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# AN ALGEBRAIC INTERPRETATION OF THE $q$-BINOMIAL COEFFICIENTS

Michael Braun

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Abstract. Gaussian numbers, also called Gaussian polynomials or $q$-binomial coefficients are the $q$-analogs of common binomial coefficients. First introduced by Euler these polynomials have played an important role in many different branches of mathematics. Sylvester discovered the unimodality of their coefficients, using invariant theory. Gauss recognized the connection of the coefficients to proper integer partitions using a combinatorial interpretation. Here in this paper we are going to point out a new algebraic interpretation of the Gaussian polynomials as orbits of the Borel group on subspaces.

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## 1. Introduction

Let $L_k(n,q)$ denote the set of $k$-dimensional subspaces of the $n$-dimensional vector space $\mathrm{GF}(q)^n$ over the finite field $\mathrm{GF}(q)$ with $q$ elements where $q$ is a prime power.

In order to determine the number of all elements of this set $L_k(n,q)$ we start with the following counting: A $k$-subspace $K$ of $\mathrm{GF}(q)^n$ has a base, consisting of $k$ linearly independent vectors of $\mathrm{GF}(q)^n$. The first vector can be an arbitrarily chosen vector unequal to zero, i. e. we have $q^n-1$ possibilities. For the second vector we can take all vectors except the $q$ multiples of the first one, i. e. we have $q^n-q$. For the third vector we can choose all except the $q^2$ linear combinations of the first ones, i. e. we have $q^n-q^2$ possible vectors. The same calculation for the remaining $k-3$ vectors yields the number $(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})$ to pick $k$ linearly independent vectors. Analogously, the term $(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})$ describes the number of bases determining the same $k$-subspace. Hence the following ratio is the number of $k$-subspaces of $\mathrm{GF}(q)^n$

$$\binom{n}{k}_q := \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})} \tag{1}$$

and it is called *Gaussian number* or *$q$-binomial coefficient*, analogous to the binomial coefficient $\binom{n}{k}$ which is the number of $k$-subsets of an $n$-element set. Canceling powers of $q$ in $\binom{n}{k}_q$ yields

$$\binom{n}{k}_q = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}.$$

The Gaussian numbers satisfy the following recursive formula

$$\binom{n}{k}_q = q^k\binom{n-1}{k}_q + \binom{n-1}{k-1}_q. \tag{2}$$

This formula together with values $\binom{n}{0}_q = \binom{n}{n}_q = 1$ can be used to prove by induction on $n$, that $\binom{n}{k}_q$ can be interpreted as a polynomial with the indeterminate $q$. Here, a simple example is

$$\binom{6}{3}_q = 1 + q + 2q^2 + 3q^3 + 3q^4 + 3q^5 + 3q^6 + 2q^7 + q^8 + q^9.$$

Therefore, Gaussian numbers can also be called *Gaussian polynomials*. First introduced by Euler [1] in his work on product series expansions, these polynomials have played an important role in many different branches of mathematics. Sylvester [5] discovered one of their deepest properties, the unimodality of their coefficients, using invariant theory. Gauss [2] recognized the connection of these coefficients to proper integer partitions using a combinatorial interpretation:

**Theorem 1.** *The Gaussian number satisfies the equation*

$$\binom{n}{k}_q = \sum_{i=0}^{(n-k)k} a_i q^i,$$

*where $a_i$ is the number of proper partitions of the number $i$, whose Ferrers diagrams fit into a rectangle with $n-k$ rows and $k$ columns.*

Here in the present paper we are going to describe a new algebraic interpretation of the coefficients of the Gaussian polynomials.

## 2. Echelon Forms

In the following we consider the vector spaces as column spaces. Now let $K$ be $k$-subspace of $\mathrm{GF}(q)^n$. A matrix $\Gamma$ having $n$ rows, $k$ columns and entries in $\mathrm{GF}(q)$ is called a *generator matrix* of $K$ if and only if the columns of $\Gamma$ yield a base of $K$, i. e.

$$K = \{\Gamma \cdot v \mid v \in \mathrm{GF}(q)^k\}.$$

There are several generator matrices for the same $k$-subspace $K$ but using elementary Gaussian transformation of the columns yields a uniquely determined generator matrix, the *canonical* generator matrix, $\Gamma_C(K)$ of the subspace $K$, having the structure which is shown in Table 1.

