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ON SUBGROUP DEPTH
(*WITH AN APPENDIX BY S. DANZ AND B. KÜLSHAMMER*)

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Abstract. We define a notion of depth for an inclusion of complex semisimple algebras, based on a comparison of powers of the induction-restriction table (and its transpose matrix) and a previous notion of depth in an earlier paper of the second author. We prove that a depth two extension of complex semisimple algebras is normal in the sense of Rieffel, and conversely. Given an extension $H \subseteq G$ of finite groups we prove that the depth of $\mathbb{C}\,H$ in $\mathbb{C}\,G$ is bounded by $2n$ if the kernel of the permutation representation of G on cosets of H is the intersection of n conjugate subgroups of H. We prove in several ways that the subgroup depth of symmetric groups $S_n \subseteq S_{n+1}$ is $2n-1$. An appendix by S. Danz and B. Külshammer determines the subgroup depth of alternating groups $A_n \subseteq A_{n+1}$ and dihedral group extensions.

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1. Introduction

Depth two is an algebraic notion for noncommutative ring extensions with an associated Galois theory [16,17]. If applied to a subalgebra pair of quantum algebras, depth two is a notion of normality that extends ordinary normality for subgroups and Hopf subalgebras [13,15,8]. A Hopf subalgebra K is normal in a finite-dimensional Hopf algebra H if and only if H is a depth two ring extension of K: see [3] for this theorem and its generalization to faithfully flat, finitely generated projective Hopf algebra extensions over a commutative base ring as well as right and left versions of normality, depth two and Hopf-Galois extension.

For M the induction-restriction table for a subalgebra pair of semisimple $\mathbb{C}$-algebras $B \subseteq A$, the depth two condition is given by a matrix inequality $MM^tM \leq qM$ for some $q \in \mathbb{Z}_+$, an observation in [9] that we build on in this paper. Recall

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that a Hopf subalgebra K is normal in a Hopf algebra H if $HK^+ = K^+H$ where K^+ is the augmentation ideal of K. A predecessor of this definition is Rieffel's definition of a normal subring: a semisimple subalgebra B in a semisimple algebra A is a normal subring if any maximal ideal in A restricts to an A-invariant ideal in B [19]. We show in Section 4 that, for a semisimple subalgebra pair $B \subseteq A$, the depth two condition is equivalent to B being a normal subring in A. As a consequence, higher depth subgroups or semisimple subalgebras may be described as being normal further along in the Jones tower of iterated endomorphism algebras (Corollary 4.8 below).

In [14] the notion of depth more than two for a Frobenius extension $B \subseteq A$ is shown to be simplified via a generalization of depth two from ring extensions to towers of three rings, where an appropriate tower of three rings is chosen in a tower of iterated endomorphism rings above $B \subseteq A$. In [9] this idea was applied to a pair of complex semisimple algebras $B \subseteq A$ with inclusion matrix M: the subalgebra pair has depth n if n is the least integer for which M^{n+1} is less than a multiple of M^{n-1}, where powers of an $r \times s$-matrix M are understood by $M^2 = MM^t$, $M^3 = MM^tM$, and so forth. As noted in Section 2 below, depth n is equivalently the point of stabilization of the zero entries of even or odd powers of M, which form a descending chain of subsets.

In [9] the generalized depth two condition on a tower of semisimple algebras $C \subseteq B \subseteq A$ with inclusion matrices N and $\tilde{M}$, respectively, is given by $N\tilde{M}\tilde{M}^t\tilde{M} \leq qN\tilde{M}$ where $q \in \mathbb{Z}_+$: let N be the identity matrix to recover the depth two condition on a subalgebra pair. Build a tower of algebras above $B \subseteq A$, where $A_1 = \operatorname{End} A_B$ and one iterates the endomorphism ring construction and embeds via left multiplication. Then $A \hookrightarrow A_1$ has inclusion matrix M^t, and the subalgebra pair $B \subseteq A$ has depth n if n is the least integer for which the tower $B \subseteq A_{n-3} \subseteq A_{n-2}$ satisfies the generalized depth two condition. For this tower of three algebras the inclusion matrices are $N = M^{n-2}$, $\tilde{M} = M$ or M^t, in the generalized depth two condition, which when substituted and simplified, becomes the depth n inequality condition, $M^{n+1} \leq qM^{n-1}$, on the inclusion matrix M.

The paper is organized as follows. In Section 2 we define a matrix M of non-negative integer coefficients with nonzero rows and columns to be of depth $n > 1$ if n is the least integer for which the $n+1$'st power of M is less than a multiple of the $n-1$'st power of M, where M^2 denotes MM^t, $M^3 = MM^tM$ and so on. For example, if M is the induction-restriction table of irreducible characters of a finite group G and a subgroup H, then M has depth two if and only if H is a normal subgroup of G [15]. If M is the inclusion matrix of a subalgebra pair of complex semisimple algebras $B \subseteq A$, then M has depth two if and only if B is a normal subring in A (in the sense of Rieffel [19]) as shown below in Theorem 4.6. We study depth three or more in Section 5, 6 and two appendices by Danz and the third coauthor. We prove that the depth of symmetric groups $S_n \subseteq S_{n+1}$ is $2n-1$. In Section 3, we make use of a well-known interpretation of the inclusion matrix M as the incidence matrix of a bicolored weighted multigraph of semisimple algebras $B \subseteq A$, showing that odd depth is one plus the diameter of the row corresponding to the simples of B; even depth is two plus the maximum graphical distance along edges of the graph from an equivalence class of simples of B, under one simple of A, to the simple of B furthest away. In Section 6, we show that a subgroup H of G has depth at most $2n$ if the largest normal subgroup of G contained in H (i.e. the core) is the intersection of n conjugates of H (and at most $2n-1$ if the core is trivial).

Throughout this paper our algebras have ground field $\mathbb{C}$, although this may be replaced by any algebraically closed field of characteristic zero with the same results. In this case semisimple algebras split into direct products of matrix algebras sometimes known as multi-matrix algebras or split semisimple algebras.

2. The depth of an irredundant matrix

In this section, we make an introductory study of irredundant matrices, which naturally arise as the induction-restriction table of irreducible $\mathbb{C}$-characters for a subgroup of a finite group. This type of matrix also occurs more generally as the induction-restriction table of simple modules, equivalently incidence matrices of inclusion diagrams, for a subalgebra pair of semisimple algebras as explained in the next section.

An $r \times s$-matrix $M = (m_{ij})$ with non-negative integer entries is called *irredundant* if each column and row vector of M is nonzero. It is called *positive* (written $M > 0$) if each m_{ij} is positive. Its (right) *square* will be the symmetric $r \times r$-matrix $\mathcal{S} := M^2 := MM^t$. The (i, j)-entry s_{ij} of $\mathcal{S}$ is the euclidean inner product of rows i and j in M. In particular, the diagonal entry s_{ii} is positive since each row in M is nonzero. Continuing, the *cube* of M is just $M^3 = MM^tM = \mathcal{S}M$, $M^4 = \mathcal{S}^2$, etc. The odd powers $M^{2n+1} = \mathcal{S}^nM$ are all of size $r \times s$, and the even powers $M^{2n} = \mathcal{S}^n$ are symmetric $r \times r$-matrices. All these powers of M are again irredundant.

Let $N = (n_{ij})$ be another irredundant $r \times s$-matrix. Then N and M are called *equivalent up to permutation* if there are permutation matrices $P \in S_r$ and $Q \in S_s$ such that $M = PNQ$. We use the ordering $M \geq N$ if $m_{ij} \geq n_{ij}$ for each $i = 1, \ldots, r$

and $j = 1, \dots, s$. If $M \geq N$ then $TM \geq TN$ and $MU \geq NU$ for every irredundant $r \times r$-matrix $T \geq 0$ and every irredundant $s \times s$-matrix $U \geq 0$. If $T > 0$ then also $TM > 0$ since the columns in M contain positive entries.

The definition below comes from considerations of what depth greater than two is for Frobenius extensions when restricted to semisimple subalgebra pairs with inclusion matrix M as outlined in the introduction. (Although not needed in this paper, the interested reader might see [16,14] for the definition of higher depth Frobenius extensions and see [9] for why semisimple $\mathbb{C}$-algebra pairs are split separable Frobenius extensions.)

Definition 2.1. An irredundant $r \times s$-matrix M has *depth* $n \geq 2$ if n is the least integer for which the following inequality (called a depth n matrix inequality) holds for some $q \in \mathbb{Z}_+$,

$$M^{n+1} \leq qM^{n-1}. \tag{1}$$

The definition depends only on the equivalence class of M up to permutation. The remarks above show that (1) implies the depth $n+1$ matrix inequality $M^{n+2} \leq qM^n$. We set $Z(M) = \{(i,j) : m_{ij} = 0\}$ and $A(M) = \{(i,j) : m_{ij} \neq 0\}$. Then

$$Z(M^{n-1}) \supseteq Z(M^{n+1}) \quad \text{and} \quad A(M^{n-1}) \subseteq A(M^{n+1})$$

for $n \geq 2$ since $M^{n+1} = \mathcal{S}M^{n-1}$ and the diagonal entries of $\mathcal{S}$ are positive.

The descending chain of subsets $Z(M) \supseteq Z(M^3) \supseteq Z(M^5) \supseteq \dots$ must stabilize at some point. The next proposition notes that the point at which these subsets become equal bounds the depth of M.

Proposition 2.2. *An irredundant matrix M satisfies a depth n inequality if and only if $Z(M^{n-1}) = Z(M^{n+1})$ if and only if $A(M^{n-1}) = A(M^{n+1})$. In particular, if $M^{n-1} > 0$ then M has depth n or less.*

Proof. Suppose that $M^{n+1} \leq qM^{n-1}$ for some $q \in \mathbb{Z}_+$. If the (i,j)-entry in M^{n-1} is zero then the (i,j)-entry in M^{n+1} is also zero, by the matrix inequality. Thus $Z(M^{n-1}) \subseteq Z(M^{n+1})$. Together with the opposite inclusion noted above, we conclude that $Z(M^{n-1}) = Z(M^{n+1})$.

Conversely, if $Z(M^{n-1}) = Z(M^{n+1})$, we may choose q to be the maximum of the entries in M^{n+1}, in which case $M^{n+1} \leq qM^{n-1}$. □

Corollary 2.3. *Let M be an irredundant matrix. If the minimum polynomial of $\mathcal{S} = MM^t$ has degree d then M has depth $2d-1$ or less. In particular, M has always finite depth.*

Proof. The minimum polynomial of $\mathcal{S}$ gives an equation of the form

$$\mathcal{S}^d + a_1\mathcal{S}^{d-1} + \cdots + a_{d-1}\mathcal{S} + a_d = 0$$

with $a_1, \ldots, a_d \in \mathbb{C}$. Thus $A(M^{2d}) = A(\mathcal{S}^d) \subseteq A(\mathcal{S}^{d-1}) = A(M^{2d-2})$, so that M has depth $2d-1$ or less. $\square$

Example 2.4. The matrix

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

has depth 4 as is easily computed (in fact, $M^5 \leq 5M^3$). Note that M is the inclusion matrix of the Sylow 2-subgroup D_8 in the symmetric group S_4 (cf. section 3).

It is interesting to compare the depths of M and M^t.

Proposition 2.5. *If an irredundant matrix M has depth n, then M^t has depth $n+1$ or less. If n is even, then M^t has depth n.*

Proof. If $M^{n+1} \leq qM^{n-1}$ for some $q \in \mathbb{Z}_+$, we multiply from the left by M^t to obtain $(M^t)^{n+2} \leq q(M^t)^n$, which shows that M^t has depth $n+1$ or less.

If n is even then the transpose of the inequality $M^{n+1} \leq qM^{n-1}$ is the inequality $(M^t)^{n+1} \leq q(M^t)^{n-1}$. $\square$

Example 2.6. The matrix M below, which is the inclusion matrix of $S_2 \subseteq S_3$ ([9]), has depth three while its transpose has depth four:

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \; MM^t = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \; M^tM = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

This is easier to see graphically; we return to this example in the next section.

3. On inclusions of complex semisimple algebras

Suppose that $B \subseteq A$ is an inclusion of complex semisimple algebras. Label the simple A-modules by $V_1, \ldots, V_s$ and the simple B-modules by $W_1, \ldots, W_r$. Restrict the j'th simple A-module V_j to B and express the result in terms of simple B-modules:

$$V_j \downarrow_B \cong \bigoplus_{i=1}^{r} m_{ij} W_i \tag{2}$$

Then $M = (m_{ij})$ is an $r \times s$-matrix, and

$$W_i \uparrow^A = W_i^A = A \otimes_B W_i \cong \bigoplus_{j=1}^{s} m_{ij} V_j \tag{3}$$

since $\mathrm{Hom}_A(A \otimes_B W_i, V_j) \cong \mathrm{Hom}_B(W_i, V_j)$ for all i, j. In other words, we have

$$[W_i \uparrow^A, V_j] = m_{ij} = [W_i, V_j \downarrow_B] \tag{4}$$

where $[X, Y] := \dim \mathrm{Hom}_A(X, Y)$ for finite-dimensional A-modules X, Y. The matrix M is known as the *inclusion matrix* of B in A [10]. It corresponds to the induction-restriction table (as it is known in group theory) for simples or their irreducible characters. It may also be viewed as the matrix of a linear mapping in K-theory, between the groups $\mathbb{Z}^r \cong K_0(B) \to K_0(A) \cong \mathbb{Z}^s$. Note that $m_{ij} \neq 0$ if and only if W_i is a constituent of $V_j \downarrow_B$.

For irreducible characters $\alpha \in \mathrm{Irr}(B)$ and $\chi \in \mathrm{Irr}(A)$, let $f_\alpha \in B$ and $e_\chi \in A$ denote the corresponding central idempotents. It is well-known that α is a constituent of $\chi \downarrow_B$ if and only if $e_\chi f_\alpha \neq 0$.

