomorphism for every R-module M, as the next result shows.

Corollary 3.3. *Let R be any (non-zero) ring and let M be any non-zero R-module. Then the R-module $M \oplus M$ is not a μ-module.*

Proof. Suppose that $M \oplus M$ is a μ-module. By Corollary 3.2, there exists $a \in R$ such that $a(M \oplus M) \subseteq M \oplus 0$ and $(1 - a)(M \oplus M) \subseteq 0 \oplus M$, so that $aM = 0$ and $(1 - a)M = 0$ giving $M = 0$, a contradiction. □

Compare the next result with Lemma 2.10

Corollary 3.4. *Let M be a μ-module for a ring R. Then for each maximal ideal P of R either $M = PM$ or there exist $m \in M$ and $p \in P$ such that $(1 - p)M \subseteq Rm$.*

Proof. Let P be a maximal ideal of R such that $M \neq PM$. Note that M/PM is a non-zero semisimple module and hence contains a maximal submodule. Let L be a maximal submodule of M such that $PM \subseteq L$. Let m be any element of M with $m \notin L$. Clearly $M = L + Rm$. By Corollary 3.2 there exists an element $p \in R$ such that $pM \subseteq L$ and $(1-p)M \subseteq Rm$. If $p \notin P$ then $R = P + Rp$ and hence $M = PM + pM \subseteq L$, a contradiction. Thus $p \in P$, as required. □

Corollary 3.5. *Let R be a ring and let M be a μ-module over R such that $M \neq PM$ for every maximal ideal P of R. Then the R-module M is a multiplication module. Moreover the mapping μ is a monomorphism.*

Proof. By Lemmas 1.3 and 2.10 and Corollary 3.4. □

We now aim to prove a partial converse of Corollary 3.5. It is proved in Theorem 2.3 that a domain is Prüfer if and only if every homomorphic image of a λ-module is a λ-module. Contrast this fact with the following result.

Proposition 3.6. *Every homomorphic image of a μ-module is a μ-module.*

Proof. Let K be any submodule of M and let $\overline{M}$ be the module M/K. Let $\overline{N}$ be any submodule of $\overline{M}$. There exists a submodule N of M containing K such that $\overline{N} = N/K$. Then

$$(\overline{N} :_R \overline{M}) = \mathrm{ann}_R(\overline{M}/\overline{N}) = \mathrm{ann}_R(M/N) = (N :_R M).$$

Apply Lemma 3.1. □

Corollary 3.7. *For any ring R, every cyclic R-module is a μ-module.*

Proof. Consider the R-module R. Let B and C be any ideals of R. Then

$$(B + C :_R R) = B + C = (B :_R R) + (C :_R R).$$

By Lemma 3.1 ${}_RR$ is a μ-module. Now apply Proposition 3.6. □

We now characterize which finitely generated modules are μ-modules.

Theorem 3.8. *Given any ring R, a finitely generated R-module is a μ-module if and only if M is a multiplication module.*

Proof. The necessity is proved in Corollary 3.5. Conversely suppose that M is a multiplication module. Let N, L be any submodules of M. Clearly $(N :_R M)+(L :_R M) \subseteq (N + L :_R M)$. Suppose that $(N :_R M) + (L :_R M) \neq (N + L :_R M)$. Let $a \in (N + L :_R M)$ such that $a \notin (N :_R M) + (L :_R M)$. Let $B = \{r \in R : ra \in (N :_R M) + (L :_R M)\}$. Then B is a proper ideal of R and hence $B \subseteq P$ for some maximal ideal P of R. Because M is a finitely generated multiplication module, Corollary 2.11 shows that there exist $m \in M$ and $p \in P$ such that $(1-p)M \subseteq Rm$.

