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RINGS WHOSE SEMIGROUP OF RIGHT IDEALS IS $\mathcal{J}$-TRIVIAL

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Abstract. A semigroup S is *$\mathcal{J}$-trivial* if any two distinct elements of S must generate distinct ideals of S. We investigate this condition for the semigroup of all right ideals of a ring under right ideal multiplication. There is a rich interplay between the underlying ring and the semigroup of all of its right ideals.

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1. Introduction

Here R is a ring. (Herein all rings are associative, not necessarily commutative, not necessarily with identity). Let $\mathbb{R}(R)$, $\mathbb{L}(R)$, and $\mathbb{I}(R)$ denote the multiplicative semigroups of right, respectively left, two-sided ideals of R. In previous works we considered these semigroups when they are bands (every element idempotent) [7, 8]. Rings for which every right ideal is idempotent are called *right weakly regular (r.w.r.)* rings, and have been studied in great detail. For a survey of r.w.r. rings, see [9].

In this paper we consider the $\mathcal{J}$-trivial condition for the semigroups $\mathbb{R}(R)$ and $\mathbb{L}(R)$ and the consequences for the underlying ring R. A semigroup S is said to be *$\mathcal{J}$-trivial* if, whenever $a, b \in S$ such that a and b generate the same ideal in S, then $a = b$. (Here S will always denote a semigroup and S^1 is the monoid obtained by adjoining an identity element 1 to S [3, p.4].) Recall that the Green's relation $\mathcal{J}$ on S is defined by: $a\mathcal{J}b$ if $a, b \in S$ and $S^1aS^1 = S^1bS^1$; i.e., a and b are $\mathcal{J}$-equivalent [3, p.48]. Semigroups which are finite and $\mathcal{J}$-trivial have arisen in the study of formal languages [12], and in the context of full transformation semigroups [13]. Saito gives conditions for a periodic semigroup to be $\mathcal{J}$-trivial [13, Lemma 1.1]. Observe that every semilattice (commutative semigroup in which every element is idempotent) is $\mathcal{J}$-trivial, and that whenever S is $\mathcal{J}$-trivial, then so is S^1 and S^0. (Here S^0 is the semigroup with zero, 0, adjoined [3, p.4].) Not all bands are $\mathcal{J}$-trivial. For example, let S be a semigroup in which $ab = b$ for all $a, b \in S$; such a semigroup is called *right zero* [3, p.37]. Any right zero semigroup with more than one element is a band that is not $\mathcal{J}$-trivial.

In this paper we show that $\mathbb{R}(R)$ is $\mathcal{J}$-trivial if R is either commutative, right duo (every right ideal of R is two-sided), or nilpotent. The paper is arranged as follows. In Section 2 we consider conditions that imply $\mathbb{R}(R)$ is $\mathcal{J}$-trivial. If R is either right duo, commutative, nilpotent, or a skewfield, then $\mathbb{R}(R)$ is $\mathcal{J}$-trivial. If $\mathbb{R}(R)$ is either 0-cancellative or has identity, then $\mathbb{R}(R)$ is $\mathcal{J}$-trivial. In Sections 3, 4, and 5 we obtain results assuming $\mathbb{R}(R)$ is $\mathcal{J}$-trivial, a hypothesis that is assumed for the remainder of this introduction. In Section 3 idempotent right ideals are shown to be ideals, maximal right ideals are considered, and the Jacobson and Brown-McCoy radicals of R are shown to be equal. In Section 4 minimal right ideals are considered, subdirectly irreducible rings are classified, and it is shown that every idempotent is central. In Section 5 it is shown that R r.w.r. implies R is strongly regular and that R π-regular implies R is strongly π-regular.

2. Conditions which imply that $\mathbb{R}(R)$ is $\mathcal{J}$-trivial

We first consider conditions on the ring R which will imply that $\mathbb{R}(R)$ is $\mathcal{J}$-trivial. For any skewfield K, the semigroup $\mathbb{R}(K)$ has only two elements, 0 and K, and K is the identity for the semigroup. So $\mathbb{R}(K)$ is $\mathcal{J}$-trivial.

Recall that a ring R is *right (left) duo* if every right (respectively, left) ideal of R is a two-sided ideal [10].

