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(n, m)-STRONGLY GORENSTEIN PROJECTIVE MODULES

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Abstract. This paper is a continuation of the papers J. Pure Appl. Algebra, 210 (2007), 437–445 and J. Algebra Appl., 8 (2009), 219–227. Namely, we introduce and study a doubly filtered set of classes of modules of finite Gorenstein projective dimension, which are called (n, m)-strongly Gorenstein projective ((n, m)-SG-projective for short)(for integers $n \geq 1$ and $m \geq 0$). We are mainly interested in studying syzygies of these modules. As consequences, we show that a module M has Gorenstein projective dimension at most m if and only if $M \oplus G$ is $(1, m)$-SG-projective for some Gorenstein projective module G. And, over rings of finite left finitistic flat dimension, that a module of finite Gorenstein projective dimension has finite projective dimension if and only if it has finite flat dimension.

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1. Introduction

Throughout this paper, R denotes a non-trivial associative ring with identity, and all modules are left R-modules. For a module M, we use $\mathrm{pd}(M)$ and $\mathrm{fd}(M)$ to denote, respectively, the classical projective and flat dimensions of M.

A module M is called *Gorenstein projective* (G-projective for short), if there exists an exact sequence of projective modules,

$$\mathbf{P} = \ \cdots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots ,$$

such that $M \cong \mathrm{Im}(P_0 \to P_{-1})$ and such that $\mathrm{Hom}(-, Q)$ leaves the sequence $\mathbf{P}$ exact whenever Q is a projective module. The exact sequence $\mathbf{P}$ is called a complete projective resolution of M.

For a positive integer n, we say that M has *Gorenstein projective dimension* at most n, and we write $\mathrm{Gpd}_R(M) \leq n$ (or simply $\mathrm{Gpd}(M) \leq n$), if there is an exact sequence of modules,

$$0 \to G_n \to \cdots \to G_0 \to M \to 0,$$

where each G_i is Gorenstein projective (suitable background materials on the notion of Gorenstein projective modules can be found in [7,8,12]).

The notion of Gorenstein projective modules was first introduced and studied by Enochs et al. [9,10,11] as a generalization of the classical notion of projective modules in the sense that a module is projective if and only if it is Gorenstein projective with finite projective dimension (see also [8,12]). In an unpublished work [7, Theorem 4.2.6 and Notes page 99], Avramov, Buchweitz, Martsinkovsky, and Reiten proved, over Noetherian rings, that finitely generated Gorenstein projective modules are just modules of Auslander's Gorenstein dimension 0 ([1], see also [2]), which are extensively studied by many others (part of the works on Gorenstein dimension is summarized in Christensen's book [7]).

The Gorenstein projective dimension has been extensively studied by many others, who proved that this dimension shares many nice properties of the classical projective dimension. In [3], Bennis and Mahdou introduced a particular case of Gorenstein projective modules, which are defined as follows:

Definition 1.1 ([3])**.** A module M is said to be strongly Gorenstein projective (SG-projective for short), if there exists an exact sequence of projective modules,

$$\mathbf{P} = \ \cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots ,$$

such that $M \cong \mathrm{Im}(f)$ and such that $\mathrm{Hom}(-,Q)$ leaves the sequence $\mathbf{P}$ exact whenever Q is a projective module.

It is proved that the class of all strongly Gorenstein projective modules is an intermediate class between the ones of projective modules and Gorenstein projective modules [3, Proposition 2.3]; i.e., we have the following inclusions

$$\begin{aligned} \{projective\ modules\} &\subseteq \{SG{-}projective\ modules\} \\ &\subseteq \{G{-}projective\ modules\} \end{aligned}$$

which are, in general, strict by [3, Examples 2.5 and 2.13]. The principal role of the strongly Gorenstein projective modules is to give the following characterization of Gorenstein projective modules [3, Theorem 2.7]: a module is Gorenstein projective if and only if it is a direct summand of a strongly Gorenstein projective module. The notion of strongly Gorenstein modules confirm that there is an analogy between the notion of Gorenstein projective modules and the notion of the usual projective modules. In fact, this is obtained because the strongly Gorenstein projective modules have simpler characterizations than their correspondent Gorenstein modules