In particular, note that $(1-p)N$ and $(1-p)L$ are both submodules of the cyclic R-module Rm. Next,

$$(1-p)aM \subseteq (1-p)(N+L) = (1-p)N + (1-p)L.$$

Because Rm is a μ-module by Corollary 3.7, we now see that

$$(1-p)a \in ((1-p)N + (1-p)L :_R Rm) = ((1-p)N :_R Rm) + ((1-p)L :_R Rm).$$

Therefore $(1-p)a = b+c$ for some $b \in ((1-p)N :_R Rm)$ and some $c \in ((1-p)L :_R Rm)$. Note that

$$(1-p)bM \subseteq bRm \subseteq (1-p)N \subseteq N,$$

and hence that $(1-p)b \in (N :_R M)$. Similarly $(1-p)c \in (L :_R M)$. But this implies that

$$(1-p)^2 a = (1-p)b + (1-p)c \in (N :_R M) + (L :_R M),$$

which in turn implies that $(1-p)^2 \in B \subseteq P$, a contradiction. It follows that $(N + L :_R M) = (N :_R M) + (L :_R M)$ for all submodules N and L of M. By Lemma 3.1 M is a μ-module. □

Corollary 3.9. *The following statements are equivalent for a module M over a ring R.*

(i) *Every finitely generated submodule of M is a μ-module.*
(ii) *Every 2-generated submodule of M is a μ-module.*
(iii) *$R = (Rx :_R Ry) + (Ry :_R Rx)$ for all elements $x, y \in M$.*
(iv) *$R + (N :_R L) + (L :_R N)$ for all finitely generated submodules N and L of M.*
(v) *Every finitely generated submodule of M is a multiplication module.*

Proof. (i) $\Leftrightarrow$ (v) By Theorem 3.8.

(i) $\Rightarrow$ (ii) Clear.

(ii) $\Rightarrow$ (iii) Let $x, y \in M$. By Lemma 3.1,

$$R = (Rx + Ry :_R Rx + Ry) = (Rx :_R Rx + Ry) + (Ry :_R Rx + Ry)$$

$$= (Rx :_R Ry) + (Ry :_R Rx).$$

(iii) $\Rightarrow$ (v) Let K be any finitely generated submodule of M. There exist a positive integer n and elements $u_i \in K$ $(1 \leq i \leq n)$ such that $K = Ru_1 + \cdots + Ru_n$. If $n = 1$ then K is a multiplication module. Suppose that $n \geq 2$. By induction on n the submodule $L = Ru_1 + \cdots + Ru_{n-1}$ is a multiplication module. Let P be any maximal ideal of R. By Corollary 2.11, there exist $v \in L$, $p \in P$ such that $(1-p)L \subseteq Rv$. Let $u = u_n$ and note that $K = L + Ru$. By hypothesis, $R = (Ru :_R Rv) + (Rv :_R Ru)$ so that either $(Ru :_R Rv) \nsubseteq P$ or $(Rv :_R Ru) \nsubseteq P$.

Suppose first that $(Ru :_R Rv) \nsubseteq P$. Then there exists $p_1 \in P$ such that $(1 - p_1)v \in Ru$. This implies that

$$(1 - p_1)(1 - p)K \subseteq (1 - p_1)(1 - p)L + Ru \subseteq Ru.$$

Now suppose that $(Rv :_R Ru) \nsubseteq P$. Then there exists $p_2 \in P$ such that $(1 - p_2)u \in Rv$. In this case,

$$(1 - p_2)(1 - p)K \subseteq (1 - p_2)(1 - p)L + (1 - p)(1 - p_2)Ru \subseteq Rv.$$

In case $n \geq 2$ we have proved that for each maximal ideal P of R there exist an element w in K and an element q in P such that $(1 - q)K \subseteq Rw$. By Lemma 2.10 K is a multiplication module.

(iv) $\Rightarrow$ (iii) Clear.

(i) $\Rightarrow$ (iv) Given any finitely generated submodules N, L of M, the submodule $N + L$ is also finitely generated and hence a μ-module. By Lemma 3.1,

$$R = (N + L :_R N + L) = (N :_R N + L) + (L :_R N + L)$$

$$= (N :_R L) + (L :_R N).$$

This completes the proof. □

Next we give examples to show that the condition that the module M be finitely generated is necessary for both implications in Theorem 3.8.