Proposition 2.1. *Let R be a ring. Then we have the following.*

(i) *If $A, B \in \mathbb{I}(R)$ and $A \neq B$, then A and B are not $\mathcal{J}$-equivalent in either $\mathbb{R}(R)$ or $\mathbb{L}(R)$.*

(ii) *$\mathbb{I}(R)$ is $\mathcal{J}$-trivial.*

(iii) *If R is right (left) duo, then $\mathbb{R}(R)$ (respectively, $\mathbb{L}(R)$) is $\mathcal{J}$-trivial.*

(iv) *If R is commutative, then $\mathbb{R}(R)$ and $\mathbb{L}(R)$ are both $\mathcal{J}$-trivial.*

Proof. Suppose $A, B \in \mathbb{I}(R)$ and A and B are $\mathcal{J}$-equivalent in $\mathbb{R}(R)$. Then either $A = B$, A = XB, A = BX, or $A = XBY$ for some $X, Y \in \mathbb{R}(R)$. In each case $A \subseteq B$. Similarly, $B \subseteq A$, so $A = B$. Proceed similarly if A, B are $\mathcal{J}$-equivalent in $\mathbb{L}(R)$. This establishes part (i). Parts (ii) and (iii) follow immediately from (i), and (iv) follows immediately from (iii). $\square$

Note that for any commutative ring A and any set Ω of commuting indeterminates, the polynomial ring $A[\Omega]$ and the ring of formal power series $A < \Omega >$ are each commutative and hence both $\mathbb{R}(A[\Omega])$ and $\mathbb{R}(A < \Omega >)$ are $\mathcal{J}$-trivial.

Proposition 2.2. *If R is nilpotent, then $\mathbb{R}(R)$ and $\mathbb{L}(R)$ are $\mathcal{J}$-trivial.*

Proof. Let $H, K \in \mathbb{R}(R)$ with $H\mathcal{J}K$. For convenience in calculation we operate in the semigroup with identity, 1, adjoined to $\mathbb{R}(R)$. So $H = XKY$ and $K = BHT$, where X, Y, B, T are each in $\mathbb{R}(R) \cup \{1\}$. A routine calculation establishes that $H = (XB)^n H(TY)^n$, for all $n \in \mathbb{N}$. If any one of $X, B, T,$ or Y is not 1, then since H is nilpotent, by choosing n large enough we get $H = 0$. So $K = 0$. If $X = Y = 1$ we get $H = K$. Thus $\mathbb{R}(R)$ is $\mathcal{J}$-trivial. Similarly, $\mathbb{L}(R)$ is $\mathcal{J}$-trivial. □

Let char $R = n > 1$. Recall that R can be embedded as an ideal in the ring R^1, where R^1 is the set $Z_n \times R$ with the operations $(\alpha, r) + (\beta, t) = (\alpha + \beta, r + t)$, $(\alpha, r)(\beta, t) = (\alpha\beta, \alpha t + \beta r + rt)$, $\alpha, \beta \in Z_n$, $r, t \in R$, and that R^1 has identity with char $R^1 = n$ [2]. Observe that right ideals of R map onto right ideals of R^1 under the embedding mapping $r \to (0, r)$. Identifying R with its image R^1 we see that $\mathbb{R}(R) \subseteq \mathbb{R}(R^1)$. We refer to this embedding process as the *Dorroh extension* of R using Z_n, since it follows a procedure first used by J. Dorroh in [5].

Corollary 2.3. *Let R be a nilpotent ring with char $R = p$, where p is a prime. Then $\mathbb{R}(R^1) = \mathbb{R}(R) \cup \{R^1\}$. Consequently, if $\mathbb{R}(R)$ is $\mathcal{J}$-trivial, then $\mathbb{R}(R^1)$ is $\mathcal{J}$-trivial.*

Proof. As described above form the Dorroh extension of R using Z_p. Then $\mathbb{R}(R) \cup \{R^1\} \subseteq \mathbb{R}(R^1)$. Let B be a nonzero right ideal of R^1 and let $\alpha 1 + r = x \in B$, where $\alpha \in Z_p$, $r \in R$. If $\alpha \neq 0$, then $\alpha^{-1}x = 1 + \alpha^{-1}r$. Since r is nilpotent, so is $\alpha^{-1}r$. Consequently $\alpha^{-1}x$ is a unit in R^1 and hence $B = R^1$. Thus $\mathbb{R}(R) \cup \{R^1\} = \mathbb{R}(R^1)$. Using this and that $\mathbb{R}(R)$ is $\mathcal{J}$-trivial it follows immediately that $\mathbb{R}(R^1)$ is $\mathcal{J}$-trivial. □

We next give an example to show that if $\mathbb{R}(R)$ is $\mathcal{J}$-trivial, then R need not be right duo.