[3, Propositions 2.9]. For instance, a module M is strongly Gorenstein projective if and only if there exists a short exact sequence of modules,

$$0 \to M \to P \to M \to 0,$$

where P is projective, and $\mathrm{Ext}(M,Q) = 0$ for any projective module Q. Using the results above, the notion of strongly Gorenstein projective modules was proven to be a good tool for establishing results on Gorenstein projective dimension (see, for instance, [4,5,6]). In [4], an extension of the notion of strongly Gorenstein projective modules is introduced as follows: for an integer $n > 0$, a module M is called n-strongly Gorenstein projective (n-SG-projective for short), if there exists an exact sequence of modules,

$$0 \to M \to P_n \to \cdots \to P_1 \to M \to 0,$$

where each P_i is projective, such that $\mathrm{Hom}(-,Q)$ leaves the sequence exact whenever Q is a projective module (equivalently, $\mathrm{Ext}^i(M,Q) = 0$ for $j+1 \leq i \leq j+n$ for some positive integer j and for any projective module Q [4, Theorem 2.8]). Then, 1-strongly Gorenstein projective modules are just strongly Gorenstein projective modules. In [4, Proposition 2.2], it is proved that an n-strongly Gorenstein projective module is projective if and only if it has finite flat dimension. In [13], Zhao and Huang, continued the study of n-strongly Gorenstein projective modules. They gave more examples and they investigated the relations between n-strongly Gorenstein projective modules and m-strongly Gorenstein projective modules whenever $n \neq m$. They also proved, for two modules M and N projectively equivalent (that is, there exist two projective modules P and Q such that $M \oplus P \cong N \oplus Q$), that M is n-strongly Gorenstein projective if and only if N is n-strongly Gorenstein projective [13, Theorem 3.14] (see Lemma 2.5 for a generalization of this result). So using this result, we prove the following lemma, which we use in the proof of the main results of this paper.
Recall, for a projective resolution of a module M,

$$\cdots \to P_1 \to P_0 \to M \to 0,$$

that the module $K_i = \mathrm{Im}(P_i \to P_{i-1})$ for $i \geq 1$, is called an i^{th} syzygy of M.

Lemma 1.2. *If M is an n-strongly Gorenstein projective module for some integer $n > 0$, then:*

(1) *Every i^{th} syzygy of M is n-strongly Gorenstein projective.*

(2) *For every complete projective resolution of* M,

$$\mathbf{P} = \ \cdots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots ,$$

every $\mathrm{Im}(P_i \to P_{i-1})$ *is* n*-strongly Gorenstein projective.*

Proof. First note that M admits a complete projective resolution

$$\mathbf{Q} = \ \cdots \to Q_2 \to Q_1 \to Q_0 \to Q_{-1} \to \cdots$$

in which all images $\mathrm{Im}(Q_i \to Q_{i-1})$ are n-strongly Gorenstein projective modules. Indeed, M is n-strongly Gorenstein projective module, then there exists an exact sequence,

$$(*) \qquad 0 \to M \to Q_{n-1} \to \cdots \to Q_0 \to M \to 0,$$

where each Q_i is a projective module, such that $\mathrm{Hom}(-, Q)$ leaves the sequence exact whenever Q is a projective module. For every $i = 1, ..., n-1$, we decompose the exact sequence $(*)$ into two short exact sequences as follows:

$$0 \to M \to Q_{n-1} \to \cdots \to Q_i \to N_i \to 0$$

and

$$0 \to N_i \to Q_{i-1} \to \cdots \to Q_0 \to M \to 0$$

Assembling these sequences so that we obtain the following exact sequence

$$0 \to N_i \to Q_{i-1} \to \cdots \to Q_0 \to Q_{n-1} \to \cdots \to Q_i \to N_i \to 0$$

This shows that each $\mathrm{Im}(Q_i \to Q_{i-1})$ is n-strongly Gorenstein projective. Then, the desired complete projective resolution $\mathbf{Q}$ is obtained by assembling the sequence $(*)$ with itself as done in the proof of [4, Proposition 2.5(2)].
Now, using the left half of $\mathbf{Q}$, $\cdots \to Q_1 \to Q_0 \to M \to 0$, and the fact that every two i^{th} syzygies of a module are projectively equivalent [14, Theorem 9.4], the assertion 1 follows from [13, Theorem 3.14].
We prove the second assertion. From 1 it remains to prove the result for the images of the right half of $\mathbf{P}$. Using [12, Proposition 1.8], a dual proof of the one of [14, Theorem 9.4] shows that the two module $\mathrm{Im}(Q_i \to Q_{i-1})$ and $\mathrm{Im}(P_i \to P_{i-1})$ are projectively equivalent for every $i \leq -1$, and therefore the result follows from [13, Theorem 3.14]. □