Example 3.10. *Let F be any field and let R denote the collection of all sequences $\{f_1, f_2, f_3, \dots\}$ of elements of F with the property that there exists a positive integer n (depending on the particular sequence) such that $f_n = f_{n+1} = f_{n+2} = \dots$. Then R is a commutative von Neumann regular ring whose socle S is a multiplication module but not a μ-module.*

Proof. It is not difficult to see that the commutative ring R is von Neumann regular. Next it is clear that S consists of all sequences $\{f_1, f_2, f_3, \dots\}$ of elements of F such that $0 = f_n = f_{n+1} = f_{n+2} = \dots$ for some positive integer n. By Proposition 2.15, the module S is a faithful multiplication module. Let S_1 denote the subset of S consisting of all sequences $\{f_1, f_2, f_3, \dots\}$ such that $f_{2k} = 0\,(k \geq 1)$ and let S_2 denote the subset of S consisting of all elements $\{f_1, f_2, f_3, \dots\}$ with $f_{2k-1} = 0\,(k \geq 1)$. Clearly S_1 and S_2 are submodules of the R-module S such that $S = S_1 \oplus S_2$, $(S_1 :_R S) = S_1$ and $(S_2 :_R S) = S_2$. By Corollary 3.2 the R-module S is not a μ-module. □

Example 3.11. *Again let $\mathbb{Z}$ denote the ring of rational integers and let M denote the Prüfer p-group for any prime p in $\mathbb{Z}$. Then the $\mathbb{Z}$-module M is a μ-module but not a multiplication module.*

Proof. If N and L are proper submodules of M then $(N :_R M) = (L :_R M) = 0$ by [7, Proposition 2.6]. Now Lemma 3.1 gives that M is a μ-module. On the other hand, if K is any proper non-zero submodule of M then $K \neq BM$ for any ideal B of $\mathbb{Z}$ and hence M is not a multiplication module. $\square$

Recall that a module M over a general ring R is called *hollow* in case M is not the sum of two proper submodules. Next we generalise Example 3.11

Proposition 3.12. *Let R be a domain which is not a field and let M be a non-zero injective R-module. Then*

(a) *The above mapping λ is a homomorphism but is neither a monomorphism nor an epimorphism.*

(b) *The mapping μ is a homomorphism if and only if M is a hollow module. However μ is neither a monomorphism nor an epimorphism.*

Proof. Note that $M = BM = \lambda(B)$ for every non-zero ideal B of R and $0 = (N :_R M) = \mu(N)$ for every proper submodule N of M by [7, Proposition 2.6].

(a) For all ideals B and C of R we have $(B \cap C)M = BM \cap CM$. By Lemma 2.1 λ is a homomorphism. Let a be any non-zero element of R such that a is not a unit in R. Then $\lambda(Ra) = M = \lambda(R)$ so that λ is not a monomorphism. For any maximal ideal P of R, $P \neq 0$ and hence $M = PM$. Thus M is not simple. If L is a non-zero proper submodule of M then $L \neq \lambda(E)$ for any ideal E of R. Thus λ is not an epimorphism.

(b) It is clear that μ is a homomorphism if and only if whenever N and L are proper submodules of M then the submodule $N + L$ is also proper, that is M is hollow. We have seen above that M is not simple and hence $\mu(L) = 0 = \mu(0)$ for every non-zero proper submodule L of M. Thus μ is not a monomorphism. Moreover if a is as before we have $Ra \neq \mu(K)$ for every submodule K of M, so that μ is not an epimorphism. $\square$

The next result is an analogue of Theorem 2.3. It is an immediate consequence of Corollary 3.9 because domains with the property that every finitely generated ideal is a multiplication module are precisely Prüfer domains.

Theorem 3.13. *The following statements are equivalent for a domain R.*

(i) *R is Prüfer.*
(ii) *Every finitely generated ideal of R is a μ-module.*
(iii) *Every 2-generated ideal of R is a μ-module.*

Proof. By Corollary 3.9. $\square$

Note that Theorem 3.13 shows that the class of μ-modules is not closed under submodules in general.

