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STANDARD TABLEAUX AND KRONECKER PROJECTIONS OF SPECHT MODULES

Jaydeep Chipalkatti and Tagreed Mohammed

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Abstract. Given a partition λ of a positive integer d, let V_λ denote the corresponding irreducible rational representation of the symmetric group $\mathfrak{S}_d$. When λ is a hook partition or a two-rowed partition, we explicitly describe the equivariant morphism $V_\lambda \otimes V_\lambda \longrightarrow V_{(d)}$ in terms of the standard tableau basis of V_λ. We give similar descriptions for the morphism $V_\lambda \otimes V_{\lambda'} \longrightarrow V_{(1^d)}$, as well as for the projection morphisms onto the irreducible factors of the tensor product $V_{(d-1,1)} \otimes V_{(d-1,1)}$. Our results can be interpreted as giving formulae for certain Clebsch-Gordan coefficients for the symmetric group.

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1. Introduction

1.1. Let $\mathfrak{S}_d$ denote the symmetric group on the set $\{1, 2, \ldots, d\}$. The finite dimensional irreducible rational representations of $\mathfrak{S}_d$ are naturally parametrised by partitions of d (see [13, Lecture 4]). Given such a partition λ, the corresponding representation V_λ (usually called a Specht module) can be constructed in several ways. In this paper we will see it as a $\mathbf{Q}$-vector space with a basis $\mathcal{B}_\lambda$ consisting of all standard Young tableaux on shape λ filled with distinct entries $\{1, 2, \ldots, d\}$ (see [12, Ch. 7] or [17, Ch. 2]).

For instance, let $d = 5$ and $\lambda = (3, 2)$. Then $V_{(3,2)}$ is a five-dimensional space with a basis:

$$T_1 = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}, \; T_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}, \; T_3 = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}, \; T_4 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}, \; T_5 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}$$

The action of $\mathfrak{S}_d$ on V_λ can be concretely described via the 'straightening algorithm'. For instance, let $g = (1\,3\,4)\,(2\,5) \in \mathfrak{S}_5$. (By our convention, g takes 1 to

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3, and 4 to 1 etc.) Replacing each element in T_1 by its image under g produces the non-standard tableau $U = \begin{array}{|c|c|c|}\hline 3 & 4 & 2 \\ \hline 5 & 1 \\ \cline{1-2}\end{array}$. The latter can be straightened into the linear combination $-T_1 + T_2 - T_5$ of standard tableaux (see §2.3 below), hence

$$g(T_1) = -T_1 + T_2 - T_5. \tag{1}$$

In particular, $V_{(d)}$ is the trivial one-dimensional representation, and $V_{(1^d)}$ is the alternating (or sign) representation. Let

$$[\![\mathrm{RT}_d]\!] = \begin{array}{|c|c|c|c|}\hline 1 & 2 & \cdots & d \\ \hline\end{array}, \quad \text{and} \quad [\![\mathrm{CT}_d]\!] = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline d \\ \hline\end{array}$$

denote the unique basis elements in $\mathcal{B}_{(d)}$ and $\mathcal{B}_{(1^d)}$ respectively.

We refer the reader to [18,20] for the basics of the representation theory of finite groups (in characteristic zero). The representations of symmetric groups have been treated in, *inter alia*, [7,12,13,14,17].

1.2. Given partitions λ and μ of d, there is an irreducible decomposition

$$V_\lambda \otimes V_\mu \simeq \bigoplus_\nu \left[V_\nu \otimes \mathbf{Q}^{C(\lambda,\mu,\nu)}\right],$$

for some nonnegative integers $C(\lambda,\mu,\nu)$. For instance, $V_{(3,2,1)} \otimes V_{(5,1)}$ decomposes into

$$V_{(4,2)} \oplus V_{(4,1,1)} \oplus V_{(3,3)} \oplus [V_{(3,2,1)} \otimes \mathbf{Q}^2] \oplus V_{(3,1,1,1)} \oplus V_{(2,2,2)} \oplus V_{(2,2,1,1)}.$$

(Such calculations can be done using the character values of the V_λ, e.g., see [13, Lecture 4].) The space of $\mathfrak{S}_d$-equivariant morphisms $\mathrm{Hom}_{\mathfrak{S}_d}(V_\lambda \otimes V_\mu, V_\nu)$ has dimension $C(\lambda,\mu,\nu)$.

The $C(\lambda,\mu,\nu)$ are called Kronecker coefficients. They occur in several disparate areas of mathematics, e.g., in invariant theory, quantum information theory and geometric complexity theory (see [6,10,15] respectively). It is a well-known open problem to give a purely combinatorial formula for calculating the Kronecker coefficients. Such formulae are known for some special classes of partitions (see [1,3,16] and the references therein).

The Kronecker coefficients are invariant under any permutation of the three partitions, i.e.,

$$C(\lambda,\mu,\nu) = C(\mu,\lambda,\nu) = C(\mu,\nu,\lambda) \quad \text{etc.}$$

1.3. For any partition λ, we have $C(\lambda, \lambda, (d)) = 1$. Hence, up to a scalar, there is a unique $\mathfrak{S}_d$-equivariant morphism

$$\mathcal{E}_\lambda : V_\lambda \otimes V_\lambda \longrightarrow V_{(d)}.$$

If we fix an isomorphism of $V_{(d)}$ with $\mathbf{Q}$ by sending $[\![\mathrm{RT}_d]\!]$ to 1, then $\mathcal{E}_\lambda(S \otimes T)$ can be identified with a rational number. The following problem is natural:

Problem 1: Give a combinatorial formula for the number $\mathcal{E}_\lambda(S \otimes T)$, where S, T are standard tableaux on shape λ.

1.4. Similarly, if λ' denotes the partition conjugate to λ, then $C(\lambda, \lambda', (1^d)) = 1$. Thus, up to a scalar, we have a unique $\mathfrak{S}_d$-equivariant morphism

$$\mathcal{F}_\lambda : V_\lambda \otimes V_{\lambda'} \longrightarrow V_{(1^d)}.$$

If we fix an isomorphism of $V_{(1^d)}$ with $\mathbf{Q}$ by sending $[\![\mathrm{CT}_d]\!]$ to 1, then $\mathcal{F}_\lambda(S \otimes T)$ can be identified with a rational number.

Problem 2: Give a combinatorial formula for the number $\mathcal{F}_\lambda(S \otimes T)$.

Problems 1 and 2 are special cases of the following problem:

Problem 3: Give an explicit construction of a $C(\lambda, \mu, \nu)$-dimensional family of morphisms

$$V_\lambda \otimes V_\mu \longrightarrow V_\nu \tag{2}$$

in terms of the tableau bases $\mathcal{B}_\lambda, \mathcal{B}_\mu, \mathcal{B}_\nu$. Such morphisms may be called Kronecker projections.

1.5. We were motivated to consider this cluster of problems by analogy with the representation theory of the general linear group GL_n. The irreducible representations of GL_n (over $\mathbf{Q}$) are in bijection with nonincreasing sequences of integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. The corresponding representation $\mathbb{S}_\alpha$ is usually called a Schur module (see [2] or [13, Lecture 6]). Given two such sequences α, β, we have an irreducible decomposition

$$\mathbb{S}_\alpha \otimes \mathbb{S}_\beta \simeq \bigoplus_\gamma \left[\mathbb{S}_\gamma \otimes \mathbf{Q}^{N(\alpha,\beta,\gamma)}\right],$$

for nonnegative integers $N(\alpha, \beta, \gamma)$, called the Littlewood-Richardson coefficients. There exist several combinatorial formulae for these coefficients (see [12, Ch. 5]); and moreover, one can explicitly describe a basis for the set of GL_n-equivariant projection morphisms $\mathbb{S}_\alpha \otimes \mathbb{S}_\beta \longrightarrow \mathbb{S}_\gamma$ (cf. [2, §IV]).

It turns out that the Littlewood-Richardson coefficients are special instances of the so-called *reduced* Kronecker coefficients; this is a consequence of the Littlewood-Murnaghan theorem (see [4]). Broadly speaking, a typical problem for representations of the symmetric group is usually harder than its counterpart for representations of the general linear group (cf. the discussion on [5, page 2]). It would not be surprising if the same were true of Problems 1–3.

1.6. Results. In this paper we solve Problem 1 for the hook partition $(d-r, 1^r)$, and the general two-rowed partition $(d-r, r)$. Moreover, we solve Problem 2 for the hook partition, and for $(d-2, 2)$.

Usually, $V_{(d-1,1)}$ is called the standard representation of $\mathfrak{S}_d$; it is of dimension $d-1$. As to Problem 3, we give formulae describing the Kronecker projections in the following case:

$$V_{(d-1,1)} \otimes V_{(d-1,1)} \simeq V_{(d-1,1)} \oplus V_{(d-2,2)} \oplus V_{(d)} \oplus V_{(d-2,1,1)}. \tag{3}$$

The paper is organized as follows. In the next section we describe the necessary notational conventions, together with some technical results on straightening and Specht polynomials. Results on the $\mathcal{E}$-morphism are given in §3, and those on the $\mathcal{F}$-morphism in §4. Results on the decomposition (3) are given in §5. A Kronecker projection is always indeterminate up to a multiplicative scalar, and all of our results are to be understood with this proviso.

1.7. Problem 3 is of interest in mathematical physics, where the entries of a matrix describing the morphism (2) are called Clebsch-Gordan coefficients for the symmetric group (see [9, Ch. 4]). (A finite system of empirically identical particles will exhibit permutation symmetries, and these coefficients play a role in the corresponding state descriptions.) The physicists customarily use a somewhat different basis for V_λ, namely the normalised Yamanouchi symbols. The relation between the two bases is explained in *loc.cit.*, hence all of our results can be interpreted as giving explicit formulae for certain Clebsch-Gordan coefficients.