The aim of this paper is to generalize the notions above to a more general context (Definition 2.1). Namely, we introduce and study a doubly filtered set of classes of modules with finite Gorenstein projective dimension, which are called (n, m)-strongly Gorenstein projective ((n, m)-SG-projective for short), (for integers $n \geq 1$ and $m \geq 0$). First, we study the relations between them (Proposition 2.2), and the stability of this new class of modules under direct sum (Proposition 2.3). Then, we set our first main result in this paper (Theorem 2.4), which shows, for an (n,m)-SG-projective module M, that $\mathrm{Gpd}(M) = k \leq m$ for some positive integer k. In particular, any i^{th} syzygy of M is $(n, m-i)$-SG-projective for $1 \leq i \leq k$, and any i^{th} syzygy of M is $(n,0)$-SG-projective for $i \geq k$. The second main purpose of the paper is to investigate the converse of the first main result. Namely, we ask: if an i^{th} syzygy of a module M is (n,m)-SG-projective, is M an $(n, m+i)$-SG-projective module? In the second main result (Theorem 2.7), we give an affirmative answer when $n = 1$ as follows: for two integers $d \geq 1$ and $m \geq 0$, if a d^{th} syzygy of a module M is $(1,m)$-SG-projective, then $\mathrm{Gpd}(M) = k \leq d+m$ for some positive integer k and M is $(1,k)$-SG-projective. These results lead to two results on modules of finite Gorenstein projective dimension:

The first one shows that $(1,m)$-SG-projective modules can serve to characterize modules of finite Gorenstein projective dimension similarly to the characterization of Gorenstein projective modules by strongly Gorenstein projective modules. Namely, we prove (Corollary 2.8): for a module M and a positive integer m, $\mathrm{Gpd}(M) \leq m$ if and only if $M \oplus G$ is $(1,m)$-SG-projective for some Gorenstein projective module G.

The second one shows, over rings of finite left finitistic flat dimension, that a module of finite Gorenstein projective dimension has finite projective dimension if and only if it has finite flat dimension (Proposition 2.10). This, in fact, holds since we establish the following extension of [4, Proposition 2.2] (Corollary 2.9): let M be an (n,m)-SG-projective module for some integers $n \geq 1$ and $m \geq 0$. Then, $\mathrm{pd}(M) < \infty$ if and only if $\mathrm{fd}(M) < \infty$.

2. Main results

In this paper, we investigate the following kind of modules:

Definition 2.1. Let $n \geq 1$ and $m \geq 0$ be integers. A module M is called (n,m)-strongly Gorenstein projective ((n,m)-SG-projective for short) if there exists an exact sequence of modules,

$$0 \to M \to Q_n \to \cdots \to Q_1 \to M \to 0,$$

where $\mathrm{pd}(Q_i) \leq m$ for $1 \leq i \leq n$, such that $\mathrm{Ext}^i(M,Q) = 0$ for any $i > m$ and for any projective module Q.

Consequently, $(1,0)$-SG-projective modules are just strongly Gorenstein projective modules (by [3, Proposition 2.9]), and, generally, $(n,0)$-SG-projective modules

are just n-strongly Gorenstein projective modules (by [4, Theorem 2.8]).
One can show easily that modules of projective dimension at most an integer m are particular examples of (n,m)-SG-projective modules for every integer $n \geq 1$. The converse is not true in general unless the (n,m)-SG-projective modules have finite flat dimension (see Corollary 2.9). To give examples of (n,m)-SG-projective modules with infinite projective dimension, we can take any $(n,0)$-SG-projective module M which is not projective (use, for instance, [4, Examples 2.4 and 2.6] and [13, Example 3.2]) and any module Q with projective dimension at most m, then we can show easily that the direct sum $M \oplus Q$ is an (n,m)-SG-projective module with infinite projective dimension.