Let $\mathcal{Z}(A)$ be the center of the algebra A. Then the algebra $\mathcal{Z}(A) \cap B$ is a (semisimple) subalgebra of $\mathcal{Z}(A)$ and $\mathcal{Z}(B)$. Thus there are partitions $\mathrm{Irr}(A) = \bigsqcup_{i=1}^t \mathcal{A}_i$ and $\mathrm{Irr}(B) = \bigsqcup_{i=1}^t \mathcal{B}_i$ such that the basis of primitive idempotents of $\mathcal{Z}(A) \cap B$ is given by

$$m_i = \sum_{\chi \in \mathcal{A}_i} e_\chi = \sum_{\alpha \in \mathcal{B}_i} f_\alpha. \tag{5}$$

Proposition 3.1. *Let $B \subseteq A$ be an inclusion of complex semisimple algebras with A free as left B-module. With the above notations it follows that*

$$\sum_{\chi \in \mathcal{A}_i} \chi(1)\chi \downarrow_B = \frac{\dim A}{\dim B} \sum_{\alpha \in \mathcal{B}_i} \alpha(1)\alpha \quad \textit{and} \quad \sum_{\alpha \in \mathcal{B}_i} \alpha(1)\alpha \uparrow^A = \sum_{\chi \in \mathcal{A}_i} \chi(1)\chi. \tag{6}$$

Proof. The regular character of A is $\rho_A = \sum_{i=1}^t \chi_i$ where $\chi_i = \sum_{\chi \in \mathcal{A}_i} \chi(1)\chi$ for $i = 1, \ldots, t$. Similarly, the regular character of B is $\rho_B = \sum_{i=1}^t \alpha_i$ where $\alpha_i = \sum_{\alpha \in \mathcal{B}_i} \alpha(1)\alpha$ for $i = 1, \ldots, t$. Since A is free as a left B-module, we also have $\rho \downarrow_B = \frac{\dim A}{\dim B} \rho_B$. Thus $\chi_i \downarrow_B = \frac{\dim A}{\dim B} \alpha_i$ for $i = 1, \ldots, t$. Since $\rho_B \uparrow^A = \rho_A$, we similarly obtain $\alpha_i \uparrow^A = \chi_i$ for $i = 1, \ldots, t$. □

We define a relation on $\mathrm{Irr}(B)$ by $\alpha \sim \beta$ if $\alpha \uparrow^G$ and $\beta \uparrow^G$ have a common irreducible constituent. This relation $\sim$ is reflexive and symmetric but not transitive in general. Its transitive closure is an equivalence relation denoted by $\approx$ or d_B^A. Thus $\alpha \approx \beta$ if and only if there are $\alpha_0, \ldots, \alpha_m \in \mathrm{Irr}(B)$ such that $\alpha = \alpha_0 \sim \alpha_1 \sim \cdots \sim$

$\alpha_m = \beta$. (This equivalence relation was considered before by Rieffel in [19]; this section and the next may be viewed as a continuation of results in his paper.)

We also define a relation on $\mathrm{Irr}(A)$ by $\chi \sim \mu$ if $\chi \downarrow_B$ and $\mu \downarrow_B$ have a common irreducible constituent. This relation $\sim$ is again reflexive and symmetric but not transitive in general. Its transitive closure is an equivalence relation denoted by $\approx$ or u_B^A. Thus $\chi \approx \mu$ if and only if there are $\mu_0, \dots, \mu_r \in \mathrm{Irr}(A)$ such that $\chi = \mu_0 \sim \mu_1 \sim \cdots \sim \mu_r = \mu$.

Let $\alpha \in \mathrm{Irr}(B)$. Then all irreducible constituents of $\alpha \uparrow^A$ are in the same equivalence class of u_B^A. If $\alpha \sim \beta \in \mathrm{Irr}(B)$ then $\alpha \uparrow^A$ and $\beta \uparrow^A$ have a common irreducible constituent $\chi \in \mathrm{Irr}(A)$. Thus all the irreducible constituents of $\alpha \uparrow^A$ and $\beta \uparrow^A$ are u_B^A-equivalent. This implies by transitivity that, for $\alpha, \beta \in \mathrm{Irr}(B)$ with $\alpha \approx \beta$, all the irreducible constituents of $\alpha \uparrow^A$ and of $\beta \uparrow^A$ are u_B^A-equivalent.

An argument similar to the one just given shows that whenever $\chi, \mu \in \mathrm{Irr}(A)$ are u_B^A-equivalent then all irreducible constituents of $\chi \downarrow_B$ are d_B^A-equivalent to all irreducible constituents of $\mu \downarrow_B$.

Proposition 3.2. *Let $B \subseteq A$ be an inclusion of complex semisimple algebras. Then $\mathcal{A}_1, \dots, \mathcal{A}_t$ are the u_B^A-equivalence classes, and $\mathcal{B}_1, \dots, \mathcal{B}_t$ are the d_B^A-equivalence classes.*

Proof. Let $\chi, \mu \in \mathrm{Irr}(A)$ such that $\chi \sim \mu$. Then $\chi \downarrow_B$ and $\mu \downarrow_B$ have a common constituent $\alpha \in \mathrm{Irr}(B)$. Thus $e_\chi f_\alpha \neq 0 \neq e_\mu f_\alpha$. Let $i \in \{1, \dots, t\}$ such that $\alpha \in \mathcal{B}_i$. Then $e_\chi m_i \neq 0 \neq e_\mu m_i$ which implies that $\chi, \mu \in \mathcal{A}_i$. This shows that each $\mathcal{A}_i$ is a union of u_B^A-equivalence classes.

Conversely, let $\mathcal{C}_1, \dots, \mathcal{C}_l$ be the u_B^A-equivalence classes of $\mathrm{Irr}(A)$, and let $e_i = \sum_{\chi \in \mathcal{C}_i} e_\chi$ for $i = 1, \dots, l$. Then $t \leq l$, $\mathrm{Irr}(A) = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_l$ and $1_A = e_1 + \cdots + e_l$.

If $\alpha \in \mathrm{Irr}(B)$ then $e_i f_\alpha \neq 0$ for some $i \in \{1, \dots, l\}$. We claim that i is unique; in fact, if also $j \in \{1, \dots, l\}$ with $e_j f_\alpha \neq 0$ then there exist $\chi \in \mathcal{C}_i$, $\mu \in \mathcal{C}_j$ such that $e_\chi f_\alpha \neq 0 \neq e_\mu f_\alpha$. Thus α is a constituent of $\chi \downarrow_B$ and $\mu \downarrow_B$. Hence $\chi \sim \mu$, and $i = j$. So our claim is proved.

We conclude that $e_i f_\alpha = e_1 f_\alpha + \cdots + e_l f_\alpha = f_\alpha$. Thus

$$e_i = \sum_{\alpha \in \mathrm{Irr}(B)} e_i f_\alpha = \sum_{\alpha \in \mathrm{Irr}(B),\ e_i f_\alpha \neq 0} e_i f_\alpha = \sum_{\alpha \in \mathrm{Irr}(B),\ e_i f_\alpha \neq 0} f_\alpha \in B.$$

In this way we obtain non-zero pairwise orthogonal idempotents $e_1, \dots, e_l$ in $\mathcal{Z}(A) \cap B$. Thus $l \leq \dim \mathcal{Z}(A) \cap B = t \leq l$, and we see that $\mathcal{A}_1, \dots, \mathcal{A}_t$ are the u_B^A-equivalence classes.

The proof of the other assertion is similar. $\square$

Corollary 3.3. *Let $B \subseteq A$ be an inclusion of complex semisimple algebras. Then the number of u_B^A-equivalence classes is the same as the number of d_B^A-equivalence classes and equals* $\dim(\mathcal{Z}(A) \cap B)$.

In the following we identify modules with their characters. Let $i, j \in \{1, \dots, r\}$ be different. We say that the distance $d(W_i, W_j)$ between W_i and W_j is m if m is the smallest number such that there are $m-1$ intermediate simple B-modules with $W_i = W_{i_0} \sim W_{i_1} \sim \cdots \sim W_{i_m} = W_j$. Thus $d(W_i, W_j) = 1$ if and only if $W_i \sim W_j$. We put $d(W_i, W_j) = -\infty$ if W_i and W_j are not d_B^A-equivalent, and $d(W_i, W_i) = 0$ for all $1 \leq i \leq r$. To any extension of complex semisimple algebras we associate the standard bipartite graph, with one edge between black and white dots for each nonzero multiplicity (or the more refined bicolored weighted multigraph called the Bratelli diagram, with as many edges as the multiplicity, see subsection 2.3 of [10]). Note that the distance defined here is half of the *graphical distance* between the black points corresponding to W_i and W_j in the standard bipartite graph.

Once again consider the inclusion matrix $M = (m_{ij})$ for an inclusion of complex semisimple algebras $B \subseteq A$ and its symmetric 'square' $\mathcal{S} = MM^t = (s_{ij})$. The entries of the powers of M (in the sense of the previous section) will be denoted by $(\mathcal{S}^m M^k)_{ij}$ where $k = 0, 1$.

Proposition 3.4. *Let $i \neq j$. Then $(\mathcal{S}^m)_{ij} > 0$ if and only if $0 < d(W_i, W_j) \leq m$. This is equivalent to the existence of a path of length $2m$ between W_i and W_j in the standard bipartite graph of $B \subseteq A$.*

Proof. Observe first that $s_{ij} > 0$ if and only if $W_i \sim W_j$ or equivalently $d(W_i, W_j) = 1$. Indeed $s_{ij} = \sum_{u=1}^{s} m_{iu} m_{ju}$. Thus $s_{ij} > 0$ if and only if there is u such that $m_{iu} > 0$ and $m_{ju} > 0$. That means that W_i and W_j are constituents of $V_u \downarrow_B$ and therefore $W_i \sim W_j$.

For $m > 1$ note that $(\mathcal{S}^m)_{ij} = \sum_{l_1, \dots, l_{m-1}} s_{il_1} s_{l_1 l_2} \cdots s_{l_{m-1} j}$. Thus $(\mathcal{S}^m)_{ij} > 0$ if and only if there are $1 \leq l_1, \dots, l_{m-1} \leq r$ such that $W_i \sim W_{l_1} \sim \cdots \sim W_{l_{m-1}} \sim W_j$, i.e. $d(W_i, W_j) \leq m$. □

We recall the notation $A(X) = \{(i, j) : x_{ij} \neq 0\}$ for an irredundant $r \times s$-matrix $X = (x_{ij})$. Then, given another irredundant $r \times s$-matrix Y, there is $q \in \mathbb{Z}_+$ such that $X \leq qY$ if and only if $A(X) \subseteq A(Y)$.

We also recall that $\mathcal{S}_{ii} > 0$ and therefore $(\mathcal{S}^p)_{ii} > 0$ for all $p > 0$. This implies that $A(\mathcal{S}^m) \subseteq A(\mathcal{S}^{m+p})$ for all $p > 0$. In terms of distance in the standard bipartite graph this says that if there is a path between W_i and W_j of length $2m$ then there is also a path of length $2(m + p)$ between the same points. For example, one can travel p-times back and forth along the last edge of the path of length $2m$.

Definition 3.5. The depth of a complex semisimple algebra inclusion $B \subseteq A$ is defined to be the depth of its inclusion matrix M (in terms of Section 2, which also notes that the definition is independent of the ordering in the basis of simples). A subgroup H of a finite group G is said to have depth n if the corresponding group algebras over $\mathbb{C}$ form an inclusion of complex semisimple algebras (via Maschke and Wedderburn theory) of depth n.

The background for this definition is given in [9,16,17,15,14,13] and their bibliographies; the definition coincides with the definition of depth introduced briefly in [9]. In the group algebra case of the definition above, the depth two condition $M^3 \leq qM$ is the same as the condition for depth two in [15, Section 3], as can be seen as follows. Suppose that $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_s\}$ and $\mathrm{Irr}(H) = \{\psi_1, \ldots, \psi_r\}$. Then $m_{ij} = \langle \psi_i \,|\, (\chi_j)_H \rangle = \langle \psi_i^G \,|\, \chi_j \rangle$ by Frobenius reciprocity, and

$$s_{ij} = \sum_{k=1}^{s} m_{ik} m_{jk} = \sum_{k=1}^{s} \langle \psi_i^G \mid \chi_k \rangle \langle \psi_j^G \mid \chi_k \rangle = \langle \psi_i^G \,|\, \psi_j^G \rangle.$$

We apply reciprocity and orthogonal expansion:

$$\begin{aligned} (\mathcal{S}M)_{ij} &= \sum_{k=1}^{r} \langle \psi_i^G \,|\, \psi_k^G \rangle \langle \psi_k \,|\, (\chi_j)_H \rangle = \sum_{k=1}^{r} \langle (\psi_i^G)_H \,|\, \psi_k \rangle \langle \psi_k \,|\, (\chi_j)_H \rangle \\ &= \langle (\psi_i^G)_H \,|\, (\chi_j)_H \rangle = \langle ((\psi_i^G)_H)^G \,|\, \chi_j \rangle \end{aligned} \tag{7}$$

For group algebras, we will in practice obtain M as follows. Both character tables of G and H will be assumed known. Restrict each of the s irreducible characters of G to H, then express each restricted character as a non-negative integer combination of the r irreducibles of H by using inner products of characters.

Theorem 3.6. *The inclusion matrix M of a complex semisimple algebra inclusion $B \subseteq A$ satisfies a depth $2m+1$ matrix inequality ($m \geq 1$) if and only if the distance between any two simple B-modules is at most m.*

Proof. Suppose that M satisfies a depth $2m+1$ matrix inequality. By Proposition 2.2, we have $A(\mathcal{S}^m) = A(\mathcal{S}^{m+p})$ for all $p > 0$. If $(\mathcal{S}^{m+p})_{ij} > 0$, then $(\mathcal{S}^m)_{ij} > 0$, so $d(W_i, W_j) \leq m$ by Proposition 3.4. It follows that $d(W_i, W_j) \leq m$ for all pairs of simples W_i, W_j over B. (The distance is $-\infty$ if two simples are not in the same connected component of the inclusion diagram).

Conversely, suppose that the distance between any two simple B-modules is at most m. By Corollary 2.2, we have to show that $A(\mathcal{S}^{m+1}) \subseteq A(\mathcal{S}^m)$. If $(\mathcal{S}^{m+1})_{ij} > 0$ then in the standard bipartite graph there is a path of length $2m+2$ between W_i and W_j. Therefore the distance between these two points is positive (not $-\infty$), and by the assumption it is at most m. Thus $(\mathcal{S}^m)_{ij} > 0$ by Proposition 3.4. □

If we define the *diameter* of a row of simples in an inclusion diagram to be the greatest graphical distance between simples (an even number), the theorem says that the minimum odd depth inequality satisfied by the inclusion matrix of $B \subseteq A$ is one plus the diameter of the simples of B in its inclusion diagram.