2. Specht modules

2.1. Let λ denote a partition of d. A *tableau* will be a filling of the Young diagram of λ with distinct entries $\{1, \dots, d\}$. The tableau is standard if the entries are increasing across rows and down the columns. Let $\mathcal{T}_\lambda$ denote the free $\mathbf{Q}$-vector space generated by all tableaux of shape λ, and define $\mathcal{R}_\lambda \subseteq \mathcal{T}_\lambda$ as the subspace generated by the following two types of elements:

(R1) If T is a tableau, and S is obtained by applying a permutation τ to any specific column of T, then $T - \operatorname{sign}(\tau)\, S \in \mathcal{R}_\lambda$. For instance,

$$\begin{ytableau} 5 & 2 \\ 3 \\ 4 \\ 1 \end{ytableau} + \begin{ytableau} 1 & 2 \\ 3 \\ 4 \\ 5 \end{ytableau} \in \mathcal{R}_{(2,1,1,1)}.$$

(R2) Fix two adjacent columns (say $c-1$ and c) in a tableau T, and let $x = (x_1, \ldots, x_r)$ be the sequence of top r elements in column c. Given a length r subsequence y of column $c-1$, obtain a new tableau S_y by exchanging the x with y pointwise. Then $T - \sum S_y \in \mathcal{R}_\lambda$, where the sum is quantified over all such subsequences y. For instance,

$$\begin{ytableau} 2 & 3 \\ 4 & 1 \\ 6 & 5 \end{ytableau} - \left\{ \begin{ytableau} 3 & 2 \\ 1 & 4 \\ 6 & 5 \end{ytableau} + \begin{ytableau} 3 & 2 \\ 4 & 6 \\ 1 & 5 \end{ytableau} + \begin{ytableau} 2 & 4 \\ 3 & 6 \\ 1 & 5 \end{ytableau} \right\} \in \mathcal{R}_{(2,2,2)}.$$

This can be seen by repeatedly exchanging the sequence $x = (3, 1)$ with every length 2 subsequence y of $(2, 4, 6)$.

Define the Specht module V_λ as the quotient $\mathcal{T}_\lambda/\mathcal{R}_\lambda$. In order to avoid unwieldy notation, we will identifiy a tableau T with its image modulo $\mathcal{R}_\lambda$. The (images of) standard tableaux form a basis $\mathcal{B}_\lambda$ of V_λ (see [12, Ch. 7]). Let

$$n_\lambda = \dim V_\lambda = \operatorname{card} \mathcal{B}_\lambda,$$

a number which is given by the well-known hook length formula (see [12, §4.3]).

2.2. It will be convenient to have a total order on each $\mathcal{B}_\lambda$. Given $T \in \mathcal{B}_\lambda$, define $w(T) = (a_d, \ldots, a_2, a_1)$, where i occurs in the a_i-th row of T. Let $S \prec T$ if $w(S)$ is lexically prior to $w(T)$. For instance, for the tableaux in §1.1, we have $w(T_3) = (2,1,1,2,1), w(T_4) = (2,1,2,1,1)$ etc., and thus $T_1 \prec T_2 \prec \cdots \prec T_5$. In general, let T_i^λ denote the i-th basis element in this order, so that

$$T_1^\lambda \prec T_2^\lambda \prec \cdots \prec T_{n_\lambda}^\lambda.$$

2.3. The straightening Algorithm. In practice one uses the relations (R1) and (R2) to rewrite a nonstandard tableau as a sum of standard ones. There are two types of moves to standardise a nonstandard tableau T. Define a *misplacement* in a tableau T as a box (r, c) such that $T(r, c-1) > T(r, c)$.

- By using (R1) if necessary, ensure that the columns of T are increasing.

- If T has no misplacements, then it is standard. If (r, c) is a misplacement, then use (R2) to rewrite T as a sum $\sum S_y$ by exchanging the top r elements of the c-th column.

Now repeat the moves on each of the S_y, and so on. The procedure eventually terminates to give a $\mathbf{Z}$-linear combination of standard tableaux, and the final result is independent of the sequence of moves chosen (cf. [12, §7.4]). Continuing the example in §1.1 (with the misplacements shown in boldface),

$$\begin{array}{|c|c|c|}\hline 3&4&2\\\hline 5&1\\\cline{1-2}\end{array} = -\begin{array}{|c|c|c|}\hline 3&1&2\\\hline 5&\mathbf{4}\\\cline{1-2}\end{array} = -\begin{array}{|c|c|c|}\hline 1&3&\mathbf{2}\\\hline 4&5\\\cline{1-2}\end{array} = -\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 4&\mathbf{2}\\\cline{1-2}\end{array} = -T_5 + \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 4&\mathbf{3}\\\cline{1-2}\end{array}$$

And now,

$$\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 4&\mathbf{3}\\\cline{1-2}\end{array} = \begin{array}{|c|c|c|}\hline 2&\mathbf{1}&5\\\hline 3&4\\\cline{1-2}\end{array} = \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array} + \begin{array}{|c|c|c|}\hline 2&3&5\\\hline 1&4\\\cline{1-2}\end{array} = T_2 - T_1.$$

2.4. Calculation of Kronecker Projections. Suppose that we are given partitions λ, μ, ν of d, and we want to calculate all $\mathfrak{S}_d$-equivariant projection morphisms $\pi : V_\lambda \otimes V_\mu \longrightarrow V_\nu$. Write

$$\pi(T_i^\lambda \otimes T_j^\mu) = \sum_k \theta(i,j,k)\, T_k^\nu,$$

for some variables $\theta(i,j,k)$. For each $g \in \mathfrak{S}_d$, there are expressions of the form $g(T_i^\lambda) = \sum\limits_\ell a_\ell\, T_\ell^\lambda$ for some integers a_ℓ. For instance, equation (1) from §1.1 shows that for $g = (1\,3\,4)\,(2\,5), \lambda = (3,2), i = 1$, we get

$$a_1 = -1,\ a_2 = 1,\ a_3 = a_4 = 0,\ a_5 = 1.$$

Now consider the commutative diagram

$$\begin{array}{ccc} V_\lambda \otimes V_\mu & \xrightarrow{\pi} & V_\nu \\ {\scriptstyle g\otimes g}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ V_\lambda \otimes V_\mu & \xrightarrow{\pi} & V_\nu \end{array}$$

where the vertical maps correspond to the action of g. A moment's reflection will show that the equation $g\circ\pi = \pi\circ(g\otimes g)$ translates into a system of $n_\lambda\, n_\mu\, n_\nu$ homogeneous linear equations for the $\theta(i,j,k)$. The combined system for the two elements $g = (1\,2)$ and $(1\,2\,\ldots\,d)$ (which together generate $\mathfrak{S}_d$) has exactly $C(\lambda,\mu,\nu)$ linearly independent solutions. We have written a set of MAPLE routines to construct these systems and find their solutions explicitly. (Another algorithm for calculating these morphisms is given in [8].)

The results in this paper were obtained by calculating a large number of such examples, and using them as data to formulate conjectures. (This is especially true of Propositions 4.1, 5.1 and 5.2.) Thus, our method is *inductive* in the sense of the word in philosophy of science.

Example 2.1. Consider the irreducible decomposition

$$V_{(3,1)} \otimes V_{(2,1,1)} \simeq V_{(3,1)} \oplus V_{(2,2)} \oplus V_{(2,1,1)} \oplus V_{(1,1,1,1)},$$

and the resulting projection $\pi : V_{(3,1)} \otimes V_{(2,1,1)} \longrightarrow V_{(2,2)}$. Given the tableau bases

$$A_1 = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}, \quad A_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad A_3 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array},$$

$$B_1 = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}, \quad B_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}, \quad B_3 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array},$$

$$C_1 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}, \quad C_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array};$$

an explicit calculation in MAPLE shows that π is given by the 2×9 matrix

$$M = \begin{bmatrix} -1 & -1 & 1 & -2 & 0 & 2 & -1 & 1 & 1 \\ 2 & 2 & 0 & 1 & 1 & -1 & 1 & 1 & 1 \end{bmatrix}.$$

This follows the convention that $\pi(A_i \otimes B_j) = x\,C_1 + y\,C_2$, where $\begin{bmatrix} x \\ y \end{bmatrix}$ is the $(3i + j - 3)$-rd column of M. For instance, $\pi(A_2 \otimes B_1) = -2\,C_1 + C_2$. Of course, it is understood that any scalar multiple of M would also give such a morphism.

2.5. Given a partition λ, define the $n_\lambda \times n_\lambda$ matrix $\mathbb{E}_\lambda$ by the formula

$$(i, j) \longrightarrow \mathcal{E}_\lambda(T_i^\lambda \otimes T_j^\lambda)/[\![\mathrm{RT}_d]\!].$$

Similarly, let $\mathbb{F}_\lambda$ denote the matrix $(i, j) \longrightarrow \mathcal{F}_\lambda(T_i^\lambda \otimes T_j^{\lambda'})/[\![\mathrm{CT}_d]\!]$. (We regard these matrices as well-defined up to scalars.) For example, if $\lambda = (3, 2)$, then the procedure in §2.4 gives

$$\mathbb{E}_{(3,2)} = \begin{bmatrix} 4 & 2 & 2 & 1 & -1 \\ 2 & 4 & 1 & 2 & 1 \\ 2 & 1 & 4 & 2 & 1 \\ 1 & 2 & 2 & 4 & 2 \\ -1 & 1 & 1 & 2 & 4 \end{bmatrix}, \quad \text{and} \quad \mathbb{F}_{(3,2)} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since taking the transpose of a tableau gives an order reversing bijection $\mathcal{B}_\lambda \simeq \mathcal{B}_{\lambda'}$, each antidiagonal entry of $\mathbb{F}_\lambda$ is of the form $\mathcal{F}_\lambda(S \otimes S^t)$.

2.6. Some straightening formulae. Define transpositions $\tau_m = (m\ m+1)$ in $\mathfrak{S}_d$ for $1 \leqslant m \leqslant d-1$; together they generate the entire symmetric group. In the cases we need, we will describe the effect of applying τ_m to an element in $\mathcal{B}_\lambda$, and rewriting the result as a sum of standard tableaux. The verifications are entirely routine, and left to the reader.

Assume $\lambda = (d-1, 1)$. We will identify an element in $\mathcal{B}_{(d-1,1)}$ only by the entry in its second row, e.g., $\begin{ytableau} 1 & 3 & 4 \\ 2 \end{ytableau}$ as [2]. Thus, $\mathcal{B}_{(d-1,1)} = \{[a] : 2 \leqslant a \leqslant d\}$. Then we have relations

$$\tau_1([a]) = \begin{cases} -[a] & \text{if } a = 2, \\ [a]-[2] & \text{if } a > 2; \end{cases} \qquad \tau_m([a]) = \begin{cases} [m+1] & \text{if } a = m, \\ [m] & \text{if } a = m+1, \\ [a] & \text{otherwise;} \end{cases} \quad \text{if } m > 1.$$

For instance,

$$\tau_1\Big(\begin{ytableau} 1 & 2 & 4 \\ 3 \end{ytableau}\Big) = \begin{ytableau} 2 & 1 & 4 \\ 3 \end{ytableau} = \begin{ytableau} 1 & 2 & 4 \\ 3 \end{ytableau} + \begin{ytableau} 2 & 3 & 4 \\ 1 \end{ytableau} = [3]-[2].$$

2.7. Assume $\lambda = (d-2, 2)$, with $d \geqslant 4$. We will abbreviate an element in $\mathcal{B}_\lambda$ by its second row, e.g., $[2,5]$ stands for $\begin{ytableau} 1 & 3 & 4 \\ 2 & 5 \end{ytableau}$ etc. Given $w = [a,b] \in \mathcal{B}_{(d-2,2)}$, let $x = \tau_m([a,b])$.