Lemma 3.14. *Given any ring R, the following statements are equivalent for an R-module M.*

(i) *$R = (N :_R L) + (L :_R N)$ for all submodules N and L of M.*
(ii) *Every submodule of M is a μ-module.*

Proof. (i) $\Rightarrow$ (ii) Let K be any submodule of M and let N and L be submodules of K. Then

$$(N + L :_R K) = R(N + L :_R K) = [(N :_R L) + (L :_R N)](N + L :_R K)$$
$$= (N :_R N + L)(N + L :_R K) + (L :_R N + L)(N + L :_R K)$$
$$\subseteq (N :_R K) + (L :_R K).$$

It is clear that $(N :_R K) + (L :_R K) \subseteq (N + L :_R K)$. By Lemma 3.1 K is a μ-module.

(ii) $\Rightarrow$ (i) Let N and L be submodules of M. Then the submodule $N + L$ is a μ-module. Lemma 3.1 then gives

$$R = (N+L :_R N+L) = (N :_R N+L)+(L :_R N+L) = (N :_R L)+(L :_R N). \quad \square$$

The next result should be compared with Proposition 2.4. Recall that a (non-zero) ring R is called *local* in case it contains precisely one maximal ideal.

Proposition 3.15. *Given any local ring R, the following statements are equivalent for an R-module M.*

(i) *M is a chain module.*
(ii) *Every submodule of M is a μ-module.*
(iii) *Every finitely generated submodule of M is a μ-module.*

Proof. (i) $\Rightarrow$ (ii) Given any submodules N and L of M either $N \subseteq L$ or $L \subseteq N$ and hence $R = (N :_R L) + (L :_R N)$. By Lemma 3.14 every submodule of M is a μ-module.

(ii) $\Rightarrow$ (iii) Clear.

(iii) $\Rightarrow$ (i) Let U and V be any submodules of M such that $U \nsubseteq V$. Let $u \in U \setminus V$. Let $v \in V$. Then $Ru + Rv$ is a μ-module and Lemma 3.1 gives that

$$R = (Ru + Rv :_R Ru + Rv)) = (Ru :_R Ru + Rv) + (Rv :_R Ru + Rv)$$
$$= (Ru :_R Rv) + (Rv :_R Ru).$$

Because R is local either $1 \in (Ru :_R Rv)$ and $v \in Ru$ or else $1 \in (Rv :_R Ru)$ and $u \in Rv \subseteq V$. By the choice of u it follows that $v \in Ru$ for all $v \in V$ and hence $V \subseteq Ru \subseteq U$. We have proved that $U \subseteq V$ or $V \subseteq U$ for any submodules U and V of M. Thus M is a chain module. $\square$

We next consider semisimple modules.

Proposition 3.16. *Let R be any ring. Then the following statements are equivalent for a semisimple R-module M.*

(i) *M is a μ-module.*

(ii) *$(N + L :_R M) = (N :_R M) + (L :_R M)$ for all submodules N and L of M with $N \cap L = 0$.*

(iii) *$R = ann_R(N) + ann_R(L)$ for all submodules N and L of M with $N \cap L = 0$.*

Proof. (i) $\Rightarrow$ (ii) Clear by Lemma 3.1.

(ii) $\Rightarrow$ (iii) Let N and L be any submodules of M with $N \cap L = 0$. There exists a submodule K of M such that $M = N \oplus L \oplus K$. By (ii),

$$R = (M :_R M) = (L \oplus K :_R M) + (N :_R M) = \mathrm{ann}_R(N) + \mathrm{ann}_R(L \oplus K)$$
$$\subseteq \mathrm{ann}_R(N) + \mathrm{ann}_R(L) \subseteq R,$$

so that $R = \mathrm{ann}_R(N) + \mathrm{ann}_R(L)$.