Corollary 3.7. *The inclusion $B \subseteq A$ has depth 3 or less if and only if $\sim$ is an equivalence relation on $\mathrm{Irr}(B)$.*

Proof. Suppose that $B \subseteq A$ has depth 3 or less. By the above theorem the distance between any two simple B-modules W_i and W_j is at most 1. If $W_i \sim W_j$ and $W_j \sim W_k$ then $d(W_i, W_k) \leq 2$. The assumption implies $d(W_i, W_k) \leq 1$, i.e. $W_i \sim W_k$. This proves that $\sim$ is transitive.

Conversely, suppose that $\sim$ is transitive. Then the distance between any two simple B-modules is at most 1. The above theorem implies that $B \subseteq A$ is of depth 3 or less. □

Let $1 \leq u \leq s$. The irreducible constituents of $V_u \downarrow_B$ are all inside of one d_B^A-equivalence class. Denote the set of these constituents by $\mathcal{V}_u$. The distance between a simple B-module W_i and $\mathcal{V}_u$ is defined as usual, as the minimal distance between W_i and an element of $\mathcal{V}_u$. Thus

$$d(W_i, \mathcal{V}_u) = \min\{d(W_i, W_j) : W_j \in \mathcal{V}_u\}. \tag{8}$$

Definition 3.8. We define $m(V_u)$ to be the maximal distance between any simple B-module W_i and the set $\mathcal{V}_u$.

Note that $m_{iu} > 0$ if and only if $W_i \in \mathcal{V}_u$.

Proposition 3.9. *Let $m \geq 1$ and $W_i \notin \mathcal{V}_u$. Then $(\mathcal{S}^m M)_{iu} > 0$ if and only if $0 < d(W_i, \mathcal{V}_u) \leq m$.*

Proof. Suppose that $0 < (\mathcal{S}^m M)_{iu} = \sum_{l=1}^{r} (\mathcal{S}^m)_{il} m_{lu}$. Then there is $1 \leq l \leq r$ such that $(\mathcal{S}^m)_{il} > 0$ and $m_{lu} > 0$. Proposition 3.4 and the remark above imply that $d(W_i, W_l) \leq m$ and $W_l \in \mathcal{V}_u$. Thus $0 < d(W_i, \mathcal{V}_u) \leq m$.

Conversely, if $0 < d(W_i, \mathcal{V}_u) \leq m$ then there is $1 \leq l \leq r$ such that $0 < d(W_i, W_l) \leq m$ and $W_l \in \mathcal{V}_u$. This implies that $(\mathcal{S}^m)_{il} > 0$ and $m_{lu} > 0$ which together give that $(\mathcal{S}^m M)_{iu} > 0$. □

Theorem 3.10. *The inclusion matrix M of $B \subseteq A$ satisfies a depth $2m$ matrix inequality (with $m \geq 2$) if and only if $m(V_u) \leq m - 1$ for any simple A-module V_u.*

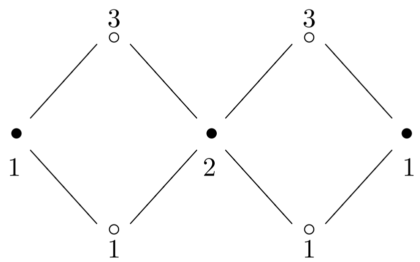
Proof. Suppose that M satisfies a depth $2m$ matrix inequality. Then $\mathcal{S}^m M \leq q\mathcal{S}^{m-1} M$ for some $q \in \mathbb{Z}_+$. By induction one can prove that $\mathcal{S}^{m+p} M \leq q^{p+1} \mathcal{S}^{m-1} M$

(multiplying with $\mathcal{S}$ on the left). Assume that $m(V_u) = m+p$ with $p \geq 0$ and some u. This implies that there is a simple B-module W_i such that $d(W_i, \mathcal{V}_u) = m+p$. Proposition 3.9 implies that $(\mathcal{S}^{m+p}M)_{iu} > 0$. Thus $(\mathcal{S}^{m-1}M)_{iu} > 0$ and Proposition 3.9 implies that $d(W_i, \mathcal{V}_u) \leq m-1$. This is a contradiction and the proof in one direction is complete.

Conversely suppose that $m(V_u) \leq m-1$ for any simple A-module V_u. By Corollary 2.2, we have to show that $A(\mathcal{S}^m M) \subseteq A(\mathcal{S}^{m-1}M)$. Let $(i,u) \in A(\mathcal{S}^m M)$. Then $(\mathcal{S}^m M)_{iu} > 0$ and Proposition 3.9 implies that $d(W_i, \mathcal{V}_u) \leq m$. Our assumption gives that $d(W_i, \mathcal{V}_u) \leq m-1$. Then Proposition 3.9 implies that $(\mathcal{S}^{m-1}M)_{iu} > 0$. Thus $(i,u) \in A(\mathcal{S}^{m-1}M)$. $\square$

We note that in terms of graphical distance, the minimal even depth matrix inequality satisfied by $B \subseteq A$ is two plus the largest graphical distance of a B-simple from an equivalence class of B-simples under one A-simple.

Example 3.11. The bipartite graph of the inclusion $B = \mathbb{C}[S_2] \subseteq A = \mathbb{C}[S_3]$ at the bottom level, joined to the graph of the semisimple pair $A \hookrightarrow E = \operatorname{End} A_B$ via λ at the top level is shown below:



Notice that the graph of $A \hookrightarrow E$ is the reflection of the graph of $B \subseteq A$ about the middle row, which is true in general by Morita theory [10]. Applying Theorem 3.6, we see from the bottom graph that the graphical distance between simples is 2, so the depth of $S_2 \subseteq S_3$ is three. Applying Theorem 3.10, we see from the top graph that the maximal distance between a simple and a set $\mathcal{V}_u$ of two simples on the middle line has graphical distance 2, so that the depth of $A \hookrightarrow E$ is four.

By simply adding dots and the same pattern of edges to the right of the diagram, we create graphs (in fact Dynkin diagrams of type A_n) for complex semisimple algebra inclusions of arbitrary odd or even depth. In terms of explicit inclusion mappings, the following inclusion $B := \mathbb{C}^n \to A := \mathbb{C} \times M_2(\mathbb{C})^{n-1} \times \mathbb{C}$ has depth $2n-1$: ($\lambda_i \in \mathbb{C}$, $n \geq 2$)

$$(\lambda_1, \ldots, \lambda_n) \mapsto (\lambda_1, \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{pmatrix}, \ldots, \begin{pmatrix} \lambda_{n-1} & 0 \\ 0 & \lambda_n \end{pmatrix}, \lambda_n)$$

while its endomorphism algebra extension $A \hookrightarrow E = M_3(\mathbb{C}) \times M_4(\mathbb{C})^{n-2} \times M_3(\mathbb{C})$ has depth $2n$: $(M_i \in M_2(\mathbb{C}))$

$$(\lambda_1, M_1, \ldots, M_{n-1}, \lambda_n) \mapsto (\begin{pmatrix} \lambda_1 & 0 \\ 0 & M_1 \end{pmatrix}, \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, \ldots, \begin{pmatrix} M_{n-1} & 0 \\ 0 & \lambda_n \end{pmatrix})$$

Remark 3.12. The definition of depth may be extended to the case depth one as follows. Define M^0 to be the $r \times r$ identity matrix I in the depth n matrix inequality condition, in which case a depth one extension of semisimple algebras $B \subseteq A$ with inclusion matrix M satisfies $\mathcal{S} \leq qI$ for $q \in \mathbb{Z}_+$. This is satisfied by a centrally projective ring extension $B \subseteq A$, defined by ${}_BA_B \oplus * \cong {}_BB^n_B$ for some n, or equivalently there are $r_i \in C_A(B)$ and $f_i \in \mathrm{Hom}\,({}_BA_B, {}_BB_B)$ such that each $a \in A$ satisfies $a = \sum_{i=1}^n r_i f_i(a)$. If A and B are the complex group algebras corresponding to $G \supseteq H$, a depth one extension is for example any subgroup of the center of G, or H is normal in G with a normal complement. This depth one condition is analyzed further in [4].

If M is the inclusion matrix of a subalgebra pair of semisimple algebras $B \subseteq A$, then M^t is the inclusion matrix of $A \hookrightarrow E = \mathrm{End}\, A_B$ (via $a \mapsto \lambda_a$ where $\lambda_a(x) = ax$ for every $a, x \in A$) by an argument that goes as follows. It is clear that the natural module A_B is finitely generated projective; it is indeed also a generator since the ground field has characteristic zero. Thus B and E are Morita equivalent algebras with context bimodules ${}_EA_B$ and $A^* := {}_B\mathrm{Hom}\,(A_B, B_B)_E$; the E-simples are then $A \otimes_B W_i$ $(i = 1, \ldots, r)$. Restricting the E-simples down to A and using Eq. (3), the columns of the inclusion matrix of $A \hookrightarrow E$ are the rows of M. We conclude that the inclusion matrix of $A \hookrightarrow E$ is M^t. Thus the following result follows from observations in Section 2.

Corollary 3.13. *The subalgebra pair of semisimple algebras $B \subseteq A$ is of depth $2n$ if and only if its endomorphism algebra extension $A \hookrightarrow E$ is of depth $2n$. If $B \subseteq A$ is of depth $2n-1$, then $A \hookrightarrow E$ is of depth $2n$ or less.*

This corollary is consistent with several general 'endomorphism ring theorems' in [14,13] and is an improvement in the semisimple case.

It is also easy to see that if $C \subseteq B$ and $B \subseteq A$ are successive subalgebra pairs of semisimple algebras with inclusion matrices M and N, respectively, then the inclusion matrix of the composite subalgebra pair $C \subseteq A$ is MN. As a simple consequence we may note an improved version of the embedding theorem [14, Corollary 8.6]. We prove that any depth n subalgebra pair may be embedded in a depth two extension, depth two being an improvement from the point of view of Galois theory (see [16,15,14,13] and papers in their bibliographies). We set up the theorem by extending the subalgebra pair and its endomorphism ring, $B \subseteq A \hookrightarrow E_1 := E = \operatorname{End} A_B$ by a tower of iterated endomorphism rings:

$$B \subseteq A \hookrightarrow E_1 \hookrightarrow E_2 \hookrightarrow \cdots \tag{9}$$

where $E_2 = \operatorname{End} E_A$, and iterate with respect to $\lambda : E_1 \hookrightarrow E_2$ to form E_3, then continuing like this. Note that E_1 is Morita equivalent to B, E_2 is Morita equivalent to A (the details are brought together in [9, Proposition 2.2]), so all E_m's are themselves semisimple algebras. Then if $B \subseteq A$ has inclusion matrix M, $A \hookrightarrow E_1$ has inclusion matrix M^t, and $E_1 \hookrightarrow E_2$ has again inclusion matrix M, and so on in alternating fashion.

Theorem 3.14. *A depth n subalgebra pair of semisimple algebras $B \subseteq A$ is embedded in the depth two subalgebra pair of semisimple algebras $B \hookrightarrow E_{n-2}$. Conversely, the subalgebra B of A has depth n or less if it embeds in a depth two extension $B \hookrightarrow E_{n-2}$.*

Proof. If the inclusion matrix of $B \subseteq A$ is M, then $M^{n+1} \leq qM^{n-1}$ for some $q \in \mathbb{Z}_+$. Since $n \geq 2$, we have $3n - 3 \geq n + 1$. Thus there is $p \in \mathbb{Z}_+$ such that $M^{3n-3} \leq pM^{n-1}$. In other words, by checking odd and even case, this is the same as

$$M^{n-1}(M^{n-1})^t M^{n-1} \leq pM^{n-1},$$

which of course is the depth two condition for the matrix $M^{n-1} = MM^tM \ldots$ ($n-1$ times M and M^t alternately). But M^{n-1} is the inclusion matrix of the composite subalgebra pair $B \hookrightarrow E_{n-2}$.

Conversely, if $B \hookrightarrow E_{n-2}$ has depth two, the zero entries of its inclusion matrix M^{n-1} satisfy $Z(M^{n-1}) = Z(M^{3n-3})$; whence $Z(M^{n-1}) = Z(M^{n+1})$ and B has depth n or less in A by Proposition 2.2. □

Given two irredundant matrices, an $r \times s$-matrix $M = (m_{ij})$ and a $p \times q$-matrix $N = (n_{ij})$, we form the tensor product $M \otimes N$ corresponding to the tensor product of linear mappings between vector spaces. In terms of block matrix representation, $M \otimes N$ is the $rp \times sq$-matrix $(m_{ij}N)$, or equivalently up to permutation (Mn_{ij}). Our interest in determining the depth of $M \otimes N$ knowing the depths of M and of N comes from the following situation in group theory. Given a subgroup $H_1 \subseteq G_1$ with inclusion matrix M and another subgroup $H_2 \subseteq G_2$ with inclusion matrix N, the inclusion matrix of $H_1 \times H_2 \subseteq G_1 \times G_2$ is $M \otimes N$.

Proposition 3.15. *If the irredundant matrices M and N have depth n and m, respectively, then $M \otimes N$ has depth* $\max\{n, m\}$ *or less.*

Proof. Suppose $n \geq m$. Note that $(M \otimes N)^t = M^t \otimes N^t$, so that $(M \otimes N)^m = M^m \otimes N^m$ in the meaning of power of non-square matrices as above. But we are given that $M^{n+1} \leq q_1 M^{n-1}$ for some $q_1 \in \mathbb{Z}_+$. Since N satisfies a depth m matrix inequality and therefore also a depth n matrix inequality, there is $q_2 \in \mathbb{Z}_+$ such that $N^{n+1} \leq q_2 N^{n-1}$. It follows that $(M \otimes N)^{n+1} \leq q_1 q_2 (M \otimes N)^{n-1}$. □

4. Depth two and normality

Let $B \subseteq A$ be an inclusion of complex semisimple algebras. We define B to be a *normal* subring in A if the restriction of every maximal ideal I (in A) to B is A-invariant, meaning that $(I \cap B)A = A(I \cap B)$ as subsets of A. This definition of normal subrings was first given in [19] and used to provide a ring-theoretic setting for Clifford theory. It is also closely related historically to the $HK^+ = K^+H$ condition of normality of a Hopf subalgebra K in a Hopf algebra H.

Let $\hat{A}$ denote the set of maximal two sided ideals of A. Similarly define $\hat{B}$. Any $I \in \hat{A}$ determines up to isomorphism a unique simple A-module denoted by V_I and a minimal (primitive) central idempotent e_I of A. Similarly any $J \in \hat{B}$ determines up to isomorphism a unique simple B-module denoted by W_J and a minimal central idempotent f_J of B.