Case $m = 1$: If $a = 2$, then $x = -[2,b]$; if $a = 3$, then $x = [3,b]-[2,b]$; if $a > 3$, then $x = [a,b]+[2,a]-[2,b]$.

Case $m = 2$: If $a = 2$, then $x = [3,b]$; if $a = 3$, then $x = [2,b]$; if $a > 3$, then $x = [a,b]+[2,a]-[3,a]$.

Case $m = 3$: If $a = 2, b \neq 4$, then $x = [2,b]-[2,4]$; if $w = [2,4]$, then $x = -[2,4]$; if $w = [3,4]$, then $x = [3,4]-[2,4]$; if $a = 3, b > 4$, then $x = [4,b]$; if $a = 4$, then $x = [3,b]$; if $a > 4$, then $x = [a,b]$.

Case $m > 3$: If $w = [m, m+1]$, then $x = [m,m+1]-[2,m+1]+[2,m]$; otherwise $x = [\tau_m(a), \tau_m(b)]$.

For instance, if $d = 5$, then $\tau_1([4,5])$ equals

$$\begin{ytableau} 2 & 1 & 3 \\ 4 & 5 \end{ytableau} = \begin{ytableau} 1 & 2 & 3 \\ 4 & 5 \end{ytableau} - \begin{ytableau} 1 & 4 & 3 \\ 2 & 5 \end{ytableau} = [4,5]-[2,5]-\begin{ytableau} 1 & 4 & 5 \\ 2 & 3 \end{ytableau} = [4,5]-[2,5]+[2,4].$$

2.8. Assume $\lambda = (2, 2, 1^{d-4})$. We will abbreviate an element in $\mathcal{B}_\lambda$ by its second column, e.g., $\{3, 6\}$ stands for $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 6 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}$ etc. Given $y = \{p, q\} \in \mathcal{B}_{(2,2,1^{d-4})}$, let $z = \tau_m(y)$.

Case $m = 1$: If $p > 2$, then $z = -y$. If $p = 2$, then

$$z = \sum_{r=2}^{q-1} (-1)^r \{r, q\} + \sum_{r=q+1}^{d} (-1)^r \{q, r\}.$$

Case $m = 2$: If $p = 2$, then $z = \{3, q\}$; if $p = 3$, then $z = \{2, q\}$; if $p > 3$, then $z = -y$.

Case $m = 3$: If $p = 2, q \neq 4$, then $z = -y$; if $y = \{3, 4\}$, then $z = -y$; if $p = 3, q > 4$, then $z = \{4, q\}$; if $p = 4$, then $z = \{3, q\}$; if $p > 4$, then $z = -y$. The case $y = \{2, 4\}$ is unusual; when each tableau in $\mathcal{B}_\lambda$ appears in z exactly once, with a sign. More precisely, $z = \sum\limits_{\{p,q\} \in \mathcal{B}_\lambda} (-1)^{p+q} \{p, q\}$.

Case $m > 3$: If $w = \{m, m + 1\}$, then $z = -y$. If p, q both differ from m or $m + 1$, then $z = -y$; otherwise $z = \{\tau_m(p), \tau_m(q)\}$.

For instance, let $d = 6$, and consider the tableau $\tau_3(\{2, 4\}) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 3 \\ \hline 5 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}$. Exchange the sequence $(2, 3)$ as in (R2), and use (R1) to arrange the columns in increasing order. The result is of the form

$$\underbrace{\sum_{p \geqslant 4} \pm \{p, q\}}_{(I)} + \sum_{q \geqslant 4} \pm \underbrace{\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & q \\ \hline \vdots \\ \cline{1-1} \\ \cline{1-1}\end{array}}_{t_q}.$$

Now exchange the 1 in each tableau t_q, which gives $\sum \pm\{2, q\} \pm \{3, q\}$, together with two occurrences of each term $\{p, q\}$ for $p \geqslant 4$ (once from t_p and once from t_q). To sum up, each term in (I) occurs thrice, but a careful accounting of the signs shows that two of them cancel each other, leaving each tableau in $\mathcal{B}_\lambda$ to occur exactly once.

2.9. Specht polynomials. Consider the polynomial ring $A = \mathbf{Q}[x_1, \ldots, x_d]$ with the natural action of $\mathfrak{S}_d$ by permuting the variables. Given a tableau T, the corresponding Specht polynomial $X_T \in A$ is defined to be the product of all terms of the form $x_i - x_j$, where j occurs below i in the same column of T. E.g., if

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 6 & \\ \cline{1-2} 5 & & \\ \cline{1-1}\end{array}\ , \text{ then}$$

$$X_T = (x_1 - x_2)\,(x_1 - x_5)\,(x_2 - x_5)\,(x_3 - x_6). \tag{4}$$

The Specht polynomials give another realisation of the irreducible representations of $\mathfrak{S}_d$ as follows. Let $\mathcal{P}_\lambda$ denote the subspace of A generated by the set of polynomials

$$\{X_T : T \text{ is a tableau on } \lambda\}.$$

Then $\mathcal{P}_\lambda$ is an irreducible subrepresentation of A, and the kernel of the natural map (in the notation of §2.1)

$$\mathcal{T}_\lambda \xrightarrow{h_\lambda} \mathcal{P}_\lambda, \quad T \longrightarrow X_T$$

is exactly $\mathcal{R}_\lambda$, which establishes an isomorphism $V_\lambda \simeq \mathcal{P}_\lambda$ (cf. [19]). For instance, continuing the example in §1.1, observe that

$$\begin{aligned} &- X_{T_1} + X_{T_2} - X_{T_5} \\ &= -(x_1 - x_2)(x_3 - x_4) + (x_1 - x_3)(x_2 - x_4) - (x_1 - x_4)(x_2 - x_5) \\ &= (x_3 - x_5)(x_4 - x_1) = X_U, \end{aligned}$$

hence the element $U - (-T_1 + T_2 - T_5)$ lies in the kernel of $h_{(3,2)}$.

If $(\mu_1, \mu_2, \ldots, \mu_r)$ denotes the conjugate of λ, then the degree of X_T (in the x_i variables) is $\prod\limits_{i=1}^{r} \binom{\mu_i}{2}$.

2.10. Write ∂_i for the differential operator $\frac{\partial}{\partial x_i}$, and consider the polynomial ring

$$A' = \mathbf{Q}[\partial_1, \ldots, \partial_d].$$

One can analogously define the Specht operator $\Delta_T \in A'$ as a product of differences $\partial_i - \partial_j$. Then the natural morphism $A' \otimes A \longrightarrow A$ (defined via differentiation) is compatible with the $\mathfrak{S}_d$-actions on both rings.

Given tableaux S, T on the same shape λ, define $\xi(S,T) = \Delta_S \circ X_T$, which evaluates to an integer. Since the pairing

$$V_\lambda \otimes V_\lambda \longrightarrow \mathbf{Q}, \quad S \otimes T \longrightarrow \xi(S,T) \tag{5}$$

is $\mathfrak{S}_d$-equivariant, it must coincide with $\mathcal{E}_\lambda$ (of course, up to a scalar). This device will be used in the next section to compute the $\mathcal{E}$-form.

2.11. If a, b are distinct integers, then we will write $x_{a,b}$ for $x_a - x_b$, and similarly for $\partial_{a,b}$. Observe that $\partial_{a,b} \circ x_{a',b'}$ is nonzero exactly when the sets $\{a, b\}, \{a', b'\}$ are not disjoint.

For instance, let $S = \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$, so that $\Delta_S = \partial_{1,3}\, \partial_{1,4}\, \partial_{3,4}\, \partial_{2,5}$. Let X_T be as in (4). In evaluating $\Delta_S \circ X_T$, the product rule dictates that we must pair each of the factors $\partial_{a,b}$ with a factor of the type $x_{a',b'}$ in such a way that the contribution is nonzero. There are only two such total pairings possible:

	$x_{1,2}$	$x_{1,5}$	$x_{2,5}$	$x_{3,6}$
(1)	$\partial_{1,3}$	$\partial_{1,4}$	$\partial_{2,5}$	$\partial_{3,4}$
(2)	$\partial_{1,4}$	$\partial_{1,3}$	$\partial_{2,5}$	$\partial_{3,4}$

Each pairing contributes $1 \times 1 \times 2 \times 1 = 2$, hence $\Delta_S \circ X_T = 4$.

2.12. Given a Specht module V_λ, let $V_\lambda^* = \mathrm{Hom}(V_\lambda, V_{(d)})$ denote the dual representation. For later use, we will define an $\mathfrak{S}_d$-equivariant isomorphism

$$e_\lambda : V_\lambda \xrightarrow{\sim} V_\lambda^*,$$

by sending a tableau S to the functional φ_S, such that $\varphi_S(T) = \mathcal{E}_\lambda(S \otimes T)$. Similarly,

$$f_\lambda : V_\lambda \xrightarrow{\sim} V_{\lambda'}^* \otimes V_{(1^d)},$$

is defined by sending S to $\psi_S \otimes [\![\mathrm{CT}_d]\!]$, where $\psi_S(T) = \mathcal{F}_\lambda(S, T)$.

3. The $\mathcal{E}$-form

Throughout this section, we will treat $\mathbf{Q}$ as the trivial $\mathfrak{S}_d$-representation by identifying 1 with $[\![\mathrm{RT}_d]\!]$.

Lemma 3.1. *The $\mathcal{E}$-form is symmetric, i.e., $\mathcal{E}_\lambda(S \otimes T) = \mathcal{E}_\lambda(T \otimes S)$.*

Proof. Define $\epsilon : V_\lambda \otimes V_\lambda \longrightarrow \mathbf{Q}$, by letting

$$\epsilon\,(S, T) = \begin{cases} 1 & \text{if } S = T, \\ 0 & \text{if } S \neq T, \end{cases}$$

for standard tableaux S, T, and extending bilinearly. Now define $\tilde{\epsilon} : V_\lambda \otimes V_\lambda \longrightarrow \mathbf{Q}$, by $\tilde{\epsilon}\,(u, v) = \frac{1}{d!} \sum\limits_{\sigma \in \mathfrak{S}_d} \epsilon\,(u^\sigma, v^\sigma)$. Since $\tilde{\epsilon}$ is $\mathfrak{S}_d$-equivariant, it must coincide with $\mathcal{E}_\lambda$ up to a scalar. Since $\tilde{\epsilon}$ is symmetric by construction, so is $\mathcal{E}_\lambda$. □

First we will determine the $\mathcal{E}$-form for the standard representation, and then later apply it to the hook partition.