Example 2.4. *Let K be any skewfield, and let $R = \begin{bmatrix} 0 & K & K \\ 0 & 0 & K \\ 0 & 0 & 0 \end{bmatrix}$. Since R is nilpotent, then $\mathbb{R}(R)$ is $\mathcal{J}$-trivial by Proposition 2.2. Further, the right ideal $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & K \\ 0 & 0 & 0 \end{bmatrix}$ is not two-sided, so that R is not right duo.*

If the skewfield in Example 2.4 has characteristic p for some prime p, then we can use Corollary 2.3 to embed the ring of Example 2.4 in a ring R^1 with identity and having that $\mathbb{R}(R^1)$ is $\mathcal{J}$-trivial.

We use $[B]$ for the ideal in the semigroup $\mathbb{R}(R)$ generated by $B \in \mathbb{R}(R)$.

Proposition 2.5. *If $\mathbb{R}(R)$ is $\mathcal{J}$-trivial and $\bar{R}$ is a homomorphic image of the ring R, then $\mathbb{R}(\bar{R})$ is $\mathcal{J}$-trivial.*

Proof. Let $\phi : R \to \bar{R}$ be a surjective ring homomorphism with $Ker\ \phi = I$. For notational convenience let $S = \mathbb{R}(\bar{R})$. For any $C \in \mathbb{R}(R)$ we use $\bar{C}$ for its image under ϕ. Consider $\bar{H}, \bar{K} \in S$ with $\bar{H}\mathcal{J}\bar{K}$. In general, from $\bar{H}\mathcal{J}\bar{K}$ we have that $\bar{H} = \alpha\bar{K}\beta$ and $\bar{K} = \gamma\bar{H}\sigma$, where $\alpha, \beta, \gamma, \sigma \in S^1$. First consider the case where $\bar{H} = \bar{X}\bar{K}\bar{Y}$ and $K = \bar{B}\bar{H}\bar{T}$. Then $H + I = (X + I)(K + I)(Y + I)$ and $K + I = (B + I)(H + I)(T + I)$. So $H + I \in [K + I]$ in $\mathbb{R}(R)$, and $K + I \in [H + I]$ in $\mathbb{R}(R)$. Since $\mathbb{R}(R)$ is $\mathcal{J}$-trivial, this yields $H + I = K + I$. Consequently $\bar{H} = \bar{K}$. The other cases, where one or more of $\alpha, \beta, \gamma,$ or σ is 1, are either similar to the first case or easier. $\square$

Example 2.6. *The homomorphic image of a $\mathcal{J}$-trivial semigroup need not be $\mathcal{J}$-trivial. Let $F =< 1, x, y >$ be the free monoid generated by x and y. This monoid is $\mathcal{J}$-trivial. Let $B =< p, q \mid pq = 1 >$ be the bicyclic semigroup. Then B is a simple monoid, and hence any two right ideals are $\mathcal{J}$-related. In particular, B is not $\mathcal{J}$-trivial. Define $\phi : F \to B$ by $\phi(1) = 1,\ \phi(x) = p,\ \phi(y) = q$. Then B is a homomorphic image of F.*

Proposition 2.7. *If $\mathbb{R}(R)$ has identity, then the identity is R and $\mathbb{R}(R)$ is $\mathcal{J}$-trivial.*

Proof. Let X be the identity of $\mathbb{R}(R)$. Let H be a right ideal of R. Then $H = HX \subseteq HR \subseteq H$ which implies that $H = HR$ and hence R is a right identity for $\mathbb{R}(R)$. So $X = R$. In this case R is right duo, and hence $\mathbb{R}(R)$ is $\mathcal{J}$-trivial by Proposition 2.1 (iii). $\square$

Note that in Proposition 2.7 one cannot replace "$\mathbb{R}(R)$ has identity" with "R has identity". Any simple ring with identity and which is not a skewfield has that $\mathbb{R}(R)$ is not $\mathcal{J}$-trivial.

The converse of Proposition 2.7 is false. In the ring of Example 2.4, the right ideal

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & K \\ 0 & 0 & 0 \end{bmatrix}$$

is not two-sided, so that R is not the identity of $\mathbb{R}(R)$. Similarly, for $n \geq 3$ one can show that, in the $n \times n$ strictly upper triangular matrix ring U over any skewfield, we have that $\mathbb{R}(U)$ is $\mathcal{J}$-trivial, but U is not the identity of $\mathbb{R}(U)$.