Proposition 4.1. *W_J is a constituent of $V_I \downarrow_B$ if and only if $I \cap B \subset J$ if and only if $f_J e_I \neq 0$.*

Proof. Suppose that W_J is a constituent of $V_I \downarrow_B$. Then $I \cap B$ annihilates W_J since I annihilates V_I. Thus $I \cap B \subseteq J$.

Suppose that $I \cap B \subseteq J$. Since $f_J \notin J$ we have $f_J \notin I \cap B$, so $f_J \notin I$. Thus $0 \neq (f_J + I)(1 + I) = (f_J + I)(e_I + I) = f_J e_I + I$; in particular, $f_J e_I \neq 0$.

If $f_J e_I \neq 0$ then W_J is a constituent of $V_i \downarrow_B$, as already observed in Section 3. □

Proposition 4.2. *Let $f \in B$ and $e \in A$ be idempotents. Then*

$$f = \sum_{I \in \hat{A},\ fe_I \neq 0} fe_I \quad \text{and} \quad e = \sum_{J \in \hat{B},\ ef_J \neq 0} ef_J.$$

The next proposition makes an improvement on [19, Proposition 2.10].

Proposition 4.3. *Suppose that for any simple A-module V the irreducible constituents of $V \downarrow_B$ form an entire equivalence class of $\approx$. Then $\sim$ is transitive and B is normal in A.*

Proof. Clearly $\sim$ is transitive. Let $I \in \hat{A}$, and suppose that $\alpha \in \mathrm{Irr}(B)$ such that $f_\alpha \in I$, in the notation of Section 3. Then $e_I f_\alpha \neq 0$.

Assume that $e_I f_\beta \neq 0$ for some $\beta \in \mathrm{Irr}(B)$ with $\beta \approx \alpha$. Then β is a constituent of $\chi\downarrow_B$ where χ is the character of V_I. By our hypothesis, α is also a constituent of $\chi\downarrow_B$. Thus $e_I f_\alpha \neq 0$, a contradiction.

This shows that $e_I f_\beta = 0$, i.e. $f_\beta \in I$, for all $\beta \in \mathrm{Irr}(B)$ with $\beta \approx \alpha$. Thus, in the notation of Section 3, we have $m_i \in I$ where $i \in \{1, \ldots, t\}$ is such that $\alpha \in \mathcal{B}_i$. We set $\mathcal{I} := \{i : 1 \leq i \leq t,\ m_i \in I\}$. Then $m := \sum_{i \in \mathcal{I}} m_i \in \mathcal{Z}(A) \cap B$ and

$$I \cap B = \bigoplus_{\alpha \in \mathrm{Irr}(B),\ f_\alpha \in I \cap B} f_\alpha B = \bigoplus_{\alpha \in \mathrm{Irr}(B),\ f_\alpha \in I} f_\alpha B = \bigoplus_{i \in \mathcal{I}} m_i B = mB.$$

Thus $(I \cap B)A = mBA = mA = Am = ABm = A(I \cap B)$. $\square$

Lemma 4.4. *We have $\mathcal{S}M \leq qM$ for some $q \in \mathbb{Z}_+$ if and only if for any simple A-module V the irreducible constituents of $V\downarrow_B$ form an entire equivalence class of $\approx$.*

Remark 4.5. With the previous notations the statement of the lemma can be rephrased as $\mathcal{S}M \leq qM$ for some $q \in \mathbb{Z}_+$ if and only if $\mathcal{V}_u$ coincides with an entire equivalence class of $\sim$ for any simple A-module V_u.

Proof. Suppose that $\mathcal{S}M \leq qM$ for some $q \in \mathbb{Z}_+$. Then $A(\mathcal{S}M) \subset A(M)$. Suppose that $W_i \sim W_j \in \mathcal{V}_u$. Then Proposition 3.4 and the remark after Definition 3.8 imply that $s_{ij} > 0$ and $m_{ju} > 0$. Thus $(\mathcal{S}M)_{iu} > 0$. This means that $(i, u) \in A(\mathcal{S}M) \subset A(M)$. Thus $m_{iu} > 0$, i.e. $W_i \in \mathcal{V}_u$.

Conversely, suppose that $\mathcal{V}_u$ coincides with an entire equivalence class of $\approx$. We need to show $\mathcal{S}M \leq qM$ for some $q \in \mathbb{Z}_+$, or $A(\mathcal{S}M) \subset A(M)$. Let $(i, u) \in A(\mathcal{S}M)$. Then there is $1 \leq l \leq r$ such that $\mathcal{S}_{il} > 0$ and $m_{lu} > 0$. This means that $W_i \sim W_l \in \mathcal{V}_u$. Since $\mathcal{V}_u$ coincides with an entire equivalence class of $\approx$ it follows that $W_i \in \mathcal{V}_u$. Thus $m_{iu} > 0$ and $(i,\ u) \in A(M)$. $\square$

Theorem 4.6. *The inclusion $B \subseteq A$ has depth 2 if and only if B is normal in A.*

Proof. If $B \subseteq A$ is normal then Proposition 2.8 in [19] implies that $\sim$ is an equivalence relation and that for any simple A-module V the irreducible constituents of $V\downarrow_B$ form an entire equivalence class. Lemma 4.4 implies that the inclusion $B \subseteq A$ has depth two.

The converse follows from Lemma 4.4 and Proposition 4.3. $\square$

This theorem provides a quick third proof of [8, main theorem, 5.1]. Recall that a Hopf subalgebra K of a Hopf algebra H is *normal* if $HK^+ = K^+H$ for K^+ the kernel of the counit $\varepsilon : K \to \mathbb{C}$.

Corollary 4.7. *A depth two Hopf subalgebra of a semisimple Hopf algebra is normal.*

Proof. A Hopf subalgebra of a semisimple Hopf algebra is known to be semisimple. If it has depth two, it is normal in the sense of Rieffel. But $H^+ \cap K = K^+$ for the kernel of the counit, the augmentation map $\varepsilon : H \to \mathbb{C}$, the kernel being of course a maximal ideal in H. The restricted ideal in K is then H-invariant, so K is a normal Hopf subalgebra. □

The following corollary puts the last theorem together with Theorem 3.14. Recall that given a subalgebra $B \subseteq A$, the first endomorphism ring is $E_1 = \mathrm{End}\, A_B$, the second $E_2 = \mathrm{End}\, (E_1)_A$, and so forth (with $A = E_0$ and $B = E_{-1}$). We embed E_n in E_{n+1} as before via left multiplication in an endomorphism tower (or Jones tower) above $B \subseteq A$. Put $E_0 = A$.

Corollary 4.8. *Let B be a semisimple subalgebra in a complex semisimple algebra A. Then the subalgebra pair $B \subseteq A$ has depth $n \geq 3$ if and only if B is normal in E_{n-2} (and not normal in E_{n-3}).*

Proof. If B is normal in E_{n-3}, the zero entries $Z(M^{n-2}) = Z(M^n)$, thus the inclusion matrix M would satisfy a depth $n-1$ inequality. □

To summarize, for semisimple algebras we have a depth two subalgebra to be normal in the overalgebra, a depth three subalgebra to be normal in E_1 and a higher depth subalgebra to be normal further along in the endomorphism tower determined by its depth.

5. Inclusions of semisimple Hopf algebras

Let $K \subseteq H$ be an inclusion of semisimple Hopf subalgebras. Let $C(H)$ and $C(K)$ be the character rings of H and K respectively. These are commutative rings if H and K are quasitriangular or cocommutative Hopf algebras such as group algebras.

If M and N are two H-modules with characters χ and μ respectively, then $m_H(M, N) := \dim \mathrm{Hom}_H(M, N)$. The same quantity is also denoted by $m_H(\chi, \mu)$. In this manner one obtains a nondegenerate symmetric bilinear form $m_H(\ ,\ )$ on the character ring $C(H)$ of H. The following result is Proposition 2 of [6]. It shows that the image of the induction map is a two sided ideal in $C(H)$. A different proof that also works in the nonsemisimple case is given below.

Lemma 5.1. *Let K be a Hopf subalgebra of a semisimple Hopf algebra H. Let M be an H-module and V a K-module. Then*

$$M \otimes V \uparrow_K^H = (M \downarrow_K^H \otimes V) \uparrow_K^H \quad \textit{and} \quad V \uparrow_K^H \otimes M = (V \otimes M \downarrow_K^H) \uparrow_K^H$$

Proof. Define

$$\phi : M \otimes (H \otimes_K V) \to H \otimes_K (M \otimes V)$$

by $m \otimes h \otimes_K v \mapsto h_{(2)} \otimes_K (S^{-1}(h_{(1)}) m \otimes v)$. It can be checked that ϕ is a well-defined morphism of H-modules. Moreover ϕ is bijective since $h \otimes_K \otimes m \otimes v \mapsto (h_{(1)} m \otimes h_{(2)} \otimes_K v)$ is its inverse map. The other isomorphism is obtained in a similar way. $\square$

In terms of characters both formulas can be written as $\chi \mathrm{Ind}(\alpha) = \mathrm{Ind}(\mathrm{Res}(\chi)\alpha)$ and $\mathrm{Ind}(\alpha)\chi = \mathrm{Ind}(\alpha \mathrm{Res}(\chi))$, or

$$\chi\alpha \uparrow = (\chi \downarrow \ \alpha) \uparrow \quad \text{and} \quad \alpha \uparrow \chi = (\alpha \ \chi \downarrow) \uparrow . \tag{10}$$

Let $T : C(K) \to C(K)$ be given by $T(\alpha) = \mathrm{Res}(\mathrm{Ind}(\alpha))$. Thus $T(\alpha) = \alpha \uparrow\downarrow$. Note that the matrix of the operator T with respect to the basis of $C(K)$ given by the irreducible characters of K is the matrix $\mathcal{S}$ defined in a previous section.

Lemma 5.2. *With the above notations one has*

$$T(\alpha T(\beta)) = T(\alpha)T(\beta) = T(T(\alpha)\beta) \tag{11}$$

for all $\alpha, \beta \in C(K)$.

Proof. One has

$$T(\alpha T(\beta)) = (\alpha(\beta \uparrow\downarrow)) \uparrow\downarrow = (\alpha((\beta \uparrow) \downarrow) \uparrow) \downarrow = (\alpha \uparrow \beta \uparrow) \downarrow = \alpha \uparrow\downarrow \beta \uparrow\downarrow$$

We have applied relation 10 for the fourth equality and the fact that Res is an algebra map in the last equality. So the first equation in the lemma is proved, and the other is obtained in a similar way. $\square$

For a Hopf algebra H, its counit (or augmentation map) is denoted by ε_H.

Proposition 5.3. *With the above notations one has*
1) $T^n(\varepsilon_K) = T(\varepsilon_K)^n$ *for all* $n \geq 1$.
2) $T^n(\alpha) = T(\alpha)T(\varepsilon_K)^{n-1}$ *for all* $n \geq 1$.

Proof. The first part is a special case of the second one. We prove the second by induction on n. The case $n = 1$ is trivial. Suppose that $T^n(\alpha) = T(\alpha)T(\varepsilon)^{n-1}$ for some n where $\varepsilon = \varepsilon_K$. Then

$$\begin{aligned} T^{n+1}(\alpha) &= T(T^n(\alpha)) = T(T(\alpha)T(\varepsilon)^{n-1}) = T(T(\alpha))T(\varepsilon)^{n-1} \\ &= T(T(\alpha)\varepsilon)T(\varepsilon)^{n-1} = T(\alpha)T(\varepsilon)^n, \end{aligned}$$

by induction and the preceding lemma. $\square$

Lemma 5.4. *Let α and β be different irreducible characters of K. Then $0 < d(\alpha,\beta) \leq m$ if and only if $m_K(\alpha, T^m(\beta)) > 0$.*

Proof. This follows from Proposition 3.4 since as noted before the matrix of the operator T with respect to the basis of $C(K)$ given by the irreducible characters of K is the matrix $\mathcal{S}$ defined in a previous section. ☐

Define $U : C(H) \to C(H)$ such that $U(\chi) = \chi \downarrow_K^H \uparrow_K^H$. It follows from Eq. (10) for $\alpha = \varepsilon_H$ that $U^m(\chi) = \chi(\varepsilon \uparrow_K^H)^m$ for all $m \geq 0$. Recall the equivalence relation u_K^H on $\mathrm{Irr}(H)$ from Section 3. One has $\chi \sim \mu$ if and only if $\chi \downarrow_K^H$ and $\mu \downarrow_K^H$ have a common constituent. Then u_K^H is the equivalence relation obtained by taking the transitive closure of $\sim$.

Remark 5.5. Note that $\chi \sim \mu$ if and only if $m_H(\chi, U(\mu)) > 0$. Inductively it can be shown that $\chi \; u_K^H \; \mu$ if and only if there is $l > 0$ such that $m_H(\chi,\ U^l(\mu)) > 0$.

Proposition 5.6. *A Hopf subalgebra K is normal in H if and only if ε_K by itself forms an equivalence class of d_K^H.*

Proof. From the decomposition of $\mathcal{Z}(H) \cap K$ it follows that the integral element Λ_K is central in H [18]. ☐

Remark 5.7. Suppose that $A = \mathbb{C}\,G$ and $B = \mathbb{C}\,H$ for finite groups $H \subseteq G$. Then $\mathcal{Z}(\mathbb{C}\,G) \cap \mathbb{C}\,H \subset \mathcal{Z}(\mathbb{C}\,N)$ where N is the core of H in G. This follows since a basis for $\mathcal{Z}(\mathbb{C}\,G)$ is given by $\sum_{g \in \mathcal{C}} g$ where $\mathcal{C}$ runs through all conjugacy classes of G.

6. Inclusion of group algebras

Let $H \subseteq G$ be an inclusion of finite groups and let $N := \mathrm{core}_G(H)$ be the core of H in G, i.e., the largest normal subgroup in G contained in H. We will use the short notations u_H^G and d_H^G for the equivalence relations $u_{\mathbb{C}\,H}^{\mathbb{C}\,G}$ and $d_{\mathbb{C}\,H}^{\mathbb{C}\,G}$ defined in an earlier section.