Proposition 3.2. *Assume $\lambda = (d-1,1)$. Then for tableaux $[a],[b] \in \mathcal{B}_{(d-1,1)}$, we have*

$$\mathcal{E}_{(d-1,1)}([a] \otimes [b]) = \begin{cases} 2 & \text{if } a = b, \\ 1 & \text{otherwise.} \end{cases} \tag{6}$$

Proof. We have $\xi([a],[b]) = \partial_{1,a} \circ x_{1,b}$, which is either 2 or 1, depending on whether a, b are equal or not. $\square$

One can use the results on straightening to give another proof as follows. *A priori*, formula (6) defines a morphism of vector spaces $V_{(d-1,1)} \otimes V_{(d-1,1)} \xrightarrow{\mathcal{E}} \mathbf{Q}$. It is enough to show that $\mathcal{E}$ is $\mathfrak{S}_d$-equivariant, i.e., the equality

$$\mathcal{E}(\tau_m([a]) \otimes \tau_m([b])) = \mathcal{E}([a] \otimes [b]) \tag{7}$$

is valid for all m. For instance, assume $a = b \neq 2$, and $m = 1$. Then the left-hand side of (7) equals

$$\begin{aligned} &\mathcal{E}(([a] - [2]) \otimes ([a] - [2])) \\ &= \mathcal{E}([a] \otimes [a]) - \mathcal{E}([a] \otimes [2]) - \mathcal{E}([2] \otimes [a]) + \mathcal{E}([2] \otimes [2]) = 2 - 1 - 1 + 2 = 2, \end{aligned}$$

which agrees with the right-hand side. Or if $a \neq 2$, then

$$\begin{aligned} &\mathcal{E}(\tau_1([a]) \otimes \tau_1([2])) = \mathcal{E}(([a] - [2]) \otimes -[2]) \\ &= -\mathcal{E}([a] \otimes [2]) + \mathcal{E}([2] \otimes [2]) = -1 + 2 = 1 = \mathcal{E}([a] \otimes [2]). \end{aligned}$$

The rest of the verifications are similar. Although such a proof would be longer, the technique is more general.

3.1. Now let $\lambda = (d-r, 1^r)$. We will identify a tableau in $\mathcal{B}_\lambda$ only by its first column from the second row onwards, e.g., $\begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array}$ will be abbreviated to $H_{(3,4,6)}$. Given two such sequences $\tilde{p} = (p_1, p_2, \dots, p_r)$ and $\tilde{q} = (q_1, q_2, \dots, q_r)$, respectively define $m(\tilde{p}, \tilde{q})$ and $n(\tilde{p}, \tilde{q})$ to be the cardinalities of the sets

$$\{i : p_i = q_i\}, \quad \{(i,j) : p_i = q_j, i \neq j\}. \tag{8}$$

In short, m (resp. n) is the number of common entries in identical (resp. different) positions. For instance, if

$$\tilde{p} = (2,\, 3,\, 4,\, 5,\, 6,\, 9), \quad \tilde{q} = (2,\, 4,\, 5,\, 6,\, 8,\, 9), \tag{9}$$

then $m(\tilde{p}, \tilde{q}) = 2$, and $n(\tilde{p}, \tilde{q}) = 3$.

Theorem 3.3. *With notation as above, we have the following formula:*

$$\mathcal{E}_{(d-r,1^r)}(H_{\tilde p} \otimes H_{\tilde q}) = \begin{cases} r+1 & \text{when } \tilde p = \tilde q, \\ (-1)^{n(\tilde p, \tilde q)} & \text{when } m(\tilde p, \tilde q) + n(\tilde p, \tilde q) = r-1, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Proof. Since the $\mathcal{E}$-form is essentially unique, it suffices to exhibit any one nonzero equivariant morphism $\mathrm{Sym}^2 V_{(d-r,1^r)} \longrightarrow \mathbf{Q}$. We will construct such a morphism and show that it is given by formula (10). There is an isomorphism (cf. [13, Exer. 4.43])

$$V_{(d-r,1^r)} \simeq \wedge^r V_{(d-1,1)}, \quad H_{\tilde p} \longrightarrow [p_1] \wedge \cdots \wedge [p_r]. \tag{11}$$

Consider the composite

$$\begin{aligned} V_{(d-r,1^r)} \otimes V_{(d-r,1^r)} &\xrightarrow{\sim} \wedge^r V_{(d-1,1)} \otimes \wedge^r V_{(d-1,1)} \to (\otimes^r V_{(d-1,1)}) \otimes (\otimes^r V_{(d-1,1)}) \\ &\to \otimes^r (V_{(d-1,1)} \otimes V_{(d-1,1)}) \to \otimes^r \mathbf{Q} \to \mathbf{Q}, \end{aligned}$$

where the second map is the tensor product of natural inclusions, the third is the regrouping map, the fourth is $\otimes^r \mathcal{E}_{(d-1,1)}$, and the last is the product map. It sends $H_{\tilde p} \otimes H_{\tilde q}$ to the sum

$$\sum_{\sigma \in \mathfrak{S}_r} \left\{ \mathrm{sign}(\sigma) \prod_{i=1}^r \mathcal{E}_{(d-1,1)}([p_i], [q_{\sigma(i)}]) \right\} = \sum_{\sigma \in \mathfrak{S}_r} \left\{ \mathrm{sign}(\sigma) \prod_{i=1}^r (1 + \delta_{p_i, q_{\sigma(i)}}) \right\}.$$

This is the $r \times r$ determinant Z, whose (i,j)-th entry is $1 + \delta_{p_i, q_j}$.

Each p_i in $\tilde p$ can contribute to at most one of the sets in (8). If $m(\tilde p, \tilde q) + n(\tilde p, \tilde q) \leqslant r-2$, then there are (at least) two numbers p_a, p_b in $\tilde p$ neither of which appears in $\tilde q$. Then the a-th and b-th rows of the determinant consist of all 1's, hence it vanishes.

Now assume $m(\tilde p, \tilde q) + n(\tilde p, \tilde q) = r-1$. Let p_a (resp. q_b) be the unique integer in $\tilde p$ (resp. $\tilde q$) which does not appear in $\tilde q$ (resp. $\tilde p$). We may assume $a \leqslant b$. Then $p_i = q_i$ for $i < a$ and $i > b$, and $p_{i+1} = q_i$ for $a \leqslant i \leqslant b-1$, hence $n(\tilde p, \tilde q) = b - a$. (The reader may wish to work out the example in (9), where $a = 2, b = 5$.) Subtract the a-th row of Z (which is all 1's) from every other row to get a new determinant Z', which has a single 1 and the rest 0's in every row except a-th. Since the 1's are on the main diagonal in rows $1, \ldots, a-1, b+1, \ldots, r$, it has the same value as the smaller determinant Z'' which is extracted from rows (and columns) numbered

$a, a+1, \dots, b$ in Z'. Now Z'' is a determinant of size $b-a+1$ with appearance $\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & & \cdots & \\ 0 & \cdots & 1 & 0 \end{vmatrix}$, which evaluates to $(-1)^{b-a}$ as claimed.

Finally, assume $\tilde{p} = \tilde{q}$, then Z is the determinant $\Delta_r = \begin{vmatrix} 2 & 1 & \dots & 1 \\ 1 & 2 & \dots & 1 \\ & \dots & \dots & \\ 1 & 1 & \dots & 2 \end{vmatrix}$. If we remove the first column and the j-th row for any $j > 1$, we end up with a determinant which has already been considered in the previous paragraph and evaluates to $(-1)^j$. Assume that $\Delta_{r-1} = r$, and expand by the first column. Then $\Delta_r = 2\,r - 1 - 1 \cdots - 1 = 2\,r - (r-1) = r+1$. This completes the proof. □

3.2. Now assume $\lambda = (d-2, 2)$. Define two distinct tableaux $[a_1, b_1], [a_2, b_2]$ to be left-aligned if $a_1 = a_2$, right-aligned if $b_1 = b_2$, misaligned if $a_1 = b_2$ or $a_2 = b_1$, and disjoint otherwise. Furthermore, we will say that condition C_2 (resp. C_3) holds if at least one of the numbers a_1, a_2 equals 2 (resp. 3). The following proposition describes the map $\mathcal{E}_{(d-2,2)}$. By §2.10, this reduces to the calculation of $\xi = \xi([a_1, b_1], [a_2, b_2])$.

Proposition 3.4. *With notation as above, the integer ξ is given by the following recipe:*

- *If the tableaux are equal, then $\xi = 4$.*
- *If left-aligned, then $\xi = 2$.*
- *If right-aligned, then: if $|a_1 - a_2| \geqslant 2$ and C_2 holds then $\xi = 1$; otherwise $\xi = 2$.*
- *If misaligned, then: if C_2 holds then $\xi = -1$; otherwise $\xi = 1$.*
- *If disjoint, then: if C_2 fails, then $\xi = 1$; if C_2 holds but C_3 fails, then $\xi = 0$; if C_2, C_3 both hold, then $\xi = 1$.*

For example,

$$\xi([2,4],[4,5]) = -1, \quad \xi([2,7],[3,7]) = 2, \quad \xi([2,5],[4,6]) = 0.$$

Proof. In each case, the condition forces a particular form on $\Delta_{[a_1,b_1]}$ and $X_{[a_2,b_2]}$, so that the value of ξ can be read off. For instance, suppose that the tableaux are misaligned and C_2 holds. We may assume that $a_1 = 2, b_1 = a_2$. Then the tableaux must be of the form $\begin{array}{|c|c|c|}\hline 1 & 3 & \cdot\cdot \\ \hline 2 & b_1 \\ \cline{1-2}\end{array}$ and $\begin{array}{|c|c|c|}\hline 1 & 2 & \cdot\cdot \\ \hline b_1 & b_2 \\ \cline{1-2}\end{array}$, hence

$$\xi = \partial_{1,2}\, \partial_{3,b_1} \circ x_{1,b_1}\, x_{2,b_2} = -1.$$

Alternately, suppose that the tableaux are right-aligned. If C_2 fails, then they must be $\begin{array}{|c|c|c|}\hline 1 & 2 & \cdot\cdot \\ \hline a_1 & b_1 \\ \cline{1-2}\end{array}$ and $\begin{array}{|c|c|c|}\hline 1 & 2 & \cdot\cdot \\ \hline a_2 & b_1 \\ \cline{1-2}\end{array}$, hence $\xi = 2$. If C_2 holds, then without loss of generality, assume $a_1 = 2$, which gives the pair $\begin{array}{|c|c|c|}\hline 1 & 3 & \cdot\cdot \\ \hline 2 & b_1 \\ \cline{1-2}\end{array}$ and $\begin{array}{|c|c|c|}\hline 1 & 2 & \cdot\cdot \\ \hline a_2 & b_1 \\ \cline{1-2}\end{array}$. Now if $a_2 = 3$, then $\xi = 2$, and if $a_2 \geqslant 4$, then $\xi = 1$. The remaining cases are similar, and are left to the reader. $\square$