(iii) $\Rightarrow$ (i) Let N and L be submodules of M Suppose first that $N \cap L = 0$. Let $\mathrm{ann}_R(N) = B$ and $\mathrm{ann}_R(L) = C$. By hypothesis, $R = B + C$. If $M = N \oplus L$ then $(N + L :_R M) = R = C + B = (N :_R M) + (L :_R M)$. Now suppose that $M \neq N \oplus L$. There exists a non-zero submodule K of M such that $M = N \oplus L \oplus K$. Note that $(N + L :_R M) = \mathrm{ann}_R(K) = D$ (say), $(N :_R M) = \mathrm{ann}_R(L \oplus K) = C \cap D$ and $(L :_R M) = \mathrm{ann}_R(N \oplus K) = B \cap D$. By hypothesis $R = B + D$ implies that $B \cap D = (B \cap D)(B + D) \subseteq BD \subseteq B \cap D$. Thus $B \cap D = BD$. Similarly $C \cap D = CD$. Combining this information gives us the following:

$$(N + L :_R M) = D = D(B + C) = CD + BD = (N :_R M) + (L :_R M).$$

Now suppose that $N \cap L$ need not be zero. There exists a submodule H of M such that $L = (N \cap L) \oplus H$. This implies that $N + L = N \oplus H$. By the above proof we have

$$(N + L :_R M) = (N \oplus H :_R M) = (N :_R M) + (H :_R M)$$
$$\subseteq (N :_R M) + (L :_R M) \subseteq (N + L :_R M).$$

Thus in any case $(N + L :_R M) = (N :_R M) + (L :_R M)$ for all submodules N and L of M. By Lemma 3.1 M is a μ-module. $\square$

Example 3.17. *Let R be the ring in Example 2.16. Then the socle S of R is a semisimple μ-module over R.*

Proof. We saw in Example 2.16 that $S = \oplus_{i \in I} U_i$. For each $i \in I$ let $\pi_i : S \to U_i$ denote the canonical projection. Let N and L be submodules of the R-module S such that $N \cap L = 0$. It is rather easy to see that $N \cap U_i = \pi_i(N)$ for all $i \in I$. Let $I_1 = \{i \in I : \pi_i(N) \neq 0\}$. Then $N = \oplus_{i \in I_1} U_i$. Similarly, if $I_2 = \{i \in I : \pi_i(L) \neq 0\}$ then $L = \oplus_{i \in I_2} U_i$. Because $N \cap L = 0$, the set $I_1 \cap I_2$ must be empty or, in other words, $I_2 \subseteq I \setminus I_1$. Let $f = \{f_i\} \in R$. Let g denote the element $\{g_i\}$ such that $g_i = f_i \, (i \in I_2)$ and otherwise $g_i = 0$. Then

$$f = g + (f - g) \in \operatorname{ann}_R(N) + \operatorname{ann}_R(L).$$

This proves that $R = \operatorname{ann}_R(N) + \operatorname{ann}_R(L)$. By Proposition 3.16 it follows that the R-module S is a μ-module. □

We saw in Corollary 2.6 that every semisimple module is a λ-module. However Proposition 3.16 shows that if M is the semisimple $\mathbb{Z}$-module $\oplus_{p \in \Pi} (\mathbb{Z}/\mathbb{Z}p)$, where Π is the collection of all primes p in $\mathbb{Z}$, then M is not a μ-module.

4. Finitely Generated Modules

Again R is a commutative ring and M a unital R-module. We begin this section with the following elementary result.

Lemma 4.1. *Let P be a maximal ideal of a ring R and let M be an R-module. Then $M \neq PM$ if and only if $P = (PM :_R M)$.*

Proof. Clear. □

The next result can be found essentially in [4, Theorem 3.1].