The main purpose of the paper is to investigate the syzygies of (n,m)-SG-projective modules. In particular, we show that (n,m)-SG-projective modules are particular examples of modules with Gorenstein projective dimension at most m. Before, we give some elementary properties of (n,m)-SG-projective modules.

Proposition 2.2. *Let M be a module and consider two integers $n \geq 1$ and $m \geq 0$. We have the following assertions:*

(1) *If M is (n,m)-SG-projective, then it is (n,m')-SG-projective for every $m' \geq m$.*
(2) *If M is (n,m)-SG-projective, then it is (nk,m)-SG-projective for every $k \geq 1$.*
In particular, every $(1,m)$-SG-projective module is (n,m)-SG-projective for every $n \geq 1$.

Proof. 1. Obvious.
2. Since M is (n,m)-SG-projective, there exists an exact sequence of modules $0 \to M \to Q_n \to \cdots \to Q_1 \to M \to 0$, where $\mathrm{pd}(Q_i) \leq m$ for $1 \leq i \leq n$, such that $\mathrm{Ext}^i(M,Q) = 0$ for any $i > m$ and for any projective module Q. Assembling this sequence with itself k times, we can show that M is also (nk,m)-SG-projective. □

Proposition 2.3. *Let $(M_i)_{i\in I}$ be a family of modules and consider the bounded families of integers $(n_i \geq 1)_{i\in I}$ and $(m_i \geq 0)_{i\in I}$.*
If, for any $i \in I$, M_i is (n_i,m_i)-SG-projective, then the direct sum $\oplus_i M_i$ is (n,m)-SG-projective, where $m = \max\{m_i\}$ and n is the least common multiple of n_i for $i \in I$.

Proof. First, note that m and n exist since the families $(n_i)_i$ and $(m_i)_i$ are bounded. Now, from Proposition 2.2, M_i is (n,m)-SG-projective for any $i \in I$.

Then, using standard arguments, we can show that the direct sum $\oplus_i M_i$ is (n, m)-SG-projective. □

Note, by [13, Example 3.13], that the family of (n, m)-SG-projective modules is not closed under direct summands. However, in Lemma 2.5 given later, we give a situation in which a direct summand of an (n, m)-SG-projective module is (n, m)-SG-projective.

Now we give our first main result, in which we study the syzygies of an (n, m)-SG-projective.
Recall, for a projective resolution of a module M,

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

that the module $K_i = \mathrm{Im}(P_i \to P_{i-1})$ for $i \geq 1$ is called an i^{th} syzygy of M.

Theorem 2.4. *If a module M is (n, m)-SG-projective for some integers $n \geq 1$ and $m \geq 0$, then:*

(1) $\mathrm{Gpd}(M) = k \leq m$ *for some positive integer* k*;*
(2) *Any i^{th} syzygy K_i of M is $(n, m-i)$-SG-projective for $1 \leq i \leq k$;*
(3) *Any i^{th} syzygy K_i of M is $(n, 0)$-SG-projective for $i \geq k$.*

Proof. 1 and 2. Since M is (n, m)-SG-projective, there exists an exact sequence of modules,

$$(*) \quad 0 \to M \to Q_n \to \cdots \to Q_1 \to M \to 0,$$

where $\mathrm{pd}(Q_i) \leq m$ for $1 \leq i \leq n$, such that $\mathrm{Ext}^i(M, Q) = 0$ for any $i > m$ and for any projective module Q. Consider a short exact sequence of modules

$$0 \to K_1 \to P_0 \to M \to 0,$$

where P_0 is projective. We prove that K_1 is $(n, m-1)$-SG-projective. First, from [14, Theorem 9.4], $\mathrm{Ext}^i(K_1, Q) = 0$ for any $i > m-1$ and for any projective module Q. Then, it remains to prove the existence of the exact sequence. For that, decompose the exact sequence above $(*)$ into short exact sequences

$$0 \to H_i \to Q_i \to H_{i-1} \to 0,$$

where $H_n = M = H_0$ and $H_i = \mathrm{Ker}(Q_i \to H_{i-1})$ for $i = 1, ..., n-1$. And consider, for $i = 0, ..., n$, a short exact sequence