3.3. This argument can be extended to any two-rowed partition.

Theorem 3.5. *Assume $\lambda = (d-r, r)$ with $r > 2$, and let S, T be a pair of standard tableaux in $\mathcal{B}_\lambda$. Then,*

(i) *the value of $\xi(S,T)$ is contained in the set*

$$\Gamma = \{0,\ \pm 1,\ \pm 2, \ldots, \pm 2^{r-2},\ 2^{r-1},\ 2^r\};$$

(ii) *all the values in Γ do occur with the following exceptions:*
- *if $d = 2r+1$, then 0 cannot occur,*
- *if $d = 2r$, then $0, \pm 1$ cannot occur.*

The result remains the same even if S, T are not assumed to be standard, except that $-2^{r-1}, -2^r$ may also occur. As we will see, the proof gives an algorithm to calculate the value of ξ. We begin with an example which illustrates all the essential points.

Example 3.6. Let $d = 15$, choose

$$S = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ \cline{1-7}\end{array}, \qquad T = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 6 & 3 & 9 & 15 & 7 & 14 & 1 & 12 \\ \hline 8 & 10 & 5 & 2 & 11 & 4 & 13 \\ \cline{1-7}\end{array},$$

and let us follow the evaluation of $\Delta_S \circ X_T$. Since x_{12} and ∂_8 are missing, some of the pairings are forced. It is clear that $\partial_{4,12}$ must pair with $x_{14,4}$ (which evaluates to -1), and $x_{6,8}$ with $\partial_{6,14}$ (which evaluates to 1). The remaining expression

$$\underbrace{\partial_{1,9}\, \partial_{5,13}}_{I}\ \underbrace{\partial_{2,10}\, \partial_{3,11}\, \partial_{7,15}}_{II} \circ \underbrace{x_{9,5}\, x_{1,13}}_{\star}\ \underbrace{x_{3,10}\, x_{15,2}\, x_{7,11}}_{\star\star}$$

factors as $[I \circ \star]\,[II \circ \star\star]$. This is so, because nothing in group II can pair with anything in group $\star$ and similarly for I and $\star\star$. Now the evaluation of $II \circ \star\star$ is tantamount to considering the tableau pair $\begin{array}{|c|c|c|}\hline 2 & 3 & 7 \\ \hline 10 & 11 & 15 \\ \hline\end{array}$ and $\begin{array}{|c|c|c|}\hline 3 & 15 & 7 \\ \hline 10 & 2 & 11 \\ \hline\end{array}$. There are two possible total pairings, $\begin{array}{|ccc} \partial_{2,10} & \partial_{3,11} & \partial_{7,15} \\ \hline x_{3,10} & x_{7,11} & x_{15,2} \\ x_{15,2} & x_{3,10} & x_{7,11} \end{array}$, each of which evaluates to -1, with a total of -2. Similarly $I \circ \star$ reduces to the tableau pair $\begin{array}{|c|c|}\hline 1 & 5 \\ \hline 9 & 13 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 9 & 1 \\ \hline 5 & 13 \\ \hline \end{array}$, which gives -2. Hence $\xi(S, T) = (-1) \times 1 \times (-2) \times (-2) = -4$.

The same pattern holds in general. After removing the forced pairings (each of which evaluates to ± 1), the rest splits into several separate calculations, each of which reduces to a tableau pair of the shape (p, p). Each such pair evaluates to ± 2, hence the final result is a power of 2 (up to a sign). If some ∂ factor cannot be paired at the initial stage, then the result is zero.

3.4. First, consider the special case $\lambda = (p, p)$. Given a tableau S on (p, p), we have an associated involution σ on the set $N_p = \{1, 2, \ldots, 2p\}$ which interchanges $S(1, i)$ and $S(2, i)$ for $1 \leqslant i \leqslant p$. Let S, T be two such tableaux with corresponding involutions σ, τ. For $x, y \in N_p$, define $x \sim y$ if either $x = \sigma(y)$ or $x = \tau(y)$. The equivalence relation generated by $\sim$ partitions N_p into disjoint equivalence classes.

Proposition 3.7. *With notation as above, assume that all of N_p is an equivalence class. Then $\xi(S, T) = \pm 2$.*

Proof. Consider the graph on vertices N_p, where for each $1 \leqslant i \leqslant p$, we introduce a single arrow from $S(2, i)$ to $S(1, i)$ and a double arrow from $T(1, i)$ to $T(2, i)$. If we disregard the directions and types of arrows, then this is merely the cycle graph C_{2p}. For instance, if

$$S = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 7 \\ \hline 2 & 4 & 6 & 8 \\ \hline \end{array}, \quad T = \begin{array}{|c|c|c|c|}\hline 2 & 1 & 3 & 8 \\ \hline 6 & 4 & 7 & 5 \\ \hline \end{array},$$

then the graph is:

$$\begin{array}{ccccccc} 1 & \leftarrow & 2 & \Rightarrow & 6 & \rightarrow & 5 \\ \Downarrow & & & & & & \Uparrow \\ 4 & \rightarrow & 3 & \Rightarrow & 7 & \leftarrow & 8 \end{array}$$

A pairing of Δ_S and X_T is specified by starting from a vertex and following a sequence $s\,d\,s\,d\ldots$ of single and double edges (but not necessarily following the sense of each arrow) so as to return to the starting point. Each subword $s\,d$ can be read as pairing a ∂ factor with an x factor. E.g., the loop $1 - 2 - 6 - \cdots - 4 - 1$ gives the total pairing

$$\begin{array}{|c|c|c|c|} \partial_{1,2} & \partial_{5,6} & \partial_{7,8} & \partial_{3,4} \\ \hline x_{2,6} & x_{8,5} & x_{3,7} & x_{1,4} \end{array}$$

The pairings corresponding to the subwords $\bullet \leftarrow \bullet \Rightarrow \bullet$ and $\bullet \rightarrow \bullet \Leftarrow \bullet$ evaluate to -1, whereas $\bullet \rightarrow \bullet \Rightarrow \bullet$ and $\bullet \leftarrow \bullet \Leftarrow \bullet$ evaluate to 1. This implies that if q is the number of times the loop goes against the sense of an arrow, then the total pairing evaluates to $(-1)^q$. The loop given above goes against the sense of the arrows

$$1 \leftarrow 2, \quad 5 \Leftarrow 8, \quad 3 \Rightarrow 7, \quad 4 \rightarrow 3, \quad 1 \Rightarrow 4,$$

hence $q = 5$.

The only possible other total pairing is given by the reverse loop, which must then evaluate to $(-1)^{2p-q}$. Hence $\xi(S,T) = (-1)^q + (-1)^{2p-q} = (-1)^q\, 2$. □

3.5. Proof of Theorem 3.5 (i). It will be convenient to write $x_{\{a,b\}}$ for $\pm x_{a,b}$ when the sign is immaterial, and similarly for $\partial_{\{a,b\}}$. First we will prove part (i), not necessarily assuming that S,T are standard.

Stage 1. This step is intended to account for all forced pairings; it arises only if $d > 2r$. Let $i = r+1$, and let $a = S(i,1)$. If no factor of the type $x_{\{a,b\}}$ occurs in X_T, then no action is necessary, and increment i by 1. If it does occur, then it can only pair with something of the form $\partial_{\{b,c\}}$ from Δ_S (since ∂_a is unavailable). If no such factor occurs in Δ_S then $\xi = 0$ and the procedure terminates. Otherwise cancel $x_{\{a,b\}}$ and $\partial_{\{b,c\}}$ from X_T and Δ_S respectively, record the appropriate sign $\partial_{\{b,c\}} \circ x_{\{a,b\}} = \pm 1$, and increment i by 1. Continue up to $i = d - 2r$. Then repeat the procedure by reversing the roles of S,T. At the end of step 1, unless the procedure has terminated with $\xi = 0$, we have recorded a sequence of entries each equal to ± 1. We are left with an expression $D \circ X$, where D (resp. X) is a product of ∂ (resp. x) factors.

Stage 2. Let A be the set of all indices a such that ∂_a occurs in D, and similarly let B be such a set for X. Then $A = B$ and card(A) is even, say $2p$. For instance, in Example 3.6 we have $A = B = \{1,2,3,5,7,9,10,11,13,15\}$.By re-labelling the elements in A as $\{1,2,\dots,2p\}$, we are reduced to the case $\lambda = (p,p)$. Now A splits into disjoint classes under the relation $\sim$, and by Proposition 3.7, each class contributes ± 2. It follows that the value of ξ lies in $\Gamma \cup \{-2^{r-1}, -2^r\}$.

Now assume that S and T are standard. If $|\xi| = 2^r$, then both tableaux must have the same columns in the same order and hence must be identical, implying $\xi = 2^r$. If $\xi = -2^{r-1}$, then at most one ∂ factor must have been cancelled at Stage 1. (Otherwise the degree of D would be $\leqslant r-2$, forcing $|\xi| \leqslant 2^{r-2}$.) First, assume that no such factor was cancelled. Then card$(A) = 2r$, all equivalence classes but one must have 2 elements, with the remaining having 4 elements. Since the tableaux are standard, the 2-element classes correspond to columns which are common to

S, T, and hence contribute 2 each. Thus the 4-element class must contribute -2. However this is impossible, since up to re-labelling it corresponds to the tableaux pair $\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}$ and $\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}$ which evaluates to 2 (and not -2).

Secondly, if one ∂ factor was cancelled at Stage 1, then a similar argument shows that S, T must share $r-1$ columns, and removing them would leave subtableaux of the form either $\begin{array}{|c|c|c|}\hline b&\cdot\cdot&a\\\hline c\\\cline{1-1}\end{array}$, $\begin{array}{|c|c|c|}\hline c&\cdot\cdot&b\\\hline a\\\cline{1-1}\end{array}$ or $\begin{array}{|c|c|c|}\hline c&\cdot\cdot&a\\\hline b\\\cline{1-1}\end{array}$, $\begin{array}{|c|c|c|}\hline a&\cdot\cdot&b\\\hline c\\\cline{1-1}\end{array}$. They respectively lead to the contradictions $b < c < b$ or $c < a < c$, hence both are impossible. This completes the proof of part (i).