Proposition 6.1. *For all $n \geq 1$ one has that $\ker_G(U^n(\varepsilon_G)) = N$.*

Proof. Note that $U(\varepsilon_G) = \varepsilon_H \uparrow_H^G$. Since ε_G is a constituent of $U(\varepsilon_G)$ it follows by induction on n that $U(\varepsilon_G)$ is a constituent of $U^n(\varepsilon_G)$ and therefore $\ker_G(U^n(\varepsilon_G)) \subseteq \ker_G(U(\varepsilon_G))$. On the other hand, in general, $\ker_G(\chi^n) \supset \ker_G(\chi)$ for any character χ of G. Since $U^n(\varepsilon_G) = (U(\varepsilon_G))^n$ we obtain that $\ker_G(U^n(\varepsilon_G)) = \ker_G(U(\varepsilon_G))$ for all $n \geq 1$. It remains to show $\ker_G(U(\varepsilon_G)) = N$. If $x \in N$ then one has that $x(g \otimes_H 1) = g(g^{-1}xg) \otimes_H 1 = g \otimes_H 1$ since $gxg^{-1} \in N \subseteq H$. Thus $N \subseteq \ker_G(U(\varepsilon_G))$. On the other hand $yg \otimes_H 1 = g \otimes_H 1$ implies that $ygH = gH$ and therefore $y \in gHg^{-1}$. Thus if $y \in \ker_G(U(\varepsilon_G))$ then $y \in \bigcap_{g \in G} gHg^{-1} = N$. ☐

Corollary 6.2. *For all $n \geq 1$ one has that* $\ker_H(T^n(\varepsilon_H)) = N$.

Proof. It follows from the previous proposition since $U^n(\varepsilon_G) \downarrow_H^G = T^n(\varepsilon_H)$. $\square$

Corollary 6.3. *Let $H \subseteq G$ be a group inclusion and N be the core of H in G. Consider the equivalence relation d_H^G on the irreducible characters of H as above. Then the equivalence class of ε_H is* $\mathrm{Irr}(H/N)$.

Proof. From the proof of the previous proposition it follows that $U^n(\varepsilon_G)$ has all irreducible constituents inside those of $U^{n+1}(\varepsilon_G)$ for all $n \geq 0$. Since $N = \ker_G(U(\varepsilon_G))$ it follows from a well known theorem of Brauer (see also Proposition 6.5 below) that $\mathrm{Irr}(H/N)$ coincides with the set of irreducible constituents of all tensor powers $U^n(\varepsilon_G)$ with $n \geq 0$. Therefore there is $m \geq 0$ such that $\mathrm{Irr}(G/N)$ coincides with the set of all irreducible constituents of $U^m(\varepsilon_G)$. If $\alpha \in \mathrm{Irr}(H/N)$ then α is a constituent of $\varepsilon_N \uparrow_N^H$. Therefore $\alpha \uparrow_H^G$ is a constituent of $\varepsilon_N \uparrow_N^G$. But the irreducible constituents $\varepsilon_N \uparrow_N^G$ are exactly those of $U^m(\varepsilon_G)$. Therefore since α is a constituent of $T(\alpha) = \alpha \uparrow_H^G \downarrow_H^G$ it follows that α is a constituent of $U^m(\varepsilon_G) \downarrow_H^G = T^m(\varepsilon_H)$. This implies that $\alpha \approx \varepsilon_H$.

Conversely suppose that $\beta \approx \varepsilon_H$. Then by definition of $\approx$ it follows that β is a constituent of $T^n(\varepsilon_H)$ for some $n \geq 0$. Therefore by Corollary 6.2 its restriction to N contains $\beta(1)$ copies of the trivial character of N. Thus $\beta \in \mathrm{Irr}(H/N)$. $\square$

Until the beginning of Subsection 6.1 we let m be the constant from the proof of previous corollary. Since $U^n(\varepsilon_G) \downarrow_H^G = T^n(\varepsilon_H)$ note that $T^m(\varepsilon_H)$ has all the possible constituents of all powers $T^n(\varepsilon_H)$ with $n \geq 0$.

Remark 6.4. Since $N \trianglelefteq G$ it is well known that $\chi\ u_N^G\ \mu$ if and only if χ and μ have exactly the same irreducible constituents viewed as N-characters by restriction. Thus $\chi\ u_N^G\ \mu$ if and only if $\chi \sim \mu$, i.e. $m_G(\chi, U_N(\mu)) > 0$. It also follows from [7] that $U_N(\chi) = \chi\varepsilon_N \uparrow_N^G$ where $U_N : C(G) \to C(G)$ is given by $U_N(\chi) = \chi \downarrow_N^G \uparrow_N^G$. Thus

$$U_N^2(\chi) = \chi(\varepsilon_N \uparrow_N^G)^2 = \frac{|G|}{|N|}\chi\varepsilon_N \uparrow_N^G = \frac{|G|}{|N|}U_N(\chi).$$

For any irreducible character $\alpha \in \mathrm{Irr}(N)$ the constituents of $\alpha \uparrow_N^G$ form an entire equivalence class under d_N^G. Similarly, for any irreducible character $\chi \in \mathrm{Irr}(G)$ the constituents of $\chi \downarrow_N^G$ form an entire equivalence class under d_N^G. Thus, by Clifford theory, the equivalence classes of $\mathrm{Irr}(N)$ under d_N^G are just the G-orbits on $\mathrm{Irr}(N)$ (under the conjugation action).

The following result follows from a well known result of Brauer and will be used in the sequel. For a proof in the context of Hopf algebras see [7].

Proposition 6.5. *Let G be a group, χ a character of G and $N = \ker_G(\chi)$. Then ε_N^G has as irreducible constituents all the possible irreducible constituents of all the powers of χ.*

Corollary 6.6. *The equivalence relation u_H^G is the same as the equivalence relation u_N^G coming from $N \trianglelefteq G$. Thus the equivalence classes of $\mathrm{Irr}(G)$ under u_H^G are in natural bijection with the G-orbits on $\mathrm{Irr}(N)$.*

Proof. Write as above $\chi \sim \mu$ if and only if $\chi \downarrow_H^G$ and $\mu \downarrow_H^G$ have a common constituent. If $\chi \sim \mu$ then clearly $\chi\ u_N^G\ \mu$. This implies that if $\chi\ u_H^G\ \mu$ then $\chi\ u_N^G\ \mu$. Conversely by Remark 5.5 we see that $\chi\ u_N^G\ \mu$ if and only if $m_G(\chi, U_N(\mu)) > 0$. Remark 6.4 implies that $U_N(\mu) = \mu\varepsilon_N \uparrow_N^G$. On the other hand, using the previous proposition it follows that $m_G(\chi,\ U_N(\mu)) = m_G(\chi, \mu\varepsilon_N^G) = m_G(\varepsilon_N \uparrow_N^G,\ \mu\chi^*) > 0$ if and only if $m_G(U^m(\varepsilon_H),\ \chi\mu^*) = m_G(\chi, U^m(\mu)) > 0$. □

Corollary 6.7. *One has that the relation $\sim$ on $\mathrm{Irr}(G)$ coming from the inclusion $H \subseteq G$ is an equivalence relation if and only if $\varepsilon_N \uparrow_N^G$ and $\varepsilon_H \uparrow_H^G$ have the same constituents.*

Proof. $\sim$ is an equivalence relation if and only if $\varepsilon_H \uparrow_H^G$ and $(\varepsilon_H \uparrow_H^G)^m$ have the same constituents. But $(\varepsilon_H \uparrow_H^G)^m$ has the same constituents as $\varepsilon_N \uparrow_N^G$. □

Proposition 6.8. *Let $N \subseteq H \subseteq G$ with $N \trianglelefteq G$. The depth of H/N inside G/N is less than or equal to the depth of H in G. If H has depth three or less in G then H/N has depth three or less in G/N.*

Proof. Let $\bar{T} : C(H/N) \to C(H/N)$ be the operator T defined as above but for the inclusion $H/N \subseteq G/N$. Since $Rep(G/N) \subseteq Rep(G)$ and $Rep(H/N) \subseteq Rep(H)$ it is easy to check that $\bar{T}$ is the restriction of T to $C(H/N)$. Indeed both restriction and induction for the inclusion $H/N \subseteq G/N$ come from the restriction and induction for the inclusion $H \subseteq G$.

Then the proposition follows from Theorem 3.6 and Theorem 3.10. □

For example, with $G = S_4$, $H = D_8$ and

$$N = \{(1), (12)(34), (13)(24), (14)(23)\} = V_4,$$

the depth of $H < G$ is four (computed in Section 2), while the depth of $H/N \cong S_2 < G/N \cong S_3$ is three (computed graphically in Section 3).

Again let G be a finite group, H a proper subgroup of G, and $N := \mathrm{core}_G(H)$ denote the core of H in G. We say $\chi \in \mathrm{Irr}(G)$ and $\psi \in \mathrm{Irr}(H)$ are *linked* if

$$0 \neq \langle \psi \uparrow^G |\chi\rangle_G = \langle \psi|\chi \downarrow_H\rangle_H. \tag{12}$$

This defines a bipartite graph Γ with vertices $\mathrm{Irr}(G) \cup \mathrm{Irr}(H)$ (the inclusion diagram of the corresponding group algebras is a weighted multigraph variant of this). As usual, we denote by $\mathrm{Irr}(G|\kappa)$ the set of all $\chi \in \mathrm{Irr}(G)$ such that $\langle \chi \downarrow_N, \kappa \rangle \neq 0$, for $\kappa \in \mathrm{Irr}(N)$.

Proposition 6.6 implies that the connected components of Γ are in bijection with the orbits of G on $\mathrm{Irr}(N)$.

6.1. A theorem with examples. Recall that the *core* $\mathrm{core}_G(H)$ of a subgroup $H < G$ is the largest normal subgroup of G contained in H. It is also defined by $\mathrm{core}_G(H) = \bigcap_{x \in G} {}^xH$ where ${}^xH$ denotes the subgroup xHx^{-1} conjugate to H.

Theorem 6.9. *Let $H \subseteq G$ be an inclusion of finite groups, and suppose that $N := \mathrm{core}_G(H)$ is the intersection of m conjugates of H. Then H has depth $\leq 2m$ in G. Moreover, if $N \subseteq \mathcal{Z}(G)$ then H has depth $\leq 2m-1$ in G.*

Proof. Let $\alpha \in \mathrm{Irr}(H)$, and let $x \in G$. Then Mackey decomposition shows that $\mathrm{Ind}^H_{H \cap xHx^{-1}}(\mathrm{Res}^{xHx^{-1}}_{H \cap xHx^{-1}}({}^x\alpha))$ is a summand of $T(\alpha) = \mathrm{Res}^G_H(\mathrm{Ind}^G_H(\alpha))$. Thus $\mathrm{Res}^G_H(\mathrm{Ind}^G_{H \cap xHx^{-1}}(\mathrm{Res}^{xHx^{-1}}_{H \cap xHx^{-1}}({}^x\alpha)))$ is a summand of

$$T^2(\alpha) = \mathrm{Res}^G_H(\mathrm{Ind}^G_H(T(\alpha))).$$

Let $y \in G$. Then, by Mackey decomposition again,

$$\mathrm{Ind}^H_{H \cap yHy^{-1} \cap yxHx^{-1}y^{-1}}(\mathrm{Res}^{yxHx^{-1}y^{-1}}_{H \cap yHy^{-1} \cap yxHx^{-1}y^{-1}}({}^{yx}\alpha))$$

is a summand of $T^2(\alpha)$. Continuing in this fashion, we see that, for $x_1 := 1, x_2, \ldots, x_m \in G$,

$$\mathrm{Ind}^H_{x_1Hx_1^{-1} \cap \ldots \cap x_mHx_m^{-1}}(\mathrm{Res}^{x_mHx_m^{-1}}_{x_1Hx_1^{-1} \cap \ldots \cap x_mHx_m^{-1}}({}^{x_m}\alpha))$$

is a summand of $T^{m-1}(\alpha)$. We can choose $x_1 = 1, x_2, \ldots, x_m =: z$ in such a way that $x_1Hx_1^{-1} \cap \ldots \cap x_mHx_m^{-1} = N$. Then $T^{m-1}(\alpha)$ has a summand of the form

$$\mathrm{Ind}^H_N(\mathrm{Res}^{zHz^{-1}}_N({}^z\alpha)) = \mathrm{Ind}^H_N({}^z\mathrm{Res}^H_N(\alpha)).$$

Let β be an irreducible constituent of $\mathrm{Res}^H_N(\alpha)$. Then $\mathrm{Ind}^G_N({}^z\beta) = \mathrm{Ind}^G_N(\beta)$ is a summand of $\mathrm{Ind}^G_H(T^{m-1}(\alpha))$. But the irreducible constituents of $\mathrm{Ind}^G_N(\beta)$ form a complete equivalence class of $\mathrm{Irr}(G)$ under u^G_H, by Corollary 6.6. Thus α has graphical distance at most $2m-1$ to any $\chi \in \mathrm{Irr}(G)$. So α has graphical distance at most $2m-2$ to any set of irreducible constituents of $\mathrm{Res}^G_H(\chi)$, for any $\chi \in \mathrm{Irr}(G)$, and the first part is proved.

Now suppose that $N \subseteq \mathcal{Z}(G)$. Then, in the notation above, $\mathrm{Ind}^H_N({}^z\beta) = \mathrm{Ind}^H_N(\beta)$ is a summand of $T^{m-1}(\alpha)$. But now the irreducible constituents of $\mathrm{Ind}^H_N(\beta)$ form a complete equivalence class of $\mathrm{Irr}(H)$ under d^G_H. This shows that any two irreducible characters of H have graphical distance at most $2m-2$, so that H has depth at most $2m-1$ in G. □

We illustrate the theorem with three examples.

Example 6.10. (1) Let $G = S_4$ and $H = D_8$, so that $N = V_4$ is the intersection of $m = 2$ conjugates of H. By the theorem, D_8 has depth ≤ 4 in S_4; indeed the depth is four by our earlier computations. In the appendix, the depth of D_{2n} in S_n is shown to be three for $n > 5$.

(2) Let $G = S_{n+1}$ and $H = S_n$ for some n. Then $N = 1$, which is the intersection of $m = n$ conjugates of H:

$$\{1\} = S_n \cap S_n^{(1\,n+1)} \cap \cdots \cap S_n^{(n-1\,n+1)}.$$

By the theorem, S_n has depth at most $2n-1$ in S_{n+1}. We will see later that $2n-1$ is precisely the depth of $S_n < S_{n+1}$.

(3) Let $G = A_6$ and $H = A_5$, so that $N = 1$ again. A computation with character tables shows that A_5 has depth 5 in A_6. However, in this case, N is *not* the intersection of 3 conjugates of H, so the bound in the theorem is not sharp here. The depth of the inclusion of alternating groups $A_n \subseteq A_{n+1}$ will be computed in the appendix.

