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BLOCK TRANSITIVE $2-(v,17,1)$ DESIGNS AND REE GROUPS

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Abstract. This article is a contribution to the study of the automorphism groups of $2-(v,k,1)$ designs. Let $\mathcal{D}$ be $2-(v,17,1)$ design, $G \leq Aut(\mathcal{D})$ be block transitive and point primitive. If G is unsolvable, then $Soc(G)$, the socle of G, is not ${}^2G_2(q)$.

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1. Introduction

A $2-(v,k,1)$ design $\mathcal{D}=(\mathcal{P},\mathcal{B})$ is a pair consisting of a finite set $\mathcal{P}$ of v points and a collection $\mathcal{B}$ of $k-$subsets of $\mathcal{P}$, called blocks, such that any 2-subsets of $\mathcal{P}$ is contained in exactly one block. We will always assume that $2<k<v$.

Let $G \leq Aut(\mathcal{D})$ be a group of automorphisms of a $2-(v,k,1)$ design $\mathcal{D}$. Then G is said to be block transitive on $\mathcal{D}$ if G is transitive on $\mathcal{B}$ and is said to be point transitive(point primitive on $\mathcal{D}$ if G is transitive (primitive) on $\mathcal{P}$. A flag of $\mathcal{D}$ is a pair consisting of a point and a block through that point. Then G is flag transitive on $\mathcal{D}$ if G is transitive on the set of flags.

The classification of block transitive $2-(v,3,1)$ designs was completed about thirty years ago (see [2]). In [3], Camina and Siemons classified $2-(v,4,1)$ designs with a block transitive, solvable group of automorphisms. Li classified $2-(v,4,1)$ designs admitting a block transitive, unsolvable group of automorphisms (see [7]). Tong and Li [11] classified $2-(v,5,1)$ designs with a block transitive, solvable group of automorphisms. Han and Li [4] classified $2-(v,5,1)$ designs with a block transitive, unsolvable group of automorphisms. Liu [9] classified $2-(v,k,1)$(where $k=6,7,8,9,10$) designs with a block transitive, solvable group of automorphisms. In [5], Han and Ma classified $2-(v,11,1)$ designs with a block transitive classical Simple groups of automorphisms.

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This article is a contribution to the study of the automorphism groups of $2-(v,k,1)$ designs. Let $\mathcal{D}$ be $2-(v,17,1)$ design, $G \leq Aut(\mathcal{D})$ be block transitive and point primitive. We prove that following theorem.

Main Theorem. Let $\mathcal{D}$ be $2-(v,17,1)$ design, $G \leq Aut(\mathcal{D})$ be block transitive and point primitive. If G is unsolvable, then $Soc(G) \ncong {}^2G_2(q)$.

2. Preliminary Results

Let $\mathcal{D}$ be a $2-(v,k,1)$ design defined on the point set $\mathcal{P}$ and suppose that G is an automorphism group of $\mathcal{D}$ that acts transitively on blocks. For a $2-(v,k,1)$ design, as usual, b denotes the number of blocks and r denotes the number of blocks through a given point. If B is a block, G_B denotes the setwise stabilizer of B in G and $G_{(B)}$ is the pointwise stabilizer of B in G. Also, G^B denotes the permutation group induced by the action of G_B on the points of B, and so $G^B \cong G_B/G_{(B)}$.

The Ree groups ${}^2G_2(q)$ form an infinite family of simple groups of Lie type, and were defined in [10] as subgroups of $GL(7,q)$. Let $GF(q)$ be finite finite field of q elements, where $q = 3^{2n+1}$ for some positive integer $n \geq 1$. Set $t = 3^{n+1}$ so that $t^2 = 3q$. We give the following information about subgroups of ${}^2G_2(q)$. For each l dividing $2n+1$, ${}^2G_2(3^l)$ denotes the subgroup of ${}^2G_2(q)$ consisting of all matrices in ${}^2G_2(q)$ with entries in subfield of 3^l. We use the symbols Q and K to note a Sylow 3-subgroup and a cyclic subgroup of order $q-1$ of ${}^2G_2(q)$, respectively.