Lemma 4.2. *Let R be any ring and let M be an R-module with annihilator A in R. If M is finitely generated then $M \neq PM$ for every maximal ideal P of R with $A \subseteq P$. Moreover, the converse holds in case M is a multiplication module.*

Proof. Suppose first that M is finitely generated. Suppose further that $M = PM$ for some maximal ideal P of R. Then the usual determinant argument gives that $(1 - p)M = 0$ for some $p \in P$. Thus $1 - p \in A$ and hence $A \nsubseteq P$. Conversely, now suppose that M is a multiplication module such that $M \neq PM$ for every maximal ideal P of R with $A \subseteq P$. We have to prove that M is finitely generated. Suppose that $R \neq \sum_{m \in M} (Rm :_R M)$. Then there exists a maximal ideal Q of R such that $\sum_{m \in M} (Rm :_R M) \subseteq Q$. Note that $A = (R0 :_R M) \subseteq Q$. By hypothesis,there exists an element x in M such that $x \notin QM$. Because M is a multiplication module, $Rx = BM$ for some ideal B of R. Clearly $B \nsubseteq Q$. But $B \subseteq (Rx :_R M) \subseteq Q$, a contradiction. Thus $R = \sum_{m \in M} (Rm :_R M)$. There exist a positive integer n and elements $m_i \in M \, (1 \leq i \leq n)$ such that $1 \in (Rm_1 :_R M) + \cdots + (Rm_n :_R M)$. In this case, we have

$$M = 1M \subseteq (Rm_1 :_R M)M + \cdots + (Rm_n :_R M)M \subseteq Rm_1 + \cdots + Rm_n \subseteq M,$$

and hence $M = Rm_1 + \cdots + Rm_n$. □

Recall that we are interested in the mapping λ from the lattice $\mathcal{L}(R)$ of ideals of a ring R to the lattice $\mathcal{L}({}_RM)$ of submodules of an R-module M defined by $\lambda(B) = BM$ for every ideal B of R and the mapping $\mu : \mathcal{L}({}_RM) \to \mathcal{L}(R)$ defined by $\mu(N) = (N :_R M)$ for every submodule N of M. Now we come to our main result.

Theorem 4.3. *Let R be a ring and let M be an R-module. Then the following statements are equivalent.*

(i) *The mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}({}_RM)$ is an isomorphism.*
(ii) *The mapping $\mu : \mathcal{L}({}_RM) \to \mathcal{L}(R)$ is an isomorphism.*
(iii) *M is a multiplication module such that $B = (BM :_R M)$ for every ideal B of R.*
(iv) *M is a finitely generated faithful multiplication module.*

Proof. (i) $\Leftrightarrow$ (ii) By Corollary 1.5.

(i) $\Rightarrow$ (iii) Because λ is an isomorphism, λ is a bijection. In particular, λ is a surjection and hence M is a multiplication module. Next Lemma 1.4 shows that $\mu\lambda = 1$ because λ is an injection. Therefore, for each ideal B of R,

$$B = \mu\lambda(B) = \mu(BM) = (BM :_R M).$$

This proves (iii).

(iii) $\Rightarrow$ (iv) Note that $\mathrm{ann}_R(M) = (0 :_R M) = 0$ by (iii) and hence M is faithful. Moreover, for any maximal ideal Q of R, $QM = M$ implies that $R = (M :_R M) = (QM :_R M) = Q$, a contradiction. Thus $M \neq QM$ for every maximal ideal M of R. By Lemma 4.2 M is finitely generated.

(iv) $\Rightarrow$ (i) By [4, Theorem 3.1], if C and D are ideals of R such that $\lambda(C) = \lambda(D)$ then $CM = DM$ and hence $C = D$. Thus λ is an injection. But M being a multiplication module gives that λ is a surjection. Being a bijection, λ is an isomorphism by Lemma 1.2. $\square$

Corollary 4.4. *Let R be a ring and let M be an R-module with annihilator A in R. Then M is a finitely generated multiplication module if and only if the mapping $\overline{\lambda} : \mathcal{L}(R/A) \to \mathcal{L}({}_RM)$ is an isomorphism.*

Proof. Suppose that M is a finitely generated multiplication module. Then the (R/A)-module M is a finitely generated faithful multiplication module (Lemma 2.9) and the mapping $\overline{\lambda}$ is an isomorphism by Theorem 4.3. Conversely, suppose that $\overline{\lambda}$ is an isomorphism. By Theorem 4.3, the (R/A)-module M is a finitely generated multiplication module. It follows that the R-module M is a finitely generated multiplication module by Lemma 2.9. $\square$