| | | | | | | |
|---|---|---|---|---|---|---|
| $*$ | $*$ | $*$ | $\cdots$ | $*$ | $*$ | |
| $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ | |
| $*$ | $*$ | $*$ | $\cdots$ | $*$ | $*$ | |
| 1 | 0 | 0 | $\cdots$ | 0 | 0 | base row $i_0$ |
| | $*$ | $*$ | $\cdots$ | $*$ | $*$ | |
| | $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ | |
| | $*$ | $*$ | $\cdots$ | $*$ | $*$ | |
| | 1 | 0 | $\cdots$ | 0 | 0 | base row $i_1$ |
| | | $*$ | $\cdots$ | $*$ | $*$ | |
| | | | $\ddots$ | | | |
| | | | | $*$ | $*$ | |
| | | | | $\vdots$ | $\vdots$ | |
| | | | | $*$ | $*$ | |
| | | | | 1 | 0 | base row $i_{k-2}$ |
| | | | | | $*$ | |
| | | | | | $\vdots$ | |
| | | | | | $*$ | |
| | | | | | 1 | base row $i_{k-1}$ |
| | | | | | 0 | |
| | | | | | $\vdots$ | |
| | | | | | 0 | |

TABLE 1. The structure of an Echelon matrix

The stars in this matrix represent elements in $\mathrm{GF}(q)$, and the rows where new steps commence are called *base rows.* These canonical forms of generator matrices are also called *Echelon forms.*

Now we introduce an equivalence relation on the set $L_k(n, q)$, which we call *Echelon equivalence*: Two $k$-subspaces $K$ and $K'$ are defined to be Echelon equivalent, abbreviated by $K \simeq_E K'$ if and only if the base rows of the canonical generator matrices $\Gamma_C(K)$ and $\Gamma_C(K')$ have the same row indices.

For instance, the two 3-subspaces of $\mathrm{GF}(5)^6$ generated by the following generator matrices are Echelon equivalent:

$$\begin{bmatrix} 2 & 1 & 1 \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 1 & 4 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & 2 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 & 0 & 3 \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 2 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

Table 2 shows all equivalence classes of Echelon forms of $\mathrm{GF}(5)^6$.

$$\begin{bmatrix} 1&0&0\\0&1&0\\0&0&1\\0&0&0\\0&0&0\\0&0&0 \end{bmatrix} \begin{bmatrix} 1&0&0\\0&1&0\\0&0&*\\0&0&1\\0&0&0\\0&0&0 \end{bmatrix} \begin{bmatrix} 1&0&0\\0&1&0\\0&0&*\\0&0&*\\0&0&1\\0&0&0 \end{bmatrix} \begin{bmatrix} 1&0&0\\0&1&0\\0&0&*\\0&0&*\\0&0&*\\0&0&1 \end{bmatrix} \begin{bmatrix} 1&0&0\\0&*&*\\0&1&0\\0&0&1\\0&0&0\\0&0&0 \end{bmatrix}$$

$$\begin{bmatrix} 1&0&0\\0&*&*\\0&1&0\\0&0&*\\0&0&1\\0&0&0 \end{bmatrix} \begin{bmatrix} 1&0&0\\0&*&*\\0&1&0\\0&0&*\\0&0&*\\0&0&1 \end{bmatrix} \begin{bmatrix} 1&0&0\\0&*&*\\0&*&*\\0&1&0\\0&0&1\\0&0&0 \end{bmatrix} \begin{bmatrix} 1&0&0\\0&*&*\\0&*&*\\0&1&0\\0&0&*\\0&0&1 \end{bmatrix} \begin{bmatrix} 1&0&0\\0&*&*\\0&*&*\\0&*&*\\0&1&0\\0&0&1 \end{bmatrix}$$

$$\begin{bmatrix} *&*&*\\1&0&0\\0&1&0\\0&0&1\\0&0&0\\0&0&0 \end{bmatrix} \begin{bmatrix} *&*&*\\1&0&0\\0&1&0\\0&0&*\\0&0&1\\0&0&0 \end{bmatrix} \begin{bmatrix} *&*&*\\1&0&0\\0&1&0\\0&0&*\\0&0&*\\0&0&1 \end{bmatrix} \begin{bmatrix} *&*&*\\1&0&0\\0&*&*\\0&1&0\\0&0&1\\0&0&0 \end{bmatrix} \begin{bmatrix} *&*&*\\1&0&0\\0&*&*\\0&1&0\\0&0&*\\0&0&1 \end{bmatrix}$$

$$\begin{bmatrix} *&*&*\\1&0&0\\0&*&*\\0&*&*\\0&1&0\\0&0&1 \end{bmatrix} \begin{bmatrix} *&*&*\\*&*&*\\1&0&0\\0&1&0\\0&0&1\\0&0&0 \end{bmatrix} \begin{bmatrix} *&*&*\\*&*&*\\1&0&0\\0&1&0\\0&0&*\\0&0&1 \end{bmatrix} \begin{bmatrix} *&*&*\\*&*&*\\1&0&0\\0&*&*\\0&1&0\\0&0&1 \end{bmatrix} \begin{bmatrix} *&*&*\\*&*&*\\*&*&*\\1&0&0\\0&1&0\\0&0&1 \end{bmatrix}$$