We say that a semigroup S is *left (right) 0-cancellative* if $sx = sy$ $(xs = ys)$ implies $x = y$ for all non-zero $s, x, y \in S$. The semigroup S is *0-cancellative* if S is both left and right 0-cancellative. See [3, p.3].

Proposition 2.8. *If $\mathbb{R}(R)$ is 0-cancellative, then $\mathbb{R}(R)$ and $\mathbb{L}(R)$ are each $\mathcal{J}$-trivial.*

Proof. Let $H, K \in \mathbb{R}(R)$ with $H\mathcal{J}K$ in $\mathbb{R}(R)$. If either H or K is zero, then both must be zero. So take H and K to be nonzero. From $H\mathcal{J}K$ we get that there exist $X, Y, B, T \in \mathbb{R}(R)^1$ such that $XHY = K$ and $BKT = H$. Then $K = XHY = X(BKT)Y \subseteq XKY = X(XHY)Y = X^2HY^2 \subseteq XHY = K$, So $K = XKY$. Thus $XKY = XHY$. Note that if either X or Y is zero, then $K = 0$. So X and Y are nonzero. If $X, Y \in \mathbb{R}(R)$, then using that $\mathbb{R}(R)$ is 0-cancellative and $XHY = XKY$ we get $K = H$. If $X = Y = 1$, then $K = H$. If $X = 1$ and $Y \in \mathbb{R}(R)$, then $KY = HY$ and hence $H = K$. Similarly, if $X \in \mathbb{R}(R)$ and $Y = 1$, we get $K = H$. Thus $\mathbb{R}(R)$ is $\mathcal{J}$-trivial. Proceed similarly to get $\mathbb{L}(R)$ is $\mathcal{J}$-trivial. □

Note that the converse of Proposition 2.8 is false, as the next example illustrates.

Example 2.9. *Let A be any commutative ring and let $R = A \oplus A$. Then $\mathbb{R}(R)$ is not 0-cancellative but $\mathbb{R}(R)$ is $\mathcal{J}$-trivial.*

Proposition 2.10. *Let R be a simple ring with $R^2 \neq 0$. Then either R is a skewfield or $\mathbb{R}(R)$ is not $\mathcal{J}$-trivial.*

Proof. Assume R is not a skewfield and let $H \in \mathbb{R}(R)$ with $0 \neq H \neq R$. If $RH = 0$, then the ideal $r(R) = \{x \mid Rx = 0\}$ is nonzero and hence $R = r(R)$, contrary to $R^2 \neq 0$. So $RH = R$. Similarly $HR \neq 0$. Then $H^2 \subseteq HR = H(RH) \subseteq H^2$ and hence $H^2 = HR$. Consequently $H^2 \in [R]$. Also, $R = RH^2$, so $R \in [H^2]$. Then $R\mathcal{J}H^2$. Since H^2 is not R we have that $\mathbb{R}(R)$ is not $\mathcal{J}$-trivial. □

Example 2.11. *In Proposition 2.2 the hypothesis "R is nilpotent" cannot be replaced by "R is nil". If R is a simple nil ring which is not nilpotent, then by Proposition 2.10 $\mathbb{R}(R)$ is not $\mathcal{J}$-trivial. Examples of such rings were first given by Smoktunowicz, see* [14].

As an immediate consequence of Proposition 2.10 we have that if R is a simple ring with identity and $M_n(R)$ is the full $n \times n$ matrix ring over R, then $\mathbb{R}(M_n(R))$ is not $\mathcal{J}$-trivial for $n > 1$.

Note that for any commutative ring A and any set Ω of commuting indeterminates, the polynomial ring $A[\Omega]$ and the ring of formal power series $A < \Omega >$ are each commutative and hence both $\mathbb{R}(A[\Omega])$ and $\mathbb{R}(A < \Omega >)$ are $\mathcal{J}$-trivial.