We obtain a corollary by recalling that G acts on the set of subgroups of G by conjugation. Let $N_G(H)$ be the normalizer of H in G, which is the stabilizer subgroup of H under conjugation. The proof is a simple application of the orbit counting theorem:

Corollary 6.11. *The depth of a subgroup H of a finite group G is bounded above by $2[G : N_G(H)]$.*

Since $N_G(H)$ contains each subgroup K in which H is normal, it follows that a subnormal subgroup H of subnormal depth in G (or defect) r (cf. [11]) has depth less than or equal to $2m^{r-1}$, where m is the maximal index of two consecutive subgroups in a subnormal series.

The following are examples of depth three or more subgroups from the literature on group theory.

Example 6.12. Brodkey's theorem (cf. Theorem 1.37 in [11]) states that if a finite group G has an abelian Sylow p-subgroup H, then the largest normal p-subgroup $\mathcal{O}_p(G) = N$ of G is the intersection of two conjugates of H. In other terms then, H is a depth four or less subgroup in G; depth three or less if $N = \{1_G\}$.

Example 6.13. If G is p-solvable, where p is odd and not a Mersenne prime, then the largest normal p-subgroup N of G is an intersection of two Sylow p-subgroups. If p is even or a Mersenne prime, then N is an intersection of three Sylow p-subgroups [5]. This in our terms implies that the Sylow p-subgroup has depth $\leq$ 4 or 6, respectively. If $N = 1$ then these numbers can be improved to 3 and 5, respectively.

Example 6.14. The theorem above implies that a subgroup H of a finite group G has at most depth three if $H \cap xHx^{-1} = 1$ for some $x \in G$. For example, a Sylow p-subgroup of $\mathrm{GL}(n, q)$ has depth three, as well as certain Borel and Weyl subgroups (for specific values of n and $q = p^r$, left as an exercise to the interested reader) [1].

The results of this paper are suited for creating a program using GAP to calculate the depth of subgroups of suitably small groups. We thank Susanne Danz for implementing such a program at the University of Jena.

In this paper we have found subgroups of depth at each odd number (the symmetric group series), at depth four (the dihedral group in S_4 with some additional examples) and a search with this program yields a subgroup of depth 6 (the 108-element normalizer subgroup of the Sylow 3-subgroup of the 432-element affine group $AGL(3, 2)$). We found no subgroups of depth an even number greater than 6.

Remark 6.15. Suppose $K < H < G$ is a tower of finite groups, where the subgroup $H < G$ is corefree and m conjugates of H have trivial intersection. Then the depth of the subgroup $K < G$ is bounded above by $2m - 1$. This follows from the same theorem since K satisfies the same core hypothesis. For example, by the results of one of the examples above, any subgroup K of S_n has depth less than or equal to $2n - 1$ in S_{n+1}.

6.2. Computations for the operator T. Suppose that H is a subgroup of a finite group G. We denote by $\mathrm{Cl}(G)$ the set of conjugacy classes of G and by $\mathrm{CF}(G)$ the ring of complex class functions on G. For a union X of conjugacy classes of G, we denote by $\gamma_{G,X}$ the characteristic function of X in $\mathrm{CF}(G)$. Then

$$\gamma_{G,X} \downarrow_H = \gamma_{H,H\cap X}.$$

Similarly, if C and D denote the conjugacy classes in G and H, respectively, of an element in H then an easy computation shows that

$$\gamma_{H,D} \uparrow^G = \frac{|G|}{|H|} \cdot \frac{|D|}{|C|} \gamma_{G,C}.$$

This implies that the eigenvectors of the linear map

$$T : \mathrm{CF}(H) \longrightarrow \mathrm{CF}(H), \quad \chi \longmapsto \chi\uparrow^G\downarrow_H,$$

corresponding to nonzero eigenvalues are precisely the class functions $\gamma_{H,C\cap H}$ ($C \in \mathrm{Cl}(G)$, $C \cap H \neq \emptyset$). Moreover, the eigenvalue of T corresponding to an eigenvector $\gamma_{C,C\cap H}$ is clearly

$$\frac{|G|}{|H|} \cdot \frac{|C \cap H|}{|C|}.$$

We denote by

$$t := |\{\frac{|G|}{|H|} \cdot \frac{|C \cap H|}{|C|} : C \in \mathrm{Cl}(G),\ C \cap H \neq \emptyset\}|$$

the number of distinct nonzero eigenvalues of T. Then the minimum polynomial of T has degree t or $t+1$. Since $\mathcal{S}$ is the matrix of T with respect to the basis $\mathrm{Irr}(H)$ of $\mathrm{CF}(H)$, Corollary 2.3 implies that H has depth $2t+1$ or less in G.

We also note that all eigenvalues of T are nonzero if and only if T is surjective. This is equivalent to the condition that two elements in H are conjugate in G if and only if they are already conjugate in H. In this case, the minimum polynomial of T has degree t. So, arguing as above, we conclude that H has depth $2t-1$ or less in G. We summarize:

Theorem 6.16. *(i) The nonzero eigenvalues of $\mathcal{S}$ are the numbers*

$$\frac{|G|}{|H|} \cdot \frac{|C \cap H|}{|C|} \qquad (C \in \mathrm{Cl}(G),\ C \cap H \neq \emptyset).$$

(ii) The subgroup H of G has depth $\leq 2t+1$ in G where t denotes the number of distinct nonzero eigenvalues of $\mathcal{S}$.

(iii) All eigenvalues of $\mathcal{S}$ are nonzero if and only if any two elements in H which are conjugate in G are already conjugate in H. In this case, H has depth $\leq 2t-1$ in G.

Example 6.17. For the inclusion of the alternating groups $A_4 < A_5$ it may be checked that the minimum polynomial of $\mathcal{S}$ is $X(X-1)(X-2)(X-5)$. By the theorem above, A_4 has depth ≤ 7 in A_5. Computing the powers of M, one sees that the subgroup $A_4 < A_5$ has depth five. The depth of the inclusion $A_n \subseteq A_{n+1}$ for arbitrary n will be computed in the appendix.

Example 6.18. Consider the inclusion $S_3 < S_4$ of symmetric groups. It can be computed that the minimal polynomial $\mathcal{S}$ in this case is given by $m(X) = X^3 - 7X^2 + 14X - 8 = (X-4)(X-2)(X-1)$. The nonzero eigenvalues of $\mathcal{S}$ are $1, 2, 4$. By the theorem above, the depth of $S_3 < S_4$ is at most five, which is the precise depth of the extension as we will see in the next subsection.

6.3. Depth of inclusions of symmetric groups. In this subsection we will prove the following:

Theorem 6.19. *For any $n \geq 2$ the standard inclusion $S_n \subseteq S_{n+1}$ has depth $2n-1$.*

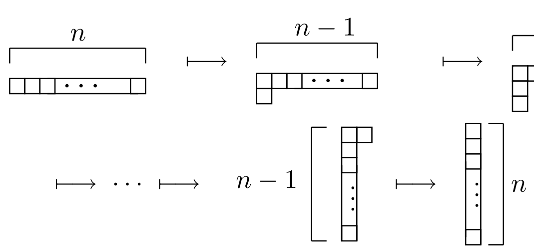
In order to prove the theorem, we recall that the irreducible characters of S_n are in bijection with partitions of n. Moreover, partitions of n can be visualized by their Young diagrams. For example, the trivial character of S_n corresponds to the trivial partition (n) of n, and the Young diagram of (n) is a row of n boxes. Similarly, the sign character of S_n corresponds to the partition $(1^n) = (1, \ldots, 1)$, and the Young diagram of (1^n) is a column of n boxes.

By the branching rules, restricting an irreducible character of S_{n+1} to S_n means removing a box from the corresponding Young diagram, and inducing an irreducible character of S_n to S_{n+1} means adding a box to the corresponding Young diagram.

By Theorem 6.9 above, the inclusion $S_n \subseteq S_{n+1}$ has depth $\leq 2n-1$. It is easy to give an alternative proof of this, based on the combinatorics of Young diagrams. These ideas are explained in more detail in the appendix where they are also used to determine the depth of the inclusion of alternating groups $A_n \subseteq A_{n+1}$.

It only remains to show that the inclusion matrix of $S_n \subseteq S_{n+1}$ does not satisfy a depth $2n-2$ inequality. For this we may argue as follows:

The sign character of S_{n+1}, denoted by V_u, restricts to the sign character σ of S_n. Thus, in the notation of Section 3, the set $\mathcal{V}_u$ consists of σ alone. It is easy to see that $d(\varepsilon, \sigma) = n-1$:



It follows that $m(V_u) = n-1$. Thus Theorem 3.10 shows that the inclusion matrix of S_n in S_{n+1} cannot satisfy a depth $2n-2$ inequality.

This result also applies to the semisimple Hecke algebras: $H(q,n)$ is depth $2n-1$ in $H(q,n+1)$, since they share the same representation theory with the permutation groups $S_n < S_{n+1}$ (see [10]).

Appendix: Depth of subgroups – some examples

Susanne Danz [1] and Burkhard Külshammer

Appendix A. Inclusions of symmetric and alternating groups

Throughout this section, let $n \geq 1$, let $\mathfrak{S}_n$ denote the symmetric group of degree n, and let $\mathfrak{A}_n$ denote the alternating group of degree n. Moreover, let $\mathcal{P}_n$ be the set of all partitions of n. By Theorem 6.19, we know that for $n \geq 2$ the ring extension $\mathbb{C}\mathfrak{S}_n \subseteq \mathbb{C}\mathfrak{S}_{n+1}$ is of depth $2n-1$. We now aim to determine the depth of the ring extension $\mathbb{C}\mathfrak{A}_n \subseteq \mathbb{C}\mathfrak{A}_{n+1}$. Moreover, we will give a combinatorial proof of Theorem 6.19. Before stating the results, we fix some further notation.

Remark A.1. (a) For $\lambda \in \mathcal{P}_n$, we denote the conjugate partition by λ'. That is, the Young diagram of λ' is obtained by transposing the Young diagram of λ. For $\lambda \in \mathcal{P}_n$, let χ^λ be the corresponding ordinary irreducible $\mathfrak{S}_n$-character. If $\lambda = \lambda'$ then $\chi^\lambda \downarrow_{\mathfrak{A}_n} = \chi^\lambda_+ + \chi^\lambda_-$, for irreducible $\mathfrak{A}_n$-characters $\chi^\lambda_+ \neq \chi^\lambda_-$. We choose our labelling in accordance with [12], Sec. 2.5. With this convention, for $\alpha' = \alpha \in \mathcal{P}_{n+1}$ and $\lambda' = \lambda \in \mathcal{P}_n$ such that $\langle \chi^\alpha \downarrow_{\mathfrak{S}_n}, \chi^\lambda \rangle \neq 0$, we have $\langle \chi^\alpha_+ \downarrow_{\mathfrak{A}_n}, \chi^\lambda_+ \rangle \neq 0 = \langle \chi^\alpha_+ \downarrow_{\mathfrak{A}_n}, \chi^\lambda_- \rangle$ and $\langle \chi^\alpha_- \downarrow_{\mathfrak{A}_n}, \chi^\lambda_- \rangle \neq 0 = \langle \chi^\alpha_- \downarrow_{\mathfrak{A}_n}, \chi^\lambda_+ \rangle$ (see [12], Theorem 2.5.13, and [2]). If $\lambda \neq \lambda'$ then $\chi^\lambda \downarrow_{\mathfrak{A}_n} = \chi^{\lambda'} \downarrow_{\mathfrak{A}_n}$ is irreducible. We may then suppose that $\lambda > \lambda'$, and set $\chi^\lambda_0 := \chi^\lambda \downarrow_{\mathfrak{A}_n}$. Here "$\geq$" denotes the usual lexicographic ordering on partitions.

(b) We consider the bipartite graphs $\Gamma(\mathfrak{S}_n)$ and $\Gamma(\mathfrak{A}_n)$. Here $\Gamma(\mathfrak{S}_n)$ has vertices $V := \mathcal{P}_n \cup \mathcal{P}_{n+1}$ and edges

$$E := \{(\alpha, \lambda) \in \mathcal{P}_{n+1} \times \mathcal{P}_n \mid \langle \chi^\alpha \downarrow_{\mathfrak{S}_n}, \chi^\lambda \rangle \neq 0\}.$$

The graph $\Gamma(\mathfrak{A}_n)$ has vertices $\widetilde{V} := V(n) \cup V(n+1)$ and edges $\widetilde{E}$ where

$$V(n) := \{[\lambda, 0] \mid \lambda \in \mathcal{P}_n,\ \lambda > \lambda'\} \cup \{[\lambda, +], [\lambda, -] \mid \lambda = \lambda' \in \mathcal{P}_n\},$$

$$V(n+1) := \{[\alpha, 0] \mid \alpha \in \mathcal{P}_{n+1},\ \alpha > \alpha'\} \cup \{[\alpha, +], [\alpha, -] \mid \alpha = \alpha' \in \mathcal{P}_{n+1}\},$$

$$\widetilde{E} := \{([\alpha, x], [\lambda, y]) \in V(n+1) \times V(n) \mid \langle \chi^\alpha_x \downarrow_{\mathfrak{A}_n}, \chi^\lambda_y \rangle \neq 0\}.$$

(c) Let $\lambda, \mu \in \mathcal{P}_n$ with corresponding Young diagrams $[\lambda]$ and $[\mu]$, respectively. We set

$$d(\lambda, \mu) := |[\lambda] \setminus [\mu]| + |[\mu] \setminus [\lambda]| = 2(n - |[\lambda] \cap [\mu]|).$$

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With this notation, we have:

Proposition A.2. *Let $n \geq 2$, and let λ, $\mu \in \mathcal{P}_n$. Then $d(\lambda,\mu)$ is the length of a shortest path from λ to μ in $\Gamma(\mathfrak{S}_n)$. In particular, the ring extension $\mathbb{C}\mathfrak{S}_n \subseteq \mathbb{C}\mathfrak{S}_{n+1}$ has depth $2n-1$.*