TABLE 2. The Echelon forms of $\mathrm{GF}(5)^6$

If $i$ is the number of stars in the matrix, then $q^i$ is the number of $k$-subspaces in the corresponding equivalence class. The maximum number of stars is $(n-k)k$. Replacing the stars by arbitrary elements of $\mathrm{GF}(q)$ yields a transversal of the equivalence classes. In order to determine the number of equivalence classes we only have to calculate the number of possibilities to choose $k$ base rows in a matrix with $n$ rows. This number is of course $\binom{n}{k}$. Hence, a well known result can now be formulated (see [4]).

**Theorem 2.** *The Gaussian number can be written as*

$$\binom{n}{k}_q = \sum_{i=0}^{(n-k)k} a_i q^i,$$

*where $a_i$ is the number of different Echelon equivalence classes with cardinality $q^i$.*

## 3. Borel Groups

We consider some certain elements of the general linear group $GL(n,q)$. But first we need some notation. If $A$ denotes an $n \times m$ matrix with entries $A_{ij}$ we use $A_{i*}$ for the $i$th row and $A_{*j}$ for the $j$th column of $A$. Furthermore, the index set for the $n$ rows is $\{0,\ldots,n-1\}$ and the index set for the $m$ columns is $\{0,\ldots,m-1\}$.

Let $\lambda \in \mathrm{GF}(q)^*$ be a non-zero element and let $0 \le k, \ell < n$. Then we define the $n \times n$ matrix $G^{\ell,k;\lambda}$ with the entries

$$G^{\ell,k;\lambda}_{ij} := \begin{cases} \lambda & \text{if } i = \ell \text{ and } j = k \\ \delta_{ij} & \text{otherwise.} \end{cases}$$

where $\delta$ denotes the Kronecker symbol

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

The multiplication of $G^{\ell,k;\lambda}$ with an $n \times m$ matrix $A$ from the left yields the matrix $B = G^{\ell,k;\lambda} \cdot A$ with the following properties:

i) $B$ only differs from $A$ in the $\ell$th row, all other rows are equal:

$$B_{i*} = A_{i*} \quad \forall i \neq \ell$$

iia) If $\ell \neq k$, then the $\ell$th row of $B$ is the sum of the $\ell$th row of $A$ and the $\lambda$ fold of the $k$th row of $A$:

$$B_{\ell*} = A_{\ell*} + \lambda A_{k*}$$

iib) If $\ell = k$ then the $\ell$th row of $B$ is the $\lambda$ fold of the $\ell$th row of $A$:

$$B_{\ell *} = \lambda A_{\ell *}$$

In other words the multiplication of elements $G^{\ell,k;\lambda}$ from the left corresponds to the elementary Gaussian transformations of the rows.

In the following let $B(n, q)$ denote the *Borel group* which is the set consisting of all invertible upper triangular matrices of $GL(n, q)$. It is obvious that this group is generated by the set $\{G^{\ell,k;\lambda} \mid \lambda \in \mathrm{GF}(q)^*, \ell \leq k\}$.

Now we consider the Borel group $B(n, q)$ acting on the set $L_k(n, q)$ of $k$-subspaces by the mapping

$$B(n, q) \times L_k(n, q) \to L_k(n, q), (G, K) \mapsto G * K,$$

where $G * K := \{G \cdot v \mid v \in K\}$. This mapping defines a group action, i. e. the equation $G * (G' * K) = (G \cdot G') * K$ holds for all $G, G' \in B(n, q)$ and $K \in L_k(n, q)$, and the equation $U * K = K$ holds if $U$ denotes the unit matrix in $B(n, q)$. This action of the Borel group defines an equivalence relation on the set of $k$-subspaces:

$$K \simeq_{B(n,q)} K' :\Longleftrightarrow \exists G \in B(n, q) : G * K = K'$$

The equivalence class of $K$ is called *orbit* and it is denoted by

$$B(n, q)(K) := \{G * K \mid G \in B(n, q)\}.$$

The set of all orbits is abbreviated by

$$B(n, q)\backslash\!\backslash L_k(n, q) := \{B(n, q)(K) \mid K \in L_k(n, q)\}.$$

A detailed description of group actions can be found in [3].