$$0 \to K_{i,1} \to P_{i,0} \to H_i \to 0,$$

where $P_{i,0}$ is projective for $i = 1, ..., n-1$, and $P_{n,0} = P_{0,0} = P_0$, and $K_{n,1} = K_{0,1} = K_1$. Applying the Horseshoe Lemma [14, Lemma 6.20], we get the following diagram for $i = n, ..., 1$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \rightarrow & H_i & \rightarrow & Q_i & \rightarrow & H_{i-1} & \rightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \rightarrow & P_{i,0} & \rightarrow & P_{i,0} \oplus P_{i-1,0} & \rightarrow & P_{i-1,0} & \rightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \rightarrow & K_{i,1} & \rightarrow & Q'_i & \rightarrow & K_{i-1,1} & \rightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Assembling these diagrams we get the following diagram :

$$\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \\
0 & \rightarrow & M & \rightarrow & Q_n & \rightarrow & \cdots & \rightarrow & Q_1 & \rightarrow & M & \rightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \\
0 & \rightarrow & P_0 & \rightarrow & P_0 \oplus P_{n-1,0} & \rightarrow & \cdots & \rightarrow & P_{1,0} \oplus P_0 & \rightarrow & P_0 & \rightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \\
0 & \rightarrow & K_1 & \rightarrow & Q'_n & \rightarrow & \cdots & \rightarrow & Q'_1 & \rightarrow & K_1 & \rightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & & & 0 & & 0 & &
\end{array}$$

It is easy to show that $\mathrm{pd}(Q'_i) \leq m-1$ for $1 \leq i \leq n$. Hence, the bottom exact sequence of the diagram is the desired sequence. Therefore, K_1 is $(n, m-1)$-SG-projective.

Then, by induction and using the same arguments above, we get that K_i is $(n, m-i)$-SG-projective for $i = 1, ..., m$. Particularly, K_m is $(n, 0)$-SG-projective, then Gorenstein projective (from [4, Proposition 2.5]), and so $\mathrm{Gpd}(M) = k \leq m$ for some positive integer k.

3. Now, we prove that any i^{th} syzygy of M is $(n, 0)$-SG-projective for $i \geq k$. Consider first K_k: a k^{th} syzygy of M. Since K_k is Gorenstein projective, we can chose a projective resolution of K_k as a left half of any of its complete projective resolution, and so we get an exact sequence

$$0 \rightarrow K'_{m-k} \rightarrow F_{m-k-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow K_k \rightarrow 0,$$

where $K'_{m-k} = \mathrm{Im}(F_{m-k} \to F_{m-k-1})$, such that $\mathrm{Hom}(-,Q)$ leaves this sequence exact whenever Q is a projective module. From the first part of the proof, K'_{m-k} is $(n,0)$-SG-projective (since it is an m^{th} syzygy of M). Then, dually to the first part of the proof, the dual version of the Horseshoe Lemma [12, Lemma 1.7] gives a raise to an exact sequence of modules of the form:

$$0 \to K_k \to L_n \to \cdots \to L_1 \to K_k \to 0,$$

where L_i is projective for $1 \leq i \leq n$. Then, with the fact that $\mathrm{Ext}^i(K_k,Q) = 0$ for any $i > 0$ and for any projective module Q (since K_k is Gorenstein projective and by [12, Proposition 2.3]), we deduce that K_k is $(n,0)$-SG-projective. Therefore, from Lemma 1.2 with [13, Theorem 3.14], we show that any i^{th} syzygy K_i of M is $(n,0)$-SG-projective for $i \geq k$. □

It is natural to ask for the converse of Theorem 2.4. Namely, we ask: if an i^{th} syzygy of a module M is (n,m)-SG-projective, is M an $(n,m+i)$-SG-projective module? In the second main result, we give an affirmative answer when $n = 1$. For that, we need the following two lemmas, which are of independent interest.

The first one gives a situation in which a direct summand of an (n,m)-SG-projective module is (n,m)-SG-projective.