Proposition 2.12. *If for some $m \in N$, $\mathbb{R}(R^m)$ is $\mathcal{J}$-trivial, then $\mathbb{R}(R)$ is $\mathcal{J}$-trivial.*

Proof. For convenience of notation let $S = \mathbb{R}(R)$ and consider $H, K \in \mathbb{R}(R)$ with $[H] = [K]$ in S. Then there exist $X, Y, B, T \in S^1$ such that $H = XKY$ and $K = BHT$. A routine calculation shows that $H = XKY = (XB)^n H(TY)^n$ for $n \in N$. Choose $n = m$ to get $H \in \mathbb{R}(R^m)$. Similarly $K \in \mathbb{R}(R^m)$. Also, $H = (XB)^m H(TY)^m = [(XB)^m X]K[Y(TY)^m$, so H is in the ideal in $\mathbb{R}(R^m)$ generated by K. Similarly, K is in the ideal in $\mathbb{R}(R^m)$ generated by H. So $H\mathcal{J}K$ in $\mathbb{R}(R^m)$. But $\mathbb{R}(R^m)$ is $\mathcal{J}$-trivial, so $H = K$. $\square$

Corollary 2.13. *If, for some $m \in N$, R^m is right duo or commutative, then $\mathbb{R}(R)$ is $\mathcal{J}$-trivial.*

Proposition 2.14. *Let $R = R_1 \oplus R_2$, where R_1 is a ring with $\mathbb{R}(R_1)$ $\mathcal{J}$-trivial and R_2 is a nilpotent ring. Then $\mathbb{R}(R)$ is $\mathcal{J}$-trivial.*

Proof. The argument is similar to that for Proposition 2.12. Since R_2 is nilpotent, some power of R is in R_1. Then H and K will be $\mathcal{J}$-equivalent in $\mathbb{R}(R_1)$, and since $\mathbb{R}(R_1)$ is $\mathcal{J}$-trivial we have $H = K$. $\square$

Corollary 2.15. *Let $R = R_1 \oplus R_2$, where R_1 is a ring such that $\mathbb{R}(R_1^m)$ is $\mathcal{J}$-trivial for some $m \in N$, and R_2 is nilpotent. Then $\mathbb{R}(R)$ is $\mathcal{J}$-trivial.*

Observe that $R = R_1 \oplus R_2$ will have $\mathbb{R}(R)$ is $\mathcal{J}$-trivial when R_2 is nilpotent and R_1^m is either commutative or right duo, for some m.

3. Maximal right ideals and radicals

Unless otherwise specified, for the remainder of the paper R will have identity.

Proposition 3.1. *Let $\mathbb{R}(R)$ be $\mathcal{J}$-trivial.*

(i) *If $H \in \mathbb{R}(R)$ and $H = H^2$, then $H \in \mathbb{I}(R)$.*
(ii) *If R is r.w.r., then $\mathbb{R}(R) = \mathbb{I}(R)$.*
(iii) *If $\mathbb{R}(R)$ is regular, then $\mathbb{R}(R) = \mathbb{I}(R)$.*

Proof. (i) We have that $H = H^2 \subseteq HR \subseteq H$, which implies that $H = HR$. Since $H = HR$ we have $H = H^2 = (HR)H = H(RH)$. Thus $H \in [RH]$, and trivially $RH \in [H]$. So $[RH] = [H]$ and since $\mathbb{R}(R)$ is $\mathcal{J}$-trivial we have $RH = H$.
(ii) This part follows immediately from part (i).
(iii) Every regular ring is r.w.r. [16, p.173]. □

Recall that a semigroup S is *periodic* if for each $s \in S$ there exists $n, m \in N, n > m$, such that $s^n = s^m$ [3, p.20].

Corollary 3.2. *Let $\mathbb{R}(R)$ be $\mathcal{J}$-trivial and periodic. If $H \in \mathbb{R}(R)$, then for some $k \in N$, H^k is an idempotent ideal. Consequently, each nonzero right ideal of R is either nilpotent or contains a nonzero idempotent ideal of R.*

Proof. Recall that each element in a periodic semigroup has a power which is an idempotent [3, p.20]. The desired result follows from this and from Proposition 3.1 (i). □

Proposition 3.3. *(i) If M is a maximal right ideal of R, then either $M^2 = M$ or M is an ideal of R.*
(ii) If $\mathbb{R}(R)$ is $\mathcal{J}$-trivial, then every maximal right ideal of R is an ideal of R.