Lemma 2.1. ([6]) *Let $T \leq {}^2G_2(q)$ and T be maximal in ${}^2G_2(q)$. Then either T is conjugate to $P_6(l) = {}^2G_2(3^l)$ for some divisor l of $2n+1$, or T is conjugate to one of the subgroups P_i in Table 1.*

Table 1: Group conjugate to T

Group	*Structure*	*Remarks*
P_1	$Q:K$	*The normaliser of Q in ${}^2G_(q)$*
P_2	$(Z_2^2 \times D_{(q+1)/2}):Z_3$	*The normaliser of a fours-group*
P_3	$Z_2 \times PSL(2,q)$	*An involution centraliser*
P_4	$Z_{q+t+1}:Z_6$	*The normaliser of Z_{q+t+1}*
P_5	$Z_{q-t+1}:Z_6$	*The normaliser of Z_{q-t+1}*

Lemma 2.2. ([8]) *Let $T = {}^2G_2(q)$ be an exceptional simple group of Lie type over $GF(q)$, and let G be a group with $T \trianglelefteq G \leq Aut(T)$. Suppose that M is a maximal subgroup of G not containing T, then one of the following holds:*

(1) $|M| < q^3|G:T|$;

(2) $T \cap M$ is a parabolic subgroup of T.

Lemma 2.3. ([5]) *Let G and $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a group and a design, and $G \leq Aut(\mathcal{D})$ be block transitive, point-primitive but not flag-transitive. Let $Soc(G) = T$. Then*

$$|T| \leq \frac{v}{\lambda} \cdot |T_\alpha|^2 \cdot |G : T|,$$

where $\alpha \in \mathcal{P}$, λ is the length of the longest suborbit of G on $\mathcal{P}$.

3. Proof of the Main Theorem

Proposition 3.1. *Let $\mathcal{D}$ be $2-(v, 17, 1)$ design, G be block transitive, point primitive but not flag transitive, then $v = 272b_2 + 1$.*

Proof. Let $b_1 = (b, v)$, $b_2 = (b, v-1)$, $k_1 = (k, v)$, $k_2 = (k, v-1)$. Obviously, $k = k_1 k_2$. Since $k = 17$, we get $k_1 = 1$. Otherwise, $k \mid v$, by [8], G is flag transitive, a contradiction. Thus $v = k(k-1)b_2 + 1 = 272b_2 + 1$. $\square$

Proposition 3.2. *Let $\mathcal{D}$ be $2-(v, 17, 1)$ design, G be block transitive, point primitive but not flag transitive and $|T|$ be even. If G be unsolvable, then $|T| \leq 137|T_\alpha|^2|G : T|$.*

Proof. Let $B = \{1, 2, \cdots, 17\} \in \mathcal{B}$. Since G is unsolvable, then the structure of G^B, the rank and subdegree of G do not occur:

Type of G^B	Rank of G	Subdegree of G
$\langle 1 \rangle$	273	$1, \overbrace{b_2, \cdots, b_2}^{272}$

Otherwise, G^B is odd and G is also odd, a contradiction with $|T|$ be even. Thus $\lambda \geq 2b_2$. By Lemma 2.3 and Proposition 3.1,

$$\frac{|T|}{|T_\alpha|^2} \leq \frac{v}{\lambda} \cdot |G : T| \leq \frac{272b_2 + 1}{2b_2} \cdot |G : T| \leq 137|G : T|.$$

$\square$

Now we may prove our main theorem.

Suppose that $Soc(G) = {}^2G_2(q) = T$, then ${}^2G_2(q) \trianglelefteq G \leq Aut({}^2G_2(q))$. We have $G = T : \langle x \rangle$, where $x \in Out(T)$, the outer automorphisms group of T which may be generated by an automorphism of field. We may assume that x is an automorphism of field. Set $\circ(x) = m$, then $m \mid (2n+1)$. Obviously, $|{}^2G_2(q)| = q^3(q^3+1)(q-1)$. By [1] and $k = 17$, G is not flag transitive. Since G is point primitive, G_α ($\alpha \in \mathcal{P}$) is the maximal subgroup of G, T is block transitive in $\mathcal{D}$. Hence $M = G_\alpha$ satisfies one of the two cases in Lemma 2.2. We will rule out these cases one by one.

Case (1) $|M| < q^3|G:T|$.

By Proposition 3.2, we have an upper bound of $|T|$,

$$|T| < 137|T_\alpha|^2|G:T| < 137q^6|G:T| = 137q^6m.$$

We get

$$q-1 < 137(2n+1).$$

Let $2n+1 = s \geq 3$, then $3^s < 138s$. Thus $s = 3, 5$.

If $s = 3$, then $|{}^2G_2(3^3)| = 3^9 \cdot 2^3 \cdot 7 \cdot 13 \cdot 19 \cdot 37$. Since $v = 272b_2 + 1$ is odd, then $2^3 \mid |T_\alpha|$. Clearly T_α is contained in some maximal subgroups of T. By Lemma 2.1, $T_\alpha \cong {}^2G_2(3)$, $(Z_2^2 \times D_{(q+1)/2}) : Z_3$ or $Z_2 \times PSL(2,q)$, where $q = 3^3$.