Lemma 2.5. *Let M and N be two modules such that $M \oplus P \cong N \oplus Q$ for some modules P and Q with finite projective dimension. Then, for two integers $n \geq 1$ and $m \geq \max\{\mathrm{pd}(P), \mathrm{pd}(Q)\}$, M is (n,m)-SG-projective if and only if N is (n,m)-SG-projective.*

Proof. By symmetry, we only need to prove the direct implication. The proof is analogous to the one of [13, Theorem 3.14]. For completeness, we give a proof here. Since M is (n,m)-SG-projective, the direct sum $M \oplus P \cong N \oplus Q$ is also (n,m)-SG-projective (by Proposition 2.3). Then, there exists for $H = N \oplus Q$ an exact sequence of modules,

$$0 \to H \to Q_n \to \cdots \to Q_1 \to H \to 0,$$

where $\mathrm{pd}(Q_i) \leq m$ for $1 \leq i \leq n$, such that $\mathrm{Ext}^i(H,L) = 0$ for any $i > m$ and for any projective module L. Then, from [14, Theorem 7.13], $\mathrm{Ext}^i(N,L) = 0$ for any $i > m$ and for any projective module L. Now, we have to construct the exact sequence associated to N. Decomposing the above sequence into three exact sequences:

$$0 \to H \to Q_n \to E \to 0,$$
$$0 \to E \to Q_{n-1} \to \cdots \to Q_2 \to F \to 0, \quad \text{and}$$
$$0 \to F \to Q_1 \to H \to 0$$

Using the first and the last short exact sequences above with, respectively, the trivial sequences $0 \to Q \to H \to N \to 0$ and $0 \to N \to H \to Q \to 0$, we get, respectively, the following pushout and pullback diagrams:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q & \longrightarrow & H & \longrightarrow & N & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q & \longrightarrow & Q_n & \dashrightarrow & G_n & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & E & = & E & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & F & = & F & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & G_1 & \dashrightarrow & Q_1 & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & H & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

From Theorem 2.4, H, E, and F have Gorenstein projective dimensions at most m. Then, from the diagrams above, G_1 and G_n have finite Gorenstein projective dimensions which are, by standard arguments, at most m. But, from the middle sequence of each diagram, G_1 and G_n have finite projective dimensions. Then, from [12, Proposition 2.27], $\mathrm{pd}(G_1) = \mathrm{Gpd}(G_1) \leq m$ and $\mathrm{pd}(G_n) = \mathrm{Gpd}(G_n) \leq m$. Finally, assembling the exact sequences:

$$0 \to N \to G_n \to E \to 0,$$
$$0 \to E \to Q_{n-1} \to \cdots \to Q_2 \to F \to 0, \quad \text{and}$$
$$0 \to F \to G_1 \to N \to 0$$

we get the following exact sequence:

$$0 \to N \to G_n \to Q_{n-1} \to \cdots \to Q_2 \to G_1 \to N \to 0.$$

This completes the proof. □

Lemma 2.6. *Let M be a module and let $n \geq 1$ and $m \geq 0$ be integers. Then,*

1. *If M is both Gorenstein projective and (n, m)-SG-projective, then it is $(n, 0)$-SG-projective.*
2. *If a d^{th} syzygy of M is (n, m)-SG-projective (for $d \geq 1$), then $\mathrm{Gpd}(M) = k \leq d + m$ for some positive integer k and any i^{th} syzygy K_i of M is $(n, 0)$-SG-projective for $i \geq k$.*

Proof. 1. The proof is analogous to the last part of the proof of Theorem 2.4.

2. Since a d^{th} syzygy of M is (n,m)-SG-projective, we can show that $\mathrm{Gpd}(M) = k \leq d+m$ for some positive integer k. Then, there exists an exact sequence of modules,

$$0 \to K_k \to P_{k-1} \to \cdots \to P_0 \to M \to 0,$$

where P_i is projective for $i = 0, ..., k-1$, and K_k is Gorenstein projective. Consider a projective resolution of K_k which is extracted from a left half of one of its complete projective resolutions:

$$0 \to K_d \to Q_{d-1} \to \cdots Q_{k+1} \to Q_k \to K_k \to 0,$$

where Q_{k+i} is projective for $i = 0, ..., d-k-1$, and K_d is Gorenstein projective. Clearly, K_d is a d^{th} syzygy of M. Hence, by hypothesis, Lemma 2.5, and since any two i^{th} syzygies of M are projectively equivalent, K_d is (n,m)-SG-projective, and then, from (1), it is $(n,0)$-SG-projective. This implies, by Lemma 1.2, that every $\mathrm{Im}(Q_i \to Q_{i-1})$ is $(n,0)$-SG-projective for $i \geq k+1$. Therefore, from Lemma 2.5, any i^{th} syzygy K_i of M is $(n,0)$-SG-projective for $i \geq k$. □