**Lemma 1.** *The Echelon equivalence and the equivalence defined by the Borel group $B(n, q)$ are equivalent:*

$$K \simeq_E K' \iff K \simeq_{B(n,q)} K'$$

**Proof.** "$\Longrightarrow$" Let $K, K' \in L_k(n, q)$ with $K \simeq_E K'$ with corresponding canonical generator matrices $A = \Gamma_C(K)$ and $B = \Gamma_C(K')$. Since $K$ and $K'$ are Echelon equivalent the matrices $A$ and $B$ have the same base row indices. Let $\{i_0, \ldots, i_{k-1}\}$ denote the set of the corresponding row indices, i. e. $A_{i*} = B_{i*}$ for all $i \in \{i_0, \ldots, i_{k-1}\}$. Furthermore we have $A_{\ell j} = B_{\ell j} = 0$ for all $0 < j < k$ and $\ell > i_j$. In the following we are going to show how to apply Borel group elements to $A$ until we get the matrix $B$. Now let $(\ell, j)$ be a position in which $A$ and $B$ are different, i. e. $A_{\ell j} \neq B_{\ell j}$. This implies $\ell < i_j$, since $A$ and $B$ are assumed to be

Echelon matrices. If we set $A' = G^{\ell,i_j;\lambda} \cdot A$ with $\lambda = B_{\ell j} - A_{\ell j}$, the matrices $A$ and $A'$ are identical except the position $(\ell, j)$ which satisfies $A'_{\ell j} = A_{\ell j} + \lambda = B_{\ell j}$, since $A_{i_j j} = 1$. By repeating this operation to all different entries of $A$ and $B$ we finally obtain the matrix $B$ from $A$ by applying Borel group elements.

"$\Longleftarrow$" We show that applying a generator of the Borel group to a canonical matrix of a subspace $K$ yields a basis of a subspace $K'$ that is Echelon equivalent to $K$. In terms of Gaussian transformations the action with a Borel group generator means either a multiplication of a row with a finite field element unequal to zero, or the addition of a multiple of a row to a row above. Considering these two possible operations we get the following cases.

ia) Multiplication of a non-base row with a non-zero element: In this case the structure of the generator matrix is still canonical with the same base row indices, since no base row is affected. Thus the resulting subspace is Echelon equivalent to the first one.

ib) Multiplication of a base row with a non-zero element: If we multiply the base row $i_j$ by an element $\lambda \neq 0$ we get a non-canonical generator matrix of a subspace $K'$, but multiplying the $j$th column by $\lambda^{-1}$ makes the generator matrix canonical again. The base row indices do not change. Hence $K'$ is Echelon equivalent to $K$.

iia) Addition of a multiple of a row to a non-base row above: The same situation applies as in case ia) since no base row was affected, the resulting subspace is Echelon equivalent.

iib) Addition of a multiple of a row to a base row above: If we add the multiple of a row to the base row $i_j$ above then this new $i_j$th row may contain elements unequal to zero, for example the element $B_{i_j k}$ with $k > i_j$. All these elements can be eliminated by adding the $-B_{i_j k}/B_{i_j j}^{-1}$-fold of the $j$th column to the $k$th column. All other elements base rows are not affected by this operation. After eliminating all elements $(i_j, k)$ unequal to zero we multiply the $j$th column by $B_{i_j j}^{-1}$ in order to obtain a canonical generator matrix. The base row indices are still the same. The resulting subspace is also Echelon equivalent to $K$. $\square$

## 4. The Algebraic Interpretation

As an immediate consequence of Lemma 1 and Theorem 2 we get the main result, that the number of Echelon forms having exactly $i$ stars is nothing but the number of orbits of the Borel group $B(n, q)$ on the set of $k$-subspaces with $q^i$ elements. More formally we can write:

**Theorem 3.** *The Gaussian number satisfies the equation*

$$\binom{n}{k}_q = \sum_{i=0}^{(n-k)k} a_i q^i,$$

*where* $a_i = \left|\left\{\Omega \in B(n,q) \backslash\backslash L_k(n,q) \,\middle|\, |\Omega| = q^i\right\}\right|$.

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**Michael Braun**
Siemens AG, Corporate Technology
D-80200 Munich, Germany
email: mic.braun@siemens.com