Proof. Let λ, $\mu \in \mathcal{P}_n$, and set $2m := d(\lambda,\mu)$. We argue with induction on m, in order to show that in $\Gamma(\mathfrak{S}_n)$ there is a path of length $2m$ from λ to μ. For $m = 0$ this is trivially true, so we may now suppose that $m \geq 1$. We construct a partition λ^1 of n such that $d(\lambda,\lambda^1) = 2$, $d(\lambda^1,\mu) = 2m-2$, and such that there is a path of length 2 from λ to λ^1. Since $\lambda \neq \mu$, we have $[\lambda] \not\subseteq [\mu] \not\subseteq [\lambda]$. Thus there is some $i \in \mathbb{N}$ such that $(i,\lambda_i) \in [\lambda] \setminus [\mu]$ and $(i+1,\lambda_i) \notin [\lambda]$. That is, (i,λ_i) is a removable node of $[\lambda]$. Analogously, there are some $r, s \in \mathbb{N}$ such that $(r,s) \in [\mu] \setminus [\lambda]$. We may suppose further that $(t,s) \in [\lambda]$, for $1 \leq t \leq r-1$, and $(r,u) \in [\lambda]$, for $1 \leq u \leq s-1$. So (r,s) is an addable node of $[\lambda]$. We define $\alpha \in \mathcal{P}_{n+1}$ with Young diagram $[\alpha] := [\lambda] \cup \{(r,s)\}$. Assume that (i,λ_i) is not a removable node of $[\alpha]$. This can happen only if $(r,s) = (i+1,\lambda_i)$ or $(r,s) = (i,\lambda_i+1)$. But, since $(r,s) \in [\mu]$, this implies also $(i,\lambda_i) \in [\mu]$, a contradiction. Therefore, $[\lambda^1] := [\alpha] \setminus \{(i,\lambda_i)\}$ is the Young diagram of a partition $\lambda^1 \in \mathcal{P}_n$ with $d(\lambda,\lambda^1) = 2$, $d(\lambda^1,\mu) = 2m-2$, and $\langle \chi^\alpha \downarrow_{\mathfrak{S}_n}, \chi^\lambda \rangle \neq 0 \neq \langle \chi^\alpha \downarrow_{\mathfrak{S}_n}, \chi^{\lambda^1} \rangle$. So there is a path of length 2 from λ to λ^1 in $\Gamma(\mathfrak{S}_n)$. By induction, there is a path of length $2m-2$ from λ^1 to μ in $\Gamma(\mathfrak{S}_n)$. So we obtain a path

$$\begin{tikzcd}
\alpha^1 \arrow[d, dash] \arrow[dr, dash] & \alpha^2 \arrow[d, dash] \arrow[dr, dash] & & \cdots & \alpha^m \arrow[d, dash] \arrow[dr, dash] & \\
\lambda = \lambda^0 & \lambda^1 & \lambda^2 & \cdots & \lambda^{m-1} & \lambda^m = \mu
\end{tikzcd}$$

of length $2m$ from λ to μ in $\Gamma(\mathfrak{S}_n)$. Conversely, let

$$\begin{tikzcd}
\beta^1 \arrow[d, dash] \arrow[dr, dash] & \beta^2 \arrow[d, dash] \arrow[dr, dash] & & \cdots & \beta^r \arrow[d, dash] \arrow[dr, dash] & \\
\lambda = \mu^0 & \mu^1 & \mu^2 & \cdots & \mu^{r-1} & \mu^r = \mu
\end{tikzcd}$$

be a shortest path in $\Gamma(\mathfrak{S}_n)$. That is, $2r \leq d(\lambda,\mu)$. We argue with induction on r to show $d(\lambda,\mu) = 2r$. If $r = 0$ then $\lambda = \mu$, and $d(\lambda,\mu) = 0 = 2r$. Next suppose that $r = 1$. Then $\lambda \neq \mu$, and $[\mu]$ is obtained by first adding a node (i,j) to $[\lambda]$, and then removing a node $(r,s) \neq (i,j)$ from $[\lambda] \cup \{(i,j)\} = [\beta^1]$. That is $d(\lambda,\mu) = 2$. Now we may suppose that $r \geq 2$. By induction, $d(\lambda,\mu^{r-1}) = 2(r-1)$ and $d(\mu^{r-1},\mu) = 2$. So $d(\lambda,\mu) \leq 2r$, and thus $2r = d(\lambda,\mu)$. This proves the first part of the statement.

Since, for any $\lambda, \mu \in \mathcal{P}_n$, we have $(1,1) \in [\lambda] \cap [\mu]$, it follows that $d(\lambda,\mu) \leq 2(n-1)$. Moreover, for $\lambda := (n)$ and $\mu := (1^n)$, we have $d(\lambda,\mu) = 2(n-1)$, so that, by what we have just shown and Theorem 3.6, the extension $\mathbb{C}\mathfrak{S}_n \subseteq \mathbb{C}\mathfrak{S}_{n+1}$ has depth $2(n-1)+1$. On the other hand, $\chi^{(n)} = \chi^{(n+1)} \downarrow_{\mathfrak{S}_n}$. Therefore, for $n > 2$, Theorem 3.10 implies that the inclusion matrix of $\mathbb{C}\mathfrak{S}_n \subseteq \mathbb{C}\mathfrak{S}_{n+1}$ cannot satisfy a depth $2(n-1)$ inequality. If $n = 2$ then $\mathbb{C}\mathfrak{S}_n \subseteq \mathbb{C}\mathfrak{S}_{n+1}$ cannot have depth $2(n-1) = 2$, since $\mathfrak{S}_2 \ntrianglelefteq \mathfrak{S}_3$. This completes the proof of the proposition. □

Lemma A.3. *Let $\lambda, \mu \in \mathcal{P}_n$. Suppose that $\mu' = \mu \neq \lambda = \lambda'$. Then $d(\lambda,\mu) \geq 4$.*

Proof. Let $\mu' = \mu \neq \lambda = \lambda'$. Then there are some $i, j \in \{1, \dots, n\}$ such that $(i, \lambda_i) \notin [\mu]$ and $(j, \mu_j) \notin [\lambda]$. Since both λ and μ are symmetric, also $(\lambda_i, i) \in [\lambda]\setminus[\mu]$ and $(\mu_j, j) \in [\mu] \setminus [\lambda]$. In particular, $i \neq \lambda_i$ or $j \neq \mu_j$, and also $i \neq j$. Hence $d(\lambda,\mu) \geq 3$, and so $d(\lambda,\mu) \geq 4$, since $d(\lambda,\mu)$ is even. □

Example A.4. Consider, for instance, the symmetric partitions $\lambda = (4,3,2,1)$ and $\mu = (5,2,1^3)$ of 10. Then we have $d(\lambda,\mu) = 2(10-4-2-1-1) = 4$.

Proposition A.5. *Let $n \geq 3$. Then the ring extension $\mathbb{C}\mathfrak{A}_n \subseteq \mathbb{C}\mathfrak{A}_{n+1}$ has depth $2(n - \lceil\sqrt{n}\rceil) + 1$.*

Proof. In consequence of Theorem 3.6 and Theorem 3.10, it suffices to show the following:

(1) For any $v, w \in V(n)$, there is a path of length at most $2(n - \lceil\sqrt{n}\rceil)$ from v to w in $\Gamma(\mathfrak{A}_n)$, and
(2) there is some $v \in V(n)$ such that in $\Gamma(\mathfrak{A}_n)$ there is no path of length less than $2(n - \lceil\sqrt{n}\rceil)$ from v to $[(n), 0]$.

For this, let $v, w \in V(n)$. Suppose first that $\lambda = \lambda' \in \mathcal{P}_n$ and that $v := [\lambda, +]$ and $w := [\lambda, -]$. Let further $\alpha \in \mathcal{P}_{n+1}$ with Young diagram $[\alpha] := [\lambda] \cup \{(1, \lambda_1 + 1)\}$. Then $\alpha > \alpha'$, and $\langle \chi_0^\alpha \downarrow_{\mathfrak{A}_n}, \chi_+^\lambda \rangle = 1 = \langle \chi_0^\alpha \downarrow_{\mathfrak{A}_n}, \chi_-^\lambda \rangle$, by [2]. Hence in $\Gamma(\mathfrak{A}_n)$ there is a path of length $2 \leq 2(n - \lceil\sqrt{n}\rceil)$ from v to w.

Therefore, from now on, we may suppose that $v = [\lambda, x]$ and $w = [\mu, y]$, for some $\lambda \geq \lambda'$ and $\mu \geq \mu'$ with $\lambda \neq \mu$, and appropriate $x, y \in \{0, +, -\}$. We set $2m := d(\lambda,\mu)$, and show that there is a path from v to w in $\Gamma(\mathfrak{A}_n)$ of length $2m$. Note that, since $\lambda \geq \lambda'$ and $\mu \geq \mu'$, we must have $\lambda_1 \geq \lceil\sqrt{n}\rceil$ and also $\mu_1 \geq \lceil\sqrt{n}\rceil$. So $2m \leq 2(n - \lceil\sqrt{n}\rceil)$, and we then get (1).

First of all, by Proposition A.2, there is a path

$$\begin{tikzcd}
\alpha^1 \arrow[d, dash] \arrow[dr, dash] & \alpha^2 \arrow[d, dash] \arrow[dr, dash] & & \cdots & \alpha^m \arrow[d, dash] \arrow[dr, dash] & \\
\lambda = \lambda^0 & \lambda^1 & \lambda^2 & \cdots & \lambda^{m-1} & \lambda^m = \mu
\end{tikzcd}$$

of length $2m$ in $\Gamma(\mathfrak{S}_n)$. Here $\lambda^0, \dots, \lambda^m \in \mathcal{P}_n$, and $\alpha^1, \dots, \alpha^m \in \mathcal{P}_{n+1}$. We now construct a path

$$\begin{tikzcd}
{[\tilde{\alpha}^1, z_1]} \arrow[d, dash] \arrow[dr, dash] & & \cdots & {[\tilde{\alpha}^m, z_m]} \arrow[d, dash] \arrow[dr, dash] & \\
\begin{array}{c}[\lambda, x] \\ = [\tilde{\lambda}^0, x_0]\end{array} & {[\tilde{\lambda}^1, x_1]} & \cdots & {[\tilde{\lambda}^{m-1}, x_{m-1}]} & \begin{array}{c}[\tilde{\lambda}^m, x_m] \\ = [\mu, y]\end{array}
\end{tikzcd}$$

in $\Gamma(\mathfrak{A}_n)$ as follows. For $i = 1, \dots, m$, we set $\tilde{\lambda}^i := \max\{\lambda^i, (\lambda^i)'\}$ and $\tilde{\alpha}^i := \max\{\alpha^i, (\alpha^i)'\}$ where the maxima are taken with respect to the lexicographic ordering on partitions. We then determine the "signs" $x_0, \dots, x_m, z_1, \dots, z_m$ inductively. Of course, $x_0 = x$. So we may suppose that $i \geq 1$ and that we have already determined $x_0, \dots, x_{i-1}$ and $z_1, \dots, z_{i-1}$. In order to fix z_i and x_i, we distinguish four cases.

Case 1. $\tilde{\lambda}^{i-1} \neq (\tilde{\lambda}^{i-1})'$ and $\tilde{\alpha}^i \neq (\tilde{\alpha}^i)'$. In this case we set $z_i := 0$ and

$$x_i := \begin{cases} +, & \text{if } \tilde{\lambda}^i = (\tilde{\lambda}^i)' \text{ and } y \in \{0, +\}, \\ -, & \text{if } \tilde{\lambda}^i = (\tilde{\lambda}^i)' \text{ and } y = -, \\ 0, & \text{if } \tilde{\lambda}^i \neq (\tilde{\lambda}^i)'. \end{cases}$$

Case 2. $\tilde{\lambda}^{i-1} \neq (\tilde{\lambda}^{i-1})'$ and $\tilde{\alpha}^i = (\tilde{\alpha}^i)'$. If $\tilde{\lambda}^i \neq (\tilde{\lambda}^i)'$ then we set $z_i := +$ and $x_i := 0$, otherwise

$$z_i := x_i := \begin{cases} +, & \text{if } y \in \{0, +\}, \\ -, & \text{if } y = -. \end{cases}$$

Case 3. $\tilde{\lambda}^{i-1} = (\tilde{\lambda}^{i-1})'$ and $\tilde{\alpha}^i \neq (\tilde{\alpha}^i)'$. We then set $z_i := 0$ and

$$x_i := \begin{cases} +, & \text{if } \tilde{\lambda}^i = (\tilde{\lambda}^i)' \text{ and } y \in \{0, +\}, \\ -, & \text{if } \tilde{\lambda}^i = (\tilde{\lambda}^i)' \text{ and } y = -, \\ 0, & \text{if } \tilde{\lambda}^i \neq (\tilde{\lambda}^i)'. \end{cases}$$

Case 4. $\tilde{\lambda}^{i-1} = (\tilde{\lambda}^{i-1})'$ and $\tilde{\alpha}^i = (\tilde{\alpha}^i)'$. Then, in particular, $\tilde{\lambda}^i \neq (\tilde{\lambda}^i)'$, by Lemma A.3. Thus we may set $z_i := x_{i-1}$ and $x_i := 0$.

Note that this construction ensures that $x_m = y$. Moreover, by [2], for $i = 1, \ldots, m$, we get a path of length 2 from $[\tilde{\lambda}^{i-1}, x_{i-1}]$ to $[\tilde{\lambda}^i, x_i]$ in $\Gamma(\mathfrak{A}_n)$, hence a path of length $2m$ from $[\lambda, x]$ to $[\mu, y]$. Note further that these considerations also show the following: in the case where $\mu = \mu'$ there are both a path of length $2m$ from $[\lambda, x]$ to $[\mu, +]$ and a path of length $2m$ from $[\lambda, x]$ to $[\mu, -]$.