(i) $T_\alpha \cong {}^2G_2(3)$. We have

$$v-1 = \frac{|T|}{|T_\alpha|} - 1 = \frac{3^9 \cdot 2^3 \cdot 7 \cdot 13 \cdot 19 \cdot 37}{3^3 \cdot 2^3 \cdot 7} - 1 = 6662330.$$

By Proposition 3.1, $156b_2 = 6662330$, a contradiction.

(ii) $T_\alpha \cong (Z_2^2 \times D_{(q+1)/2}) : Z_3$. We have

$$v-1 = \frac{|T|}{|T_\alpha|} - 1 = \frac{3^9 \cdot 2^3 \cdot 7 \cdot 13 \cdot 19 \cdot 37}{3 \cdot 2^3 \cdot 7} - 1 = 59960978.$$

By Proposition 3.1, $156b_2 = 59960978$, a contradiction.

(iii) $T_\alpha \cong Z_2 \times PSL(2,q)$. We have

$$v-1 = \frac{|T|}{|T_\alpha|} - 1 = \frac{3^9 \cdot 2^3 \cdot 7 \cdot 13 \cdot 19 \cdot 37}{3^3 \cdot 2^3 \cdot 7 \cdot 13} - 1 = 512486.$$

By Proposition 3.1, $156b_2 = 512486$, a contradiction.

If $s = 5$, then $|{}^2G_2(3^5)| = 3^{15} \cdot (3^{15}+1)) \cdot (3^5-1)$. Since $v = 272b_2 + 1$ is odd, then $2^3 \mid |T_\alpha|$. Clearly T_α is contained in some maximal subgroups of T. By Lemma 2.1,

$$T_\alpha \cong {}^2G_2(3),\ (Z_2^2 \times D_{(q+1)/2}) : Z_3 \text{ or } Z_2 \times PSL(2,q),$$

where $q = 3^5$. It is not difficult to exclude them by Proposition 3.1.

Case (2) $T \cap M$ is a parabolic subgroup of T.

By Lemma 2.1, the parabolic subgroup of ${}^2G_2(q)$ is conjugate to QK. Then the order of parabolic subgroup is $q^3(q-1)$ and $v = q^3 + 1$. By Proposition 3.1, we have $q^3 = v - 1 = 272b_2$ and so $272 \mid q^3$, a contradiction.

This completes the proof the Main Theorem.

References

[1] F. Buekenhout, A. Delandtsheer, J. Doyen, P. Kleidman, M. W. Liebeck and J. Saxl, *Linear spaces with flag transitive automorphism groups*, Geom. Dedicata, 36 (1990), 89-94.

[2] P. C. Calpham, *Steiner system with block transitive automorphism groups*, Discrete Math., 14 (1976), 121-131.

[3] A. Camina and J. Siemons, *Block-transitive automorphism groups of 2-(v, k, 1) design*, J. Comb. Theory A, 51 (1989), 268-276.

[4] G. G. Han and H. L. Li, *Unsolvable block transitive automorphism groups of* $2-(v,5,1)$ *designs*, J. Comb. Theory A, 114 (2007),77-96.

[5] G. G. Han and C. G. Ma, *Block transitive* $2-(v,11,1)$ *design and classical simple groups*, Adv Math., 39 (3)(2010), 319-330.

[6] P. B. Kleidman, *The maximal subgroups of the Chevally groups* ${}^2G_2(q)$ *with q odd, the Ree groups* ${}^2G_2(q)$*, and their automorphism groups*, J. Algebra, 117 (1988), 30-71.

[7] H. Li, *On block-transitive* $2-(v,4,1)$ *designs*, J. Combin. Theory Ser. A, 69 (1995), 115-124.

[8] M. Liebeck and J. Saxl, *On the orders of maximal subgroups of the finite exceptional groups of Lie type*, Proc. Lond. Math. Soc., 55 (1987), 299-330.

[9] W. J. Liu, Block transitive $2-(v,k,1)$ designs, Ph.D. Thesis, Zhejiang University, 1998.

[10] R. Ree, *A family of simple groups associated with the simple Lie algebra of type* (G_2), Amer. J. Math., 83 (1961), 432-462.

[11] W. Tong and H. L. Li, *Solvable block transitive automorphism groups of* $2-(v,5,1)$ *designs*, Discrete Math., 260 (2003), 267-273.

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