3.6. Proof of (ii). Assume $d = 2r+1$. If $a = S(r+1, 1)$ and $x_{\{a,b\}}$ occurs in X_T, then b must occur somewhere in the first r columns of S, and hence a factor of the type $\partial_{\{b,c\}}$ must occur in Δ_S. Of course, a similar reasoning applies to $T(r+1, 1)$. Thus ξ cannot be zero in this case. If $d = 2r$, then each equivalence class in N_r contributes ± 2, hence $0, \pm 1$ cannot occur. It remains to show that all the values in Γ do occur (subject to the stated exceptions). Let $\mathbb{S}$ denote the standard tableau on $\lambda = (d-r, r)$ such that $\mathbb{S}(1, i) = 2i - 1$ and $\mathbb{S}(2, i) = 2i$ for $1 \leqslant i \leqslant r$, and $\mathbb{S}(1, i) = r + i$ for $r+1 \leqslant i \leqslant d-r$. For instance, $\mathbb{S} = \begin{array}{|c|c|c|c|c|}\hline 1&3&5&7&8\\\hline 2&4&6\\\cline{1-3}\end{array}$.

Evidently, $\xi(\mathbb{S}, \mathbb{S}) = 2^r$. Fix a k in the range $1 \leqslant k \leqslant r-1$, and interchange the position of $2i$ with $2i+1$ in $\mathbb{S}$ for $1 \leqslant i \leqslant k$. This gives a new standard tableau $\mathbb{T}_{r-k}$, and it is easy to check that $\xi(\mathbb{S}, \mathbb{T}_{r-k}) = 2^{r-k}$. If $d > 2r$, then the same recipe works also for $k = r$.

Now assume $d > 2r+1$, and let $\mathbb{U}$ denote the standard tableau with the sequence $[r+1, r+2, \dots, 2r-2, 2r+1, 2r+2]$ as its second row. For instance, $\mathbb{U} = \begin{array}{|c|c|c|c|c|}\hline 1&2&3&5&6\\\hline 4&7&8\\\cline{1-3}\end{array}$. Then $\mathbb{U}(1, r+1) = 2r-1, \mathbb{U}(1, r+2) = 2r$, which forces $\xi(\mathbb{S}, \mathbb{U}) = 0$, since the factor $\partial_{2r-1,2r}$ in $\Delta_{\mathbb{S}}$ cannot be paired. So far we have accounted for all nonnegative values in Γ. The remaining constructions are a little more complicated.

Assume $r > 2$. Given a t in the range $r \leqslant t \leqslant 2r-1$, let $\mathbb{V}_t$ denote the standard tableau of size (r, r) whose first row is $[1, 2, \dots, r-1, t]$. Now define

$$\tau(r) = \begin{cases} 2r-2 & \text{if } r \equiv 0 \pmod 4, \\ r+1 & \text{if } r \equiv 1 \pmod 4, \\ 2r-3 & \text{if } r \equiv 2 \pmod 4, \\ r & \text{if } r \equiv 3 \pmod 4. \end{cases}$$

For instance, $\mathbb{V}_{\tau(5)} = \mathbb{V}_6 = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 6 \\ \hline 5 & 7 & 8 & 9 & 10 \\ \hline \end{array}$.

Lemma 3.8. *If $\mathbb{S}$ is the standard tableau on (r, r) defined above, then*

$$\xi(\mathbb{S}, \mathbb{V}_{\tau(r)}) = -2.$$

Proof. The proof is difficult to motivate, because the numbers $\tau(r)$ were obtained after a lot of trial and error. They are so constructed as to satisfy two requirements:

(1) The equivalence relation imposed on N_r by $\mathbb{S}$ and $\mathbb{V}_{\tau(r)}$ gives only one equivalence class.

(2) Suppose we construct the graph as in the proof of Proposition 3.7, and choose the loop which starts with $\bullet \to 1 \Rightarrow \bullet$. If a is the unique integer in the set $\{1, 3, 4, 6\}$ such that $r \equiv a \pmod 4$, then the loop goes against the sense of the arrows precisely $q = \frac{r-a}{2} + 1$ times (an odd number), and hence $\xi = -2$.

For instance, assume $r \equiv 1 \pmod 4$. Then $\mathbb{V}_{r+1}$ has $[1, 2, \dots, r-1, r+1]$ as its first row, and $[r, r+2, \dots, 2r]$ as the second row. Now an examination of the graph shows that the loop goes against the sense of the following arrows:

$$r + 1 \to r, \quad \text{and} \quad 2k \Rightarrow 2k + r \ \text{ for } 1 \leqslant k \leqslant \frac{r-1}{2};$$

hence $q = \frac{r-1}{2} + 1$. (The reader may wish to work out the case $r = 9$.) Or, if $r \equiv 3 \pmod 4$, then the loop goes against the arrows $2k \Rightarrow 2k + r$ $(1 \leqslant k \leqslant \frac{r-3}{2} + 1)$ and no others. The remaining cases are entirely similar. □

Assume that $\mathbb{S}$ is of size $(d - r, r)$ as before. For $3 \leqslant k \leqslant r$, define a standard tableau $\widetilde{T}_k$ by replacing the first k columns of $\mathbb{S}$ by $\mathbb{V}_{\tau(k)}$, and leaving columns $k + 1$ through r (if there are any such) unchanged. Then it is immediate that $\xi(\mathbb{S}, \widetilde{T}_k) = -2 \times 2^{r-k} = -2^{r-k+1}$.

It only remains to exhibit -1 as a possible value. Assume $d > 2r$. Given a p in the range $r + 1 \leqslant p \leqslant 2r$, let $\mathbb{W}_p$ denote the standard tableau of size $(d - r, r)$ whose first row is $[1, 2, \dots, r, p, 2r + 2, \dots, d]$. Now define

$$\pi(r) = \begin{cases} r + 2 & \text{if } r \equiv 0 \pmod 4, \\ 2r - 1 & \text{if } r \equiv 1 \pmod 4, \\ r + 1 & \text{if } r \equiv 2 \pmod 4, \\ 2r & \text{if } r \equiv 3 \pmod 4. \end{cases}$$

For instance, if $\lambda = (7, 5)$, then $\mathbb{W}_{\pi(5)} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 & 9 & 12 \\ \hline 6 & 7 & 8 & 10 & 11 & \multicolumn{2}{c}{} \\ \cline{1-5} \end{array}$.

Lemma 3.9. *With notation as above, we have* $\xi(\mathbb{S}, \mathbb{W}_{\pi(r)}) = -1$.

Proof. We will truncate $\mathbb{S}$ and $\mathbb{W}$ after their first r columns, which does not affect the calculation of ξ. For instance, assume $r \equiv 1 \pmod 4$. Now $\mathbb{S}$ and $\mathbb{W}$ are both comprised of the same entries, except that $2r-1$ is missing from $\mathbb{W}$ but not $\mathbb{S}$, and $2r+1$ is missing from $\mathbb{S}$ but not $\mathbb{W}$. Due to this manufactured mismatch, the pairing between the ∂ and x factors is completely forced. Indeed, $\partial_{2r-1,2r}$ must necessarily pair with $x_{r-1,2r}$ (since x_{2r-1} is not available), which determines the rest: $\partial_{r,r+1}$ with $x_{r,2r+1}$ (evaluating to 1), $\partial_{2k-1,2k}$ with $x_{2k-1,2k+r-1}$ for $1 \leqslant k \leqslant \frac{r-1}{2}$ (each evaluating to 1), and $\partial_{2k+r,2k+r+1}$ with $x_{2k,2k+r}$ for $1 \leqslant k \leqslant \frac{r-3}{2}$ (each evaluating to -1). Thus $\xi = (-1)^{\frac{r-3}{2}} = -1$.

Alternately, if $r \equiv 3 \pmod 4$, then there are $\frac{r-1}{2}$ forced pairs each evaluating to -1, and the rest to 1; hence $\xi = (-1)^{\frac{r-1}{2}} = -1$. The remaining cases are entirely similar.

This completes the proof of Theorem 3.5. □

4. The $\mathcal{F}$-form

In this section we will calculate the $\mathcal{F}$-form for the hook partition, and the special two-rowed partition $(d-2,2)$. Throughout this section, we will treat $\mathbf{Q}$ as the alternating $\mathfrak{S}_d$-representation by identifying 1 with $[\![\mathrm{CT}_d]\!]$.

4.1. Assume $\lambda = (d-r, 1^r)$, then $\lambda' = (r+1, 1^{d-r-1})$. Combined with the identification (11) from §3.1, the map $\mathcal{F}_\lambda$ is the exterior multiplication

$$\wedge^r V_{(d-1,1)} \otimes \wedge^{d-r-1} V_{(d-1,1)} \longrightarrow \wedge^{d-1} V_{(d-1,1)} \simeq V_{(1^d)}. \tag{12}$$

This gives the following description: given increasing sequences $\tilde{p}, \tilde{q}$ of lengths r and $d-r-1$ respectively, $\mathcal{F}_\lambda(H_{\tilde{p}}, H_{\tilde{q}})$ is zero if $\tilde{p}, \tilde{q}$ have a common intersection. If not, it is the sign of the permutation obtained by the concatenation $\tilde{p} \circ \tilde{q}$.

E.g., assume $d = 6, r = 3$. Then $\mathcal{F}_\lambda(H_{(2,4,5)}, H_{(3,4)}) = 0$, and

$$\mathcal{F}_\lambda(H_{(2,4,6)}, H_{(3,5)}) = \mathrm{sign}\,(2\,4\,6\,3\,5) = -1.$$

4.2. Now assume $\lambda = (d-2,2)$. Given tableaux $w = [a,b]$ and $y = \{p,q\}$ in $\mathcal{B}_\lambda, B_{\lambda'}$ respectively, write $f = \mathcal{F}_{(d-2,2)}(w \otimes y)$. The following proposition describes the map $\mathcal{F}_{(d-2,2)}$.