Proof. (i) Since RM is an ideal of R and $M \subseteq RM$ we have that either $RM = M$, and hence M is a two-sided ideal of R, or $RM = R$. If the latter holds, then $M^2 = (MR)M = M(RM) = MR = M$.
(ii) Let $\mathbb{R}(R)$ be $\mathcal{J}$-trivial and let M be a maximal right ideal of R. Suppose M is not an ideal of R. Then $RM = R$. Hence $R \in [M]$. So $[R] \subseteq [M]$, but, because R has identity, $M = MR \in [R]$, which implies $[M] \subseteq [R]$. So $[R] = [M]$, and since $\mathbb{R}(R)$ is $\mathcal{J}$-trivial we have $R = M$, a contradiction. □

It is worth noting that from Proposition 3.3 (i) we see that in a ring with identity a maximal right ideal which is nilpotent must be a two-sided ideal.

Recall that because R has identity the Jacobson radical of R, denoted by $J(R)$, is the intersection of all maximal right ideals of R, and the Brown-McCoy radical of R, denoted by $B(R)$, is the intersection of all maximal ideals of R [15]. Neither of these results need hold for rings without identity [15].

Corollary 3.4. *If $\mathbb{R}(R)$ is $\mathcal{J}$-trivial, then $J(R) = B(R)$. If $J(R) = 0$, then R is isomorphic to the subdirect product of skewfields.*

Proof. That $J(R) = B(R)$ follows immediately from Proposition 3.3(ii). If $J(R) = 0$, then $B(R) = 0$ and R is isomorphic to a subdirect product of rings with identity of the form R/M, where the ideal M is also maximal as a right ideal of R. So R/M has no proper nonzero right ideals and hence is a skewfield. □

4. Minimal right ideals

Recall that an idempotent e is *left semicentral* if $ere = re$ for all $r \in R$ [1].

Proposition 4.1. *If $\mathbb{R}(R)$ is $\mathcal{J}$-trivial, then any idempotent in R is central.*

Proof. Let $e \in E(R)$. Since $e \in ReR$ we have $eR \subseteq ReR$ and hence $eR \subseteq eReR \subseteq eR$, so $eR = (eR)^2$. Then by Proposition 3.1(i) we have $eR = ReR$. Then $Re = Ree \subseteq ReR = eR$. So for each $r \in R$ there exists $y \in R$ such that $re = ey$. Then $ere = e^2y = ye = re$. Thus e is left semicentral and consequently $1-e$ is left semicentral. Let $f \in E(R)$. Then $(ef-fe)e = 0$ and $(ef-fe)(1-e) = ef-fe-(ef-fe)e = ef-fe$. Thus $ef-fe = (ef-fe)(1-e) = (1-e)(ef-fe)(1-e) = 0$. So e commutes with every idempotent of R. It is well-known that this implies e is central in R. □

Proposition 4.2. *Let $\mathbb{R}(R)$ be $\mathcal{J}$-trivial. If B is a minimal right ideal of R and $B^2 \neq 0$, then we have the following.*

(i) *B is an ideal of R,*

(ii) *there exists a central idempotent $e \in R$ such that $B = eR$ and $eR = Re = eRe$,*

(iii) *$R = eR \oplus (1-e)R = eRe \oplus (1-e)R$ and eRe is a skewfield, so $(1-e)R$ is an ideal of R which is maximal as a right (left) ideal of R.*

Proof. (i) Since $0 \neq B^2 \subseteq B$, by minimality of B we get $B^2 = B$. So by Proposition 3.1(i), B is an ideal of R.
(ii) It is well-known that any non-nilpotent minimal right ideal is generated by an idempotent [11, Section 31]. So there exists $e \in E(R)$ such that $B = eR$. By Proposition 4.1, e is central.
(iii) Since eR is a minimal right ideal of R we have that eRe is a skewfield [11, Theorem 3.16]. Using the Pierce decomposition with e we have $R = eR \oplus (1-e)R$, and this is a direct sum of two-sided ideals of R. From $eRe = eR \cong R/(1-e)R$, and since eRe is a skewfield, then $(1-e)R$ is maximal as a right (left) ideal of R. □

Corollary 4.3. *Let $\mathbb{R}(R)$ be $\mathcal{J}$-trivial. If R has a minimal right ideal which is not nilpotent, then $R = R_1 \oplus R_2$ where $\mathbb{R}(R_1)$ and $\mathbb{R}(R_2)$ are $\mathcal{J}$-trivial.*