In order to prove (2), let conversely $\lambda, \mu \in V(n)$ such that $2 \leq d(\lambda, \mu) =: 2m$, let $x, y \in \{0, +, -\}$ be appropriate signs, and let

$$\begin{tikzcd}[column sep=small]
{[\beta^1, z_1]} \arrow[d, dash] \arrow[dr, dash] & & \cdots & {[\beta^r, z_r]} \arrow[d, dash] \arrow[dr, dash] & \\
{\begin{array}{c}[\lambda, x]\\ = [\lambda^0, x_0]\end{array}} & {[\lambda^1, x_1]} & \cdots & {[\lambda^{r-1}, x_{r-1}]} & {\begin{array}{c}[\lambda^r, x_r]\\ = [\mu, y]\end{array}}
\end{tikzcd}$$

be a shortest path from $[\lambda, x]$ to $[\mu, y]$ in $\Gamma(\mathfrak{A}_n)$. Then $r \leq m$, by what we have shown above. We also observe that the partitions $\lambda^0, \lambda^1, \ldots, \lambda^r$ must be pairwise different. To see this, assume that $\lambda^i = \lambda^j$, for some $0 \leq i < j \leq r$. Then we may suppose that $x_i = +$ and $x_j = -$. As we have seen above, there is a path of length 2 from $[\lambda^i, x_i]$ to $[\lambda^j, x_j]$ so that $j = i + 1$, since the given path is as short as possible. If $j < r$ then $\lambda^{i+2} \neq \lambda^i = \lambda^{i+1}$, by the minimality of r. But, since there is a path of length 2 from $[\lambda^{i+1}, x_{i+1}] = [\lambda^i, x_{i+1}]$ to $[\lambda^{i+2}, x_{i+2}]$, there is also a path of length 2 from $[\lambda^i, x_i]$ to $[\lambda^{i+2}, x_{i+2}]$, as we have proved above. But this contradicts the minimality of r. If $i + 1 = r$ then $i > 0$ and $\lambda^{i-1} \neq \lambda^i = \lambda^{i+1}$. This implies that there is a path of length 2 from $[\lambda^{i-1}, x_{i-1}]$ to $[\lambda^{i+1}, x_{i+1}]$, which is again a contradiction to the minimality of r.

We now set $\tilde{\lambda}^0 := \lambda^0 = \lambda$. Since $\langle \chi^{\beta^1}_{z_1} \downarrow_{\mathfrak{A}_n}, \chi^{\lambda}_{x} \rangle \neq 0 \neq \langle \chi^{\beta^1}_{z_1} \downarrow_{\mathfrak{A}_n}, \chi^{\lambda^1}_{x_1} \rangle$, there is some $\tilde{\beta}^1 \in \{\beta^1, (\beta^1)'\}$ such that $\langle \chi^{\tilde{\beta}^1} \downarrow_{\mathfrak{S}_n}, \chi^{\lambda} \rangle \neq 0$. Having fixed $\tilde{\beta}^1$, we can find $\tilde{\lambda}^1 \in \{\lambda^1, (\lambda^1)'\}$ such that also $\langle \chi^{\tilde{\beta}^1} \downarrow_{\mathfrak{S}_n}, \chi^{\tilde{\lambda}^1} \rangle \neq 0$. Let now $i \geq 2$. We may argue inductively, and suppose that we have a path

$$\begin{tikzcd}[column sep=small]
\tilde{\beta}^1 \arrow[d, dash] \arrow[dr, dash] & \tilde{\beta}^2 \arrow[d, dash] \arrow[dr, dash] & & \cdots & \tilde{\beta}^{i-1} \arrow[d, dash] \arrow[dr, dash] & \\
\lambda = \tilde{\lambda}^0 & \tilde{\lambda}^1 & \tilde{\lambda}^2 & \cdots & \tilde{\lambda}^{i-2} & \tilde{\lambda}^{i-1}
\end{tikzcd}$$

in $\Gamma(\mathfrak{S}_n)$, for appropriate $\tilde{\beta}^j \in \{\beta^j, (\beta^j)'\}$, $\tilde{\lambda}^j \in \{\lambda^j, (\lambda^j)'\}$, and all $j = 1, \ldots, i-1$. We then choose $\tilde{\beta}^i \in \{\beta^i, (\beta^i)'\}$ and $\tilde{\lambda}^i \in \{\lambda^i, (\lambda^i)'\}$ such that $\langle \chi^{\tilde{\beta}^i} \downarrow_{\mathfrak{S}_n}, \chi^{\tilde{\lambda}^{i-1}} \rangle \neq 0 \neq \langle \chi^{\tilde{\beta}^i} \downarrow_{\mathfrak{S}_n}, \chi^{\tilde{\lambda}^i} \rangle$. In this way we obtain in $\Gamma(\mathfrak{S}_n)$ a path of length $2r$ from λ to μ, or from λ to μ'.

Now let $\lambda := (n)$, and let $\mu \in \mathcal{P}_n$ be such that

$$\{1, \ldots, \lfloor \sqrt{n} \rfloor\} \times \{1, \ldots, \lfloor \sqrt{n} \rfloor\} \subseteq [\mu] \cap [\mu'],$$
$$[\mu] \cup [\mu'] \subseteq \{1, \ldots, \lceil \sqrt{n} \rceil\} \times \{1, \ldots, \lceil \sqrt{n} \rceil\},$$

and such that $\mu \geq \mu'$. Then $2 \leq d(\lambda, \mu) = 2(n - \lceil \sqrt{n} \rceil)$, and $2(n - \lceil \sqrt{n} \rceil) \leq d(\lambda, \mu') \leq 2(n - \lceil \sqrt{n} \rceil) + 2$. Assume that in $\Gamma(\mathfrak{A}_n)$ there is a path of length $2r < 2(n - \lceil \sqrt{n} \rceil)$ from $[(n), 0]$ to $[\mu, y]$, for some admissible $y \in \{0, +, -\}$. Then, by the above considerations, in $\Gamma(\mathfrak{S}_n)$ there is a path of length $2r$ from (n) to μ, or from (n) to μ'. But, since $2r < \min\{d((n), \mu), d((n), \mu')\}$, this is impossible, by Proposition A.2. Therefore, we have now also shown (2), and the assertion of the proposition follows. $\square$

Remark A.6. Note that the ring extensions $\mathbb{C}\mathfrak{S}_1 \subseteq \mathbb{C}\mathfrak{S}_2$ and $\mathbb{C}\mathfrak{A}_2 \subseteq \mathbb{C}\mathfrak{A}_3$ are clearly of depth 2, since $\mathfrak{S}_1 \trianglelefteq \mathfrak{S}_2$ and $\mathfrak{A}_2 \trianglelefteq \mathfrak{A}_3$.

Appendix B. Inclusions of dihedral groups in symmetric groups

Lemma B.1. *Let $n \geq 4$, let $G := \mathfrak{S}_n$, and let $H := \langle a \rangle$ where a is an n-cycle in G. Then the ring extension $\mathbb{C}H \subseteq \mathbb{C}G$ has depth 3.*

Proof. We may suppose that $a = (1, 2, \ldots, n)$. Since H is not normal in G, $\mathbb{C}H \subseteq \mathbb{C}G$ is not of depth 2. We set $g := (2, n-2)(n-1, n)$ and $\tilde{a} := a^g = (1, n-2, 3, 4, \ldots, n-3, 2, n, n-1)$. In the case where $n = 4$, this means $g = (3, 4)$ and thus $\tilde{a} = (1, 2, 4, 3)$. It suffices to show that $\langle a \rangle \cap \langle \tilde{a} \rangle = 1$. For $n = 4$ this is obviously true. Let now $n > 4$. Assume, for a contradiction, that $\langle a \rangle \cap \langle \tilde{a} \rangle \neq 1$. Then there are some $1 \neq l \in \mathbb{N}$, some $k \in \mathbb{N}$, and some $i \in \{1, \ldots, l-1\}$ such that $n = kl$ and $a^{ki} = \tilde{a}^k \neq 1$. If $k \in \{2, \ldots, n-4\}$ then $1 + ki = a^{ki}(1) = \tilde{a}^k(1) = 1 + k$. Thus $i = 1$, and if $k < n-4$ then we have the contradiction $2 + k = \tilde{a}^k(n-2) = a^k(n-2) \equiv n - 2 + k \pmod{n}$. If $k = n-4$ then $2 = \tilde{a}^k(n-2) = a^k(n-2) \equiv n - 2 + k \pmod{n}$, hence $k = 4$ and $n = 8$. But then $a^4(n) = 4 \neq 3 = \tilde{a}^4(n)$, a contradiction. If $k = 1$ then $1 + i = a^i(1) = \tilde{a}(1) = n - 2$, thus $i = n - 3$. But $a^{n-3}(2) = n - 1 \neq n = \tilde{a}(2)$, a contradiction. Finally, let $k \in \{n-1, n-2, n-3\}$. Since $n = kl \geq 2k \geq 2(n-3) = 2n - 6$, we get $n \leq 6$. Thus $n = 6$, $k = 3$, $l = 2$, $i = 1$, and we have a contradiction. $\square$

Proposition B.2. *Let $n > 5$, and let $H := D_{2n}$ be the dihedral subgroup of $\mathfrak{S}_n$ of order $2n$, with generators $a := (1, 2, \dots, n)$ and $b := (1, n)(2, n-1)(3, n-2) \cdots (\lfloor \frac{n}{2} \rfloor, \lceil \frac{n+2}{2} \rceil)$. Then the ring extension $\mathbb{C}H \subseteq \mathbb{C}\mathfrak{S}_n$ has depth 3.*

Proof. Since H is not normal in $\mathfrak{S}_n$, the extension $\mathbb{C}H \subseteq \mathbb{C}\mathfrak{S}_n$ is not of depth 2. Again we set $g := (2, n-2)(n-1, n)$, $\tilde{a} := a^g = (1, n-2, 3, \dots, n-3, 2, n, n-1)$, and $\tilde{b} := b^g = (1, n-1)(n-2, n)(2, 3)(4, n-3) \cdots (\lfloor \frac{n}{2} \rfloor, \lceil \frac{n+2}{2} \rceil)$. Also here it suffices to show that $H \cap H^g = 1$. For $n = 6$ this is obviously true. Thus, for the remainder of the proof, let $n \geq 7$. Assume, for a contradiction, that there is some $1 \neq x \in H \cap H^g$. Note that $H = \{a^i, a^i b \mid i = 0, \dots, n-1\}$ and $H^g = \{\tilde{a}^i, \tilde{a}^i \tilde{b} \mid i = 0, \dots, n-1\}$. We distinguish between four cases.

Case 1. $x = a^i = \tilde{a}^j$, for some $i, j \in \{1, \dots, n-1\}$. Then the proof of Lemma B.1 leads to a contradiction.

Case 2. $x = a^i = \tilde{a}^j \tilde{b}$, for some $i, j \in \{0, \dots, n-1\}$. That is $a^{2i} = \tilde{a}^j \tilde{b} \tilde{a}^j \tilde{b} = \tilde{a}^j \tilde{a}^{-j} = 1$, hence n is even, and $i = n/2$. In particular,

$$5 \leq 1 + \frac{n}{2} = a^i(1) = \tilde{a}^j(\tilde{b}(1)) = \tilde{a}^j(n-1)$$

which implies $j = 1 + n/2$. So $4 + n/2 = a^i(4) = \tilde{a}^j(\tilde{b}(4)) = \tilde{a}^j(n-3)$. But, on the other hand, $\tilde{a}^j(8-3) = 6$, $\tilde{a}^j(10-3) = 3$, and $\tilde{a}^j(n-3) = (n-4)/2$ for $n > 10$. In any case this is not equal to $4 + n/2$, a contradiction.

Case 3. $x = a^i b = \tilde{a}^j$, for some $i, j \in \{0, \dots, n-1\}$. But then $x^g = a^j = \tilde{a}^i \tilde{b}$ which is impossible, by the considerations in case 2 above.

Case 4. $x = a^i b = \tilde{a}^j \tilde{b}$, for some $i, j \in \{0, \dots, n-1\}$. Therefore, $1 + i = a^i(1) = a^i(b(n)) = \tilde{a}^j(\tilde{b}(n)) = \tilde{a}^j(n-2)$. Suppose that $1 \leq j \leq n-5$ so that $1 + i = \tilde{a}^j(n-2) = a^j(2) = 2 + j$. That is $i = j + 1 \in \{2, \dots, n-4\}$. However, if $i = j + 1 = 2$ then $\tilde{a}^j(\tilde{b}(1)) = 1 \neq 2 = a^i(b(1))$, if $i = j + 1 = 3$ then $\tilde{a}^j(\tilde{b}(1)) = n - 2 \neq 3 = a^i(b(1))$, and if $3 < i = j + 1 \leq n - 4$ then $\tilde{a}^j(\tilde{b}(1)) = j \neq j + 1 = i = a^i(b(1))$. Moreover, if $j = 0$ then we have $1 + i = n - 2$, thus $i = n - 3$. But $\tilde{a}^0(\tilde{b}(1)) = n - 1 \neq n - 3 = a^{n-3}(b(1))$. Consequently, we must have $j \in \{n-4, n-3, n-2, n-1\}$.

Suppose $j = n - 1$ so that $a^i b = \tilde{a}^{-1}\tilde{b}$. Then $1 + i = a^i(1) = a^i(b(n)) = \tilde{a}^{-1}(\tilde{b}(n)) = \tilde{a}^{-1}(n-2) = 1$, thus $i = 0$ and $b = \tilde{a}^{-1}\tilde{b}$. But $b(n-1) = 2 \neq n - 1 = \tilde{a}^{-1}(\tilde{b}(n-1))$, a contradiction.

Next suppose that $j = n-2$ where $a^i b = \tilde{a}^{-2}\tilde{b}$, and $1+i = a^i(b(n)) = \tilde{a}^{-2}(\tilde{b}(n)) = n - 1$. Hence $i = n - 2$ which gives the contradiction $a^{-2}(b(2)) = a^{-2}(n-1) = n - 3 \neq 1 = \tilde{a}^{-2}(3) = \tilde{a}^{-2}(\tilde{b}(2))$.

If $j = n - 3$ then $a^i b = \tilde{a}^{-3}\tilde{b}$, and so $i = n - 1$. But this time we get the contradiction $1 = \tilde{a}^{-3}(4) = \tilde{a}^{-3}(\tilde{b}(n-3)) = a^{-1}(b(n-3)) = a^{-1}(4) = 3$.

Lastly, assume that $j = n - 4$ so that $a^i b = \tilde{a}^{-4}\tilde{b}$, and $i = 1$. But, for $n \neq 8$, this implies $n - 6 = \tilde{a}^{-4}(2) = \tilde{a}^{-4}(\tilde{b}(3)) = a(b(3)) = a(n-2) = n-1$, and for $n = 8$ we get $6 = \tilde{a}^{-4}(2) = \tilde{a}^{-4}(\tilde{b}(3)) = a(b(3)) = a(6) = 7$. Hence we have again a contradiction.

To summarize, neither of the four cases above can occur, and the assertion of the proposition follows. □

Remark B.3. (a) By Example 2.4, we know that the ring extension $\mathbb{C}D_8 \subseteq \mathbb{C}\mathfrak{S}_4$ has depth 4.

(b) The inclusion matrix of the groups $D_{10} < \mathfrak{S}_5$ is

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix},$$

which has depth 5. Hence the same applies to the ring extension $\mathbb{C}D_{10} \subseteq \mathbb{C}\mathfrak{S}_5$.

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