Proposition 4.1. *The value of f is given by the following rule:*

$$f = \begin{cases} (-1)^{b-a+1} & \text{if } a = p, b = q, \\ (-1)^q & \text{if } p = 2, a = q, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Proof. Define a morphism of vector spaces

$$V_{(d-2,2)} \otimes V_{(2,1,\dots,1)} \xrightarrow{\mathcal{F}} \mathbf{Q}, \quad w \otimes y \longrightarrow f$$

by the formula above. It is enough to show that $\mathcal{F}$ is $\mathfrak{S}_d$-equivariant, i.e., in all cases we have an equality

$$\mathcal{F}_\lambda(\tau_m(w) \otimes \tau_m(y)) = -\mathcal{F}_\lambda(w \otimes y). \tag{14}$$

(Recall that τ_m is an odd permutation.) For instance, assume $w = [a,b]$ and $y = \{2,a\}$, so that $\mathcal{F}(w \otimes y) = (-1)^a$. Further assume $m = 1$. Then (since necessarily $a > 3$),

$$\tau_1(w) = [a,b] + [2,a] - [2,b], \quad \tau_1(y) = \sum_{r<a} (-1)^r \{r,a\} + \sum_{r>a} (-1)^r \{a,r\}.$$

The left-hand side of (14) is the sum of terms

$$[a,b] \otimes \{2,a\} \to (-1)^a, \quad [a,b] \otimes \{a,b\} \to (-1)^{b-a+1}(-1)^b, \quad [2,a] \otimes \{2,a\} \to (-1)^{a-1},$$

which is $-(-1)^a$ as required.

As a second instance, assume $w = [4,b], y = \{2,4\}$; then the right-hand side of (14) is -1. Further assume $m = 3$, then the left-hand side is

$$\sum (-1)^{p+q} \mathcal{F}([3,b] \otimes \{p,q\}),$$

where the sum is quantified over all $\{p,q\} \in \mathcal{B}_{(2,2,1^{d-4})}$. The only nonzero term comes from $p = 3, q = b$, which gives $(-1)^{b+3} \times (-1)^{b-3+1} = -1$. The remaining cases are similar. □

We should like to reassure the reader that wherever he is called upon to check the remaining cases, we have already done so.

4.3. Notice the following properties of the matrix $\mathbb{F}_{(d-2,2)}$ whose entries are given by this proposition:

(1) All nonzero entries are ± 1.
(2) Each antidiagonal entry is nonzero.

(3) Amongst its $\frac{1}{4}\, d\,(d-3)\,(d^2-3d-2)$ entries away from the antidiagonal, only $\frac{1}{2}\,(d-3)(d-4)$ are nonzero; that is to say, the matrix is very sparse away from the antidiagonal.

These observations seems to hold true of other partitions as well. E.g., an explicit calculation of $\mathbb{F}_{(4,2,1)}$ shows that properties (1) and (2) are still true, and only 25 (out of a possible 1190) entries away from the anti-diagonal are nonzero. It should be worthwhile to formulate (and prove) a precise conjecture along these lines. (This would entail making a careful choice of normalisation for the map $\mathcal{F}_\lambda$.)

By construction, $\mathbb{F}_\lambda^t = \mathbb{F}_{\lambda'}$ (up to a scalar), and from the chain of isomorphisms

$$V_{\lambda'} \xrightarrow{\sim} V_\lambda^* \otimes V_{(1^d)} \xrightarrow{\sim} V_\lambda \otimes V_{(1^d)} \xrightarrow{\sim} V_{\lambda'}^*$$

one deduces that, up to a scalar,

$$\mathbb{E}_{\lambda'} = \mathbb{F}_\lambda^t\, \mathbb{E}_\lambda^{-1}\, \mathbb{F}_\lambda.$$

Applying this formula to $\lambda = (d-2,2)$ gives an indirect description of the matrix $\mathbb{E}_{(2,2,1^{d-4})}$.

4.4. Suppose that λ is self-conjugate, i.e., $\lambda' = \lambda$. Then $V_{(1^d)}$ must be a subrepresentation of exactly one of the two summands in the decomposition

$$V_\lambda \otimes V_\lambda \simeq \operatorname{Sym}^2 V_\lambda \oplus \wedge^2 V_\lambda,$$

i.e., $\mathbb{F}_\lambda$ must be either symmetric or skew-symmetric. It would be of interest to know which possibility holds. One can computationally determine this as follows: given the character formula $\chi_{\operatorname{Sym}^2 V_\lambda}(g) = \frac{1}{2}\,[\,\chi_\lambda(g)^2 + \chi_\lambda(g^2)\,]$, calculate the inner product

$$\langle \chi_{(1^d)}, \chi_{\operatorname{Sym}^2 V_\lambda} \rangle = \frac{1}{d!} \sum_{g \in \mathfrak{S}_d} \operatorname{sign}(g)\, \chi_{\operatorname{Sym}^2 V_\lambda}(g).$$

(The value of χ_λ at a conjugacy class in $\mathfrak{S}_d$ is given by the Frobenius character formula – see [12, §7.3].) The inner product is 1 if $V_{(1^d)}$ occurs in $\operatorname{Sym}^2 V_\lambda$, and 0 otherwise. Mike Zabrocki has checked all such cases for $d \leqslant 14$. He has proposed the following conjecture based on the outcome of the data. Given a self-conjugate partition λ, define

$$\operatorname{flank}(\lambda) = \sum_{i \geqslant 1} \max\,(\lambda_i - i, 0).$$

This can be visualised as the number of boxes lying to any one side of the diagonal of its Young diagram. For instance, $(4,4,2,2) = \begin{array}{|c|c|c|c|} \hline & * & * & * \\ \hline & & * & * \\ \hline & & \multicolumn{2}{c}{} \\ \cline{1-2} & & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$ has flank $= 5$.

Conjecture 4.2. *The matrix $\mathbb{F}_\lambda$ is symmetric if flank(λ) is even, and skew-symmetric otherwise.*

The hook $\lambda_r = (d-r, 1^r)$ is self-conjugate exactly when $d = 2r+1$. In that case, the map in (12) is symmetric if r is even, and skew-symmetric otherwise. Since $\operatorname{flank}(\lambda_r) = d - r - 1 = r$, this is consistent with the conjecture.

5. The square of the standard representation

5.1. We begin by giving a short proof of identity (3) from §1. It is enough to show that both sides have the same character. If $C \subseteq \mathfrak{S}_d$ is a conjugacy class whose typical element has i_r cycles of length r (for $r = 1, 2, \dots$), then

$$\chi_{(d-1,1)}(C) = i_1 - 1, \qquad \chi_{(d-2,1,1)}(C) = \tfrac{1}{2}\,(i_1-1)(i_1-2) - i_2,$$
$$\chi_{(d-2,2)}(C) = \tfrac{1}{2}\, i_1\,(i_1 - 3) + i_2,$$

by the formulae in [11, p. 157]. Now the result follows from the equality

$$(i_1 - 1)^2 = (i_1 - 1) + \frac{1}{2}\, i_1\,(i_1 - 3) + i_2 + 1 + \frac{1}{2}\,(i_1 - 1)(i_1 - 2) - i_2. \quad \square$$

Let

$$\pi_\nu : V_{(d-1,1)} \otimes V_{(d-1,1)} \longrightarrow V_\nu$$

denote the Kronecker morphisms coming from the decomposition (3). Of course, $\pi_{(d)}$ is the same as $\mathcal{E}_{(d-1,1)}$, and $\pi_{(d-2,1,1)}$ is the natural map

$$V_{(d-1,1)} \otimes V_{(d-1,1)} \longrightarrow \wedge^2 V_{(d-1,1)}.$$

Hence, $\pi_{(d-2,1,1)}([a] \otimes [b])$ is equal to $H_{(a,b)}, -H_{(b,a)}$, or zero, according to whether $a < b$, $a > b$ or $a = b$.

5.2. For the case $\nu = (d-1, 1)$, let us write $\pi_{(d-1,1)}([a] \otimes [b]) = \sum\limits_{2 \leqslant c \leqslant d} \mu(a,b,c)\,[c]$.

Proposition 5.1. *The coefficients μ are given by the following formula:*

$$\mu(a,b,c) = \begin{cases} 2-d & \text{if } a = b = c, \\ 2 & \text{if } a = b, \text{ but } c \neq a, \\ 1 & \text{if } a \neq b. \end{cases} \tag{15}$$

Proof. The formula as given defines a vector space morphism $V_{(d-1,1)}^{\otimes 2} \longrightarrow V_{(d-1,1)}$, which we must show to be equivariant. This involves checking that the equality

$$\tau_m \circ \pi_{(d-1,1)}([a] \otimes [b]) = \pi_{(d-1,1)}(\tau_m([a]) \otimes \tau_m([b])) \tag{16}$$

always holds. We will verify two instances, and leave the rest to the reader. Let $z = \sum_{2 \leqslant [c] \leqslant d} [c]$ denote the sum of all elements in $\mathcal{B}_{(d-1,1)}$.

Assume $a = b \neq 2$ and $m = 1$. Then $\pi([a] \otimes [b]) = (2-d)\,[a] + 2 \sum_{c \neq a} [c]$. Applying τ_1 gives

$$(2-d)([a]-[2]) - 2\,[2] + 2 \sum_{c \neq a,2} ([c]-[2]) = 2\,z - d\,[2] - d\,[a].$$

On the right-hand side,

$$\begin{aligned} \pi(([a]-[2]) \otimes ([a]-[2])) &= \pi([a] \otimes [a]) + \pi([2] \otimes [2]) - 2\,\pi([a] \otimes [2]) \\ &= (2\,z - d\,[a]) + (2\,z - d\,[2]) - 2\,z = 2\,z - d\,[2] - d\,[a], \end{aligned}$$

as it should be.

Alternately, assume $a \neq b$, and $m > 1$. Then $\pi([a] \otimes [b]) = z$, and $\tau_m(z) = z$. On the right-hand side, $\tau_m([a]) = [a']$ and $\tau_m([b]) = [b']$ for some $a' \neq b'$, hence $\pi([a'] \otimes [b']) = z$. □

5.3. Now assume $\nu = (d-2,2)$, and write

$$\pi_{(d-2,2)}([a] \otimes [b]) = \sum_{\{p,q\} \in \mathcal{B}_{(d-2,2)}} \vartheta(a,b;p,q)\,[p,q].$$

It seems difficult to guess a formula for the ϑ directly, indeed we could detect no clear pattern even after computing several examples. Instead, it turns out to be easier to describe the map

$$\gamma : V_{(d-1,1)} \otimes V_{(d-2,2)} \longrightarrow V_{(d-1,1)}.$$

We will then use the description of γ together with the $\mathcal{E}$-forms for $(d-1,1)$ and $(d-2,2)$ to find the coefficients in $\pi_{(d-2,2)}$. Write

$$\gamma\,([a] \otimes [p,q]) = \sum_{2 \leqslant b \leqslant d} \eta(a;p,q;b)\,[b].$$

The rule for determining the η is rather complicated. The possible values are $0, \pm 1, \pm 2$. There are 8 cases depending on the triple (a,p,q), and each case branches further depending on b.