Corollary 4.5. *Let R be a domain and let M be an R-module. Then the mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}({}_RM)$ is an isomorphism if and only if M is a faithful multiplication module.*

Proof. Suppose that M is a (non-zero) faithful multiplication module. By [4, Lemma 4.3 and Theorem 4.4] M is finitely generated. Now apply Theorem 4.3. □

5. Isomorphisms of Lattices

Let R be a ring and M an R-module. Suppose that $\rho : \mathcal{L}(R) \to \mathcal{L}(R)$ is a lattice isomorphism (in particular, ρ could be the lattice isomorphism induced from an automorphism of the ring R) and suppose that $\sigma : \mathcal{L}({}_RM) \to \mathcal{L}({}_RM)$ is a lattice isomorphism(in particular, σ could be the lattice isomorphism induced from an R-isomorphism from M to M). Then $\sigma\lambda\rho : \mathcal{L}(R) \to \mathcal{L}({}_RM)$ is a mapping which is an isomorphism if and only if λ is an isomorphism. Thus in general there can be many mappings from $\mathcal{L}(R)$ to $\mathcal{L}({}_RM)$ each of which may or may not be an isomorphism.

Now let R be a ring which is not Hopfian, that is there exists a ring epimorphism $\nu : R \to R$ which is not a monomorphism. Let A denote the non-zero kernel of ν. Then $R \cong R/A$ so that $\mathcal{L}(R)$ is isomorphic to $\mathcal{L}(R/A)$ which in turn is isomorphic to $\mathcal{L}({}_R(R/A))$. However in this case $\lambda : \mathcal{L}(R) \to \mathcal{L}({}_R(R/A))$ is not an isomorphism because R/A is not a faithful R-module. For a specific example, let R denote the polynomial ring $F[x_1, x_2, x_3, \dots]$ in indeterminates $x_i\,(i \geq 1)$ over a field F. Let n be any positive integer. The linear mapping ν from the F-vector space R to R defined by $\nu(x_i) = 0$ if $1 \leq i \leq n$ and $\nu(x_i) = i - n$ if $i \geq n+1$ induces a ring epimorphism with non-zero kernel $A_n = Rx_1 + \cdots + Rx_n$.

This leads us to ask the following question.

Question 5.1. *Let R be a ring and let M be an $R-$module such that the lattices $\mathcal{L}(R)$ and $\mathcal{L}({}_RM)$ are isomorphic. Then what can one say about the module M? For example, need M be a multiplication module?*

As a contribution towards the answer to this question we offer some modest results. We look at Noetherian rings and modules. Any Noetherian module has Krull dimension (see, for example, [6, Lemma 6.2.3]). For the definition and properties of Krull dimension see [6, Chapter 6]. We shall denote the Krull dimension of the ring R by $\mathrm{kdim}(R)$ and of an R-module M by $\mathrm{kdim}({}_RM)$, if either exists.

Lemma 5.2. *Let R be a Noetherian domain and let M be an R-module such that the lattices $\mathcal{L}(R)$ and $\mathcal{L}({}_RM)$ are isomorphic. Then M is a finitely generated faithful R-module.*

Proof. Let $A = \mathrm{ann}_R(M)$. Clearly the module M is Noetherian and $\mathrm{kdim}(R) = \mathrm{kdim}({}_RM)$. But M is a finitely generated (R/A)-module. By [6, Lemma 6.2.5],

$\mathrm{kdim}(M) \leq \mathrm{kdim}(R/A)$. Thus $\mathrm{kdim}(R) \leq \mathrm{kdim}(R/A)$. Finally, by [6, Proposition 6.3.11], $A = 0$ and hence M is faithful. □

Recall that a ring R is called *semilocal* provided it contains only a finite number of maximal ideals.