Proof. From Proposition 4.2(iii) we have $R = eR \oplus (1-e)R$, where eR and $(1-e)R$ are ideals of R. Use $R/eR \cong (1-e)R$ and Proposition 2.5 to get that $\mathbb{R}((1-e)R)$ is $\mathcal{J}$-trivial. Similarly, $\mathbb{R}(eR)$ is $\mathcal{J}$-trivial. □

Proposition 4.4. *Let R be a subdirectly irreducible ring (not necessarily having identity) and let H be the heart of R. Assume $H^2 \neq 0$ and that $\mathbb{R}(R)$ is $\mathcal{J}$-trivial. If R contains a minimal right ideal B of R with $B \subseteq H$, then R is a skewfield.*

Proof. It is well-known that the non-nilpotent heart of a subdirectly irreducible ring must itself be a simple ring [4, p.135]. So H is a simple ring. If $B^2 = 0$, then the ring H must contain a non-zero nilpotent ideal. Consequently this ideal is H itself, contrary to $H^2 \neq 0$. So $B^2 \neq 0$. Use Proposition 4.2 to get that H is a skewfield. So the ring H has an identity element, which forces $H = R$, and hence R is a skewfield. □

Corollary 4.5. *Let R be a subdirectly irreducible ring (not necessarily having identity) with heart H, $H^2 \neq 0$. If $\mathbb{R}(R)$ is $\mathcal{J}$-trivial and R is right Artinian, then R is a skewfield.*

Proof. The chain condition yields the existence of a minimal right ideal B of R with $B \subseteq H$. □

Example 4.6. *The ring in Example 2.4 is subdirectly irreducible with heart* $H = \begin{bmatrix} 0 & 0 & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

5. Regularity conditions

Let $E(R)$ denote the set of idempotents of R. Recall that a ring R is *strongly regular* if R is regular and every idempotent of R is central [6].

Theorem 5.1. *If R is r.w.r. and $\mathbb{R}(R)$ is $\mathcal{J}$-trivial, then R is strongly regular.*

Proof. Let $B \in \mathbb{R}(R)$. Then $B = B^2 = (BR)R = B(RB)$. So $B \in [RB]$. Since trivially RB is in $[B]$, we then have $[B] = [RB]$ and consequently $B = RB$. So each right ideal of R is a two-sided ideal. It is known that a r.w.r. ring with this property is a regular ring [7]. By Proposition 4.1 we have that every idempotent of R is central. Therefore, R is strongly regular. □

Corollary 5.2. *The following are equivalent:*

(i) *R is r.w.r. and $\mathbb{R}(R)$ is $\mathcal{J}$-trivial,*

(ii) *R is regular and $\mathbb{R}(R)$ is $\mathcal{J}$-trivial,*

(iii) *R is strongly regular,*

(iv) *$\mathbb{R}(R)$ is a semilattice.*

Proof. The equivalence of (i), (ii), and (iii) is clear from the proof of Theorem 5.1. The equivalence of (iii) and (iv) is given in [7]. Any semilattice is a band and is $\mathcal{J}$-trivial, so (iv) implies (i), completing the logical circuit. □

Note that for a skewfield K, the ring is $M_n(K)$ is regular, and hence r.w.r., but for $n > 1$, $\mathbb{R}(M_n(K))$ is not $\mathcal{J}$-trivial.

Recall that R is *π-regular* if for each $r \in R$ there exists $b \in R$ such that $r^n br^n$, and R is *strongly π-regular* if for each $r \in R$ there exists $m \in N$ such that $r^n = r^{n+1}y$ for some $y \in R$ [16, Section 23]. It is known that every strongly π-regular ring is π-regular, but there are π-regular rings that are not strongly π-regular [16, Theorem 23.4].

Proposition 5.3. *Let $\mathbb{R}(R)$ be $\mathcal{J}$-trivial. Then R is π-regular if and only if R is strongly π-regular.*

Proof. Since all strongly π-regular rings are π-regular, it suffices to show that π-regular implies strongly π-regular when $\mathbb{R}(R)$ is $\mathcal{J}$-trivial. Let R be π-regular and let $r \in R$. Then $r^n = r^n br^n$, for some $n \in N$, $b \in R$. Observe that $r^n b$ is idempotent, so by Proposition 4.1, $r^n b$ is central and hence $r^n = r^{2n}b \in r^{n+1}R$. So R is strongly π-regular. □

Note that the hypothesis that R is π-regular and $\mathbb{R}(R)$ is $\mathcal{J}$-trivial does not imply that R is r.w.r., as the example of any nonzero nilpotent ring shows.

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