Proposition 5.2. *With notation as above,*

(c1) *If $a = p = 2$; then $\eta = 2, -2$ resp. for $b = 3, q$.*
(c2) *If $a = p, a \neq 2$; then $\eta = 2, -2$ resp. for $b = 2, q$.*
(c3) *If $a \neq p, a = 2$; then $\eta = 1, 1, -1$ resp. for $b = 2, p, q$.*
(c4) *If $a = q, p = 2$; then $\eta = -1, 1, -1$ resp. for $b = 2, 3, q$.*
(c5) *If $a = q, p \neq 2$; then $\eta = 1, -1, -1$ resp. for $b = 2, p, q$.*

(c6) *If* $a = 3, p = 2$; *then* $\eta = 1, 1, -1$ *resp. for* $b = 2, 3, q$.

(c7) *If* $a \notin \{2, 3, q\}, p = 2$; *then* $\eta = 1, -1$ *resp. for* $b = 3, q$.

(c8) *If* $a \notin \{2, p, q\}, p \neq 2$; *then* $\eta = 1, -1$ *resp. for* $b = 2, q$.

In cases not covered by the above, $\eta = 0$.

For instance,

$$\eta(5; 2, 5; 5) = -1 \text{ by (c4)}, \quad \eta(6; 3, 5; 2) = 1 \text{ by (c8)}, \text{ and } \eta(4; 4, 7; 5) = 0.$$

Proof. We are required to check all cases of the equality

$$\tau_m \circ \gamma\,([a] \otimes [p, q]) = \gamma\,(\tau_m([a]) \otimes \tau_m([p, q])). \tag{17}$$

For instance, assume $a = p > 3$, and $m = 1$. Then by (c2),

$$\tau_1 \circ \gamma\,([p] \otimes [p, q]) = \tau_1\,(2\,[2] - 2\,[q]) = -2\,[2] - 2\,([q] - [2]) = -2\,[q].$$

On the right-hand side,

$$\gamma(\tau_1([p]) \otimes \tau_1([p, q])) = \gamma\,(([p] - [2]) \otimes ([p, q] + [2, p] - [2, q])),$$

which breaks up into

$$\begin{aligned} [p] \otimes [p, q] &\to 2\,[2] - 2\,[q], & [p] \otimes [2, p] &\to -\,[2] + [3] - [p], \\ -\,[p] \otimes [2, q] &\to -\,[3] + [q], & -\,[2] \otimes [p, q] &\to -[2] - [p] + [q], \\ -[2] \otimes [2, p] &\to -\,2\,[3] + 2\,[p], & [2] \otimes [2, q] &\to 2\,[3] - 2\,[q]. \end{aligned}$$

After some cancellation, this reduces to $-2\,[q]$.

As another instance, assume $a = 2, p > 3$ and $m = 1$. Then by (c3), the left-hand side is

$$\tau_1([2] + [p] - [q]) = -[2] + [p] - [2] - [q] + [2] = [p] - [q] - [2].$$

The right-hand side is $\gamma\,(-[2] \otimes ([p, q] + [2, p] - [2, q]))$. This breaks up into

$$-[2] \otimes [p, q] \to -[2] - [p] + [q], \quad -[2] \otimes [2, p] \to -2\,[3] + 2\,[p], \quad [2] \otimes [2, q] \to 2\,[3] - 2\,[q],$$

and the addition is again $[p] - [q] - [2]$. The remaining cases are left to the reader. □

5.4. Now γ gives rise to a morphism

$$\begin{aligned} \widetilde{\gamma} : V_{(d-1,1)} \otimes V^*_{(d-1,1)} &\longrightarrow V^*_{(d-2,2)}, \\ [a] \otimes \varphi &\longrightarrow \{\, [p, q] \to \varphi \circ \gamma\,([a] \otimes [p, q])\,\}\,. \end{aligned}$$

Then $\pi_{(d-2,2)}$ is simply the composite

$$V_{(d-1,1)} \otimes V_{(d-1,1)} \xrightarrow{1 \otimes e_{(d-1,1)}} V_{(d-1,1)} \otimes V^*_{(d-1,1)} \xrightarrow{\widetilde{\gamma}} V^*_{(d-2,2)} \xrightarrow{e^{-1}_{(d-2,2)}} V_{(d-2,2)},$$

where the e_λ are as in §2.12. This gives the following rule for calculating $\vartheta(a,b;p,q)$ in terms of the η coefficients and the matrices $\mathbb{E}_{(d-1,1)}, \mathbb{E}_{(d-2,2)}$. We will construct three matrices M_1, M_2, M_3 respectively of sizes

$$1 \times (d-1), \quad (d-1) \times \frac{d(d-3)}{2}, \quad \frac{d(d-3)}{2} \times 1.$$

The i-th entry of the row-matrix M_1 is $\mathcal{E}_{(d-1,1)}([a] \otimes [i+1])$. If $[p_j, q_j]$ denotes the j-th basis element in $\mathcal{B}_{(d-2,2)}$ under the total order in §2.2, then the (i,j)-th entry of M_2 is $\eta(b; p_j, q_j; i{+}1)$. Finally, define M_3 to be the r-th column of $\mathbb{E}^{-1}_{(d-2,2)}$, where $[p,q] = [p_r, q_r]$. Altogether we have the following description of the ϑ coefficients.

Proposition 5.3. *With notation as above, $\vartheta(a,b;p,q)$ is the entry in the 1×1 matrix $M_1\, M_2\, M_3$.* □

5.5. Concluding Remarks. It appears that Problem 3 (as stated on page 125) in its full generality is very difficult; indeed, this is already a consensus view regarding the presumably simpler problem of finding a combinatorial formula for the Kronecker coefficients. On the other hand, since the Kronecker coefficient is known be to be unity in case of Problems 1 and 2, one expects them to be more tractable, at least for special classes of partitions.

If each of the partitions λ or μ is either a hook or has two parts, then the $C(\lambda, \mu, \nu)$ have been calculated in [16]. It would be of interest to have explicit descriptions of the corresponding Kronecker projections $V_\lambda \otimes V_\mu \longrightarrow V_\nu$ for these cases.

5.6. The following conjecture for a 'near-hook' shape was obtained by computational experimentation in MAPLE. Let $\lambda = (2^s, 1^{d-2s})$, and consider the set $\Gamma = \{\mathcal{E}_\lambda(S \otimes T) : S, T \in \mathcal{B}_\lambda\}$ of values attained by the $\mathcal{E}$-form.

Conjecture 5.4. *There exist integers $a_1, \dots, a_t, b$ such that $\Gamma = \{\pm a_1, \dots, \pm a_t, b\}$.*

For instance, for $\lambda = (2^1, 1^4)$, we have (up to a rescaling) $\Gamma = \{\pm 2, \pm 5, 30\}$.

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References

[1] B. Adsul and K. V. Subrahmanyam, *A geometric approach to the Kronecker problem–I. The two row case*, Proc. Indian Acad. Sci. Math. Sci., 118(2) (2008), 213–226.

[2] K. Akin, D. Buchsbaum and J. Weyman, *Schur functors and Schur complexes*, Adv. Math., 44(3) (1982), 207–278.

[3] C. Bessenrodt and A. Kleshchev, *On Kronecker products of complex representations of the symmetric and alternating groups*, Pacific J. Math., 190(2) (1999), 201–223.

[4] E. Briand, R. Orellana and M. Rosas, *The stability of the Kronecker product of Schur functions*, J. Algebra, 331(1) (2011), 11–27.

[5] P. Bürgisser and C. Ikenmeyer, The complexity of computing Kronecker coefficients, 20th Annual International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2008), 357–368, Discrete Math. Theor. Comput. Sci. Proc., Nancy, 2008.

[6] F. Caselli, *Diagonal invariants and the refined multimahonian distribution*, J. Algebraic Combin., 30(2) (2009), 193–213.

[7] T. Ceccherini-Silberstein, F. Scarabotti and F. Tolli, Representation Theory of the Symmetric Groups, Cambridge University Press, 2010.

[8] J.-Q. Chen, M.-J. Gao, *The Clebsch-Gordan coefficients of permutation groups* $S(2) - S(6)$, J. Phys. A, 18(2) (1985), 189–213.

[9] J.-Q. Chen, J. Ping and F. Wang, Group Representation Theory for Physicists, World Scientific Publishing Co., Singapore, 2002.

[10] M. Christandl and G. Mitchison, *The spectra of density operators and the Kronecker coefficients of the symmetric group*, Comm. Math. Phys., 261(3) (2006), 789–797.

[11] G. Frobenius, Über die Charaktere der symmetrischen Gruppe, pp. 148–166 in vol. III of Gesammelte Abhandlungen, Springer–Verlag, Heidelberg, 1968.

[12] W. Fulton, Young Tableaux, London Mathematical Society Student Texts, Cambridge University Press, 1997.

[13] W. Fulton and J. Harris, Representation Theory, A First Course, Graduate Texts in Mathematics. Springer–Verlag, New York, 1991.

[14] G. D. James and A. Kerber, The Representation Theory of the Symmetric Group, Addison- Wesley, 1981.

[15] K. Mulmuley and M. Sohoni, Geometric Complexity Theory IV: quantum group for the Kronecker problem, CoRR abs/cs/0703110, 2007.

[16] M. Rosas, *The Kronecker product of Schur functions indexed by two-row shapes or hook shapes*, J. Algebraic Combin., 14(2) (2001), 153–173.

[17] B. Sagan, The Symmetric Group, Graduate Texts in Mathematics, Springer–Verlag, New York, 2001.

[18] J.-P. Serre, Linear Representations of Finite Groups, Graduate Texts in Mathematics, Springer–Verlag, 1977.

[19] T. Terasoma and H. Yamada, *Higher Specht polynomials for the symmetric group*, Proc. Japan Acad. Ser. A Math. Sci., 69 (1993), 41–44.

[20] S. Weintraub, Representation Theory of Finite Groups: Algebra and Arithmetic, American Mathematical Society, Providence, RI, 2003.

Jaydeep Chipalkatti and Tagreed Mohammed
Machray Hall, Department of Mathematics
University of Manitoba
Winnipeg, MB R3T 2N2, Canada
e-mails: chipalka@cc.umanitoba.ca (J. Chipalkatti)
tagreed.mohammed@gmail.com (T. Mohammed)