Theorem 5.3. *Let R be a ring and let M be a finitely generated R-module such that the lattices $\mathcal{L}(R)$ and $\mathcal{L}({}_RM)$ are isomorphic. Suppose further that either (a) R is a local ring or (b) R is a semilocal ring and M is faithful. Then M is cyclic.*

Proof. (a) Suppose that R has unique maximal ideal P. Then the lattice $\mathcal{L}({}_RM)$ must have only one maximal submodule which must be the submodule PM. Let $m \in M \setminus PM$. Then $M = Rm + PM$. It follows that $M/Rm = P(M/Rm)$. But M is finitely generated so that Nakayama's Lemma gives that $M = Rm$.

(b) Let $Q_1, \ldots, Q_n$ denote the maximal ideals of R, for some positive integer n. If $M = Q_iM$, for some $1 \leq i \leq n$, then by the usual determinant argument we have $(1-q)M = 0$ for some $q \in Q_i$, contradicting the fact that M is faithful. Thus $M \neq Q_iM$ $(1 \leq i \leq n)$. Because the lattices $\mathcal{L}(R)$ and $\mathcal{L}({}_RM)$ are isomorphic it follows that the maximal submodules of M are precisely the submodules Q_iM $(1 \leq i \leq n)$. Let $Q = Q_1 \cap \cdots \cap Q_n$, the Jacobson radical of R. Then QM is the Jacobson radical of M and hence $QM = Q_1M \cap \cdots \cap Q_nM$. The lattice isomorphism then gives $M/QM \cong R/Q$, which is a cyclic R-module. The proof of (a) then gives that M is cyclic. □

Corollary 5.4. *Let R be a semilocal Noetherian domain. Then the following statements are equivalent for a non-zero R-module M.*

(i) *The above mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}({}_RM)$ is an isomorphism.*
(ii) *The lattices $\mathcal{L}(R)$ and $\mathcal{L}({}_RM)$ are isomorphic.*
(iii) *$M \cong R$.*

Proof. (i) ⇒ (ii) Clear.

(ii) ⇒ (iii) By Lemma 5.2 M is faithful and by Theorem 5.3 M is cyclic. Therefore $M \cong R$.

(iii) ⇒ (i) By Theorem 4.3. □

Next we give another situation where we can settle when there is an isomorphism from the lattice of ideals of a ring R to the lattice of submodules of an R-module M.

Theorem 5.5. *Let R be a Dedekind domain (which is not a field). Then the following statements are equivalent for a non-zero R-module M.*

(i) *The above mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}({}_RM)$ is an isomorphism.*
(ii) *The lattices $\mathcal{L}(R)$ and $\mathcal{L}({}_RM)$ are isomorphic.*

(iii) *$M \cong B$ for some non-zero ideal B of R.*

Proof. (i) $\Rightarrow$ (ii) Clear.

(ii) $\Rightarrow$ (iii) Suppose that $\tau : \mathcal{L}({}_RM) \to \mathcal{L}(R)$ is an isomorphism. Because R is a Noetherian ring, the R-module M must be Noetherian. Next note that if N and L are submodules of M such that $N \cap L = 0$ then $\tau(N) \cap \tau(L) = 0$. This implies that $\tau(N)\tau(L) = 0$ and hence $\tau(N) = 0$ or $\tau(L) = 0$. Thus $N = 0$ or $L = 0$. It follows that the R-module M is uniform. Since R has zero socle it follows that the R-module M has zero socle and thus M is a torsion-free R-module. Putting this information together gives us that M is isomorphic to a non-zero ideal of R.

(iii) $\Rightarrow$ (i) Every non-zero ideal of R is invertible and hence is a multiplication module. By Corollary 4.5, (i) follows. $\square$

Combining Corollaries 4.5 and 5.4 and Theorem 5.5 we see that if R is a semilocal Noetherian domain or a Dedekind domain then the lattices $\mathcal{L}(R)$ and $\mathcal{L}({}_RM)$ are isomorphic if and only if M is a faithful multiplication module, and in this case the R-module M is finitely generated and the mappings λ and μ are both isomorphisms.

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