Now, we can prove the second main result:

Theorem 2.7. *Consider two integers $d \geq 1$ and $m \geq 0$. If a d^{th} syzygy of a module M is $(1,m)$-SG-projective, then $\mathrm{Gpd}(M) = k \leq d+m$ for some positive integer k and M is $(1,k)$-SG-projective.*

Proof. By Lemma 2.6 (2), $\mathrm{Gpd}(M) = k \leq d+m$ for some positive integer k and any i^{th} syzygy K_i of M is $(1,0)$-SG-projective for $i \geq k$. In particular, we have an exact sequence of modules,

$$0 \to K_k \to P_{k-1} \to \cdots \to P_0 \to M \to 0,$$

where P_i is projective for $i = 0, ..., k-1$, and the k^{th} syzygy K_k of M is $(1,0)$-SG-projective. Then, there exists an exact sequence of modules,

$$0 \to K_k \to P \to K_k \to 0,$$

where P is projective. Then, by [4, Proposition 2.5(1) and its proof], K_k is $(k,0)$-SG-projective such that, by assembling the short exact sequence above with itself k times, we have an exact sequence of the form $0 \to K_k \to P \to \cdots P \to K_k \to 0$. Then, using the same proof as the one of [12, Theorem 2.10], we get the following exact sequence:

$$0 \to Q_k \to Q_{k-1} \to \cdots Q_1 \to G \to M \to 0,$$

where $Q_k = P$, $Q_i = P \oplus P_{i-1}$ for $i = 1, ..., k-1$, and $G = K_k \oplus P_0$. The module $G = K_k \oplus P_0$ is $(1,0)$-SG-projective with a short exact sequence $0 \to G \to Q \to G \to 0$, where $Q = P \oplus P_0 \oplus P_0$. Then, from the Horseshoe Lemma [14, Lemma 6.20], we get the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \uparrow & & & & \uparrow & & \\
0 & \to & M & & & & M & \to & 0 \\
 & & \uparrow & & & & \uparrow & & \\
0 & \to & G & \to & Q & \to & G & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & Q_1 & \to & Q_1 \oplus Q_1 & \to & Q_1 & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & Q_{k-1} & \to & Q_{k-1} \oplus Q_{k-1} & \to & Q_{k-1} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & Q_k & \to & Q'_k & \to & Q_k & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Since Q_k is projective, Q'_k is projective. Putting the cokernel into this diagram, we obtain an exact sequence $0 \to M \to P' \to M \to 0$ such that, by the middle exact sequence, $\mathrm{pd}(P') \leq k$. Therefore, M is $(1,k)$-SG-projective. □

As consequences of the two main results, we get some results on modules with finite Gorenstein projective dimension.

The first one extends the role of strongly Gorenstein projective modules (i.e., $(1,0)$-SG-projective modules), which serve to characterize Gorenstein projective modules, to the setting of $(1,m)$-SG-projective modules as follows:

Corollary 2.8. *Let M be a module and let m be a positive integer. Then,* $\mathrm{Gpd}(M) \leq m$ *if and only if there exists a Gorenstein projective module G such that the direct sum $M \oplus G$ is $(1,m)$-SG-projective.*

Proof. $\Leftarrow$. Follows from Theorem 2.4(1) and [12, Proposition 2.19].

$\Rightarrow$. Since $\mathrm{Gpd}(M) \leq m$, there exists an exact sequence of modules,

$$(*) \quad 0 \to K_m \to P_{m-1} \to \cdots \to P_0 \to M \to 0,$$

where P_i is projective for $i = 0, ..., m-1$ and K_m is Gorenstein projective. Then, from [3, Theorem 2.7], there exists a Gorenstein projective module G' such that $K_m \oplus G'$ is $(1, 0)$-SG-projective. From the right half of a complete projective resolution of G', we get an exact sequence,

$$0 \to G' \to Q_{m-1} \to \cdots \to Q_0 \to G \to 0,$$

where Q_i is projective for $i = 0, ..., m-1$ and G is Gorenstein projective. Adding this sequence with the sequence $(*)$, we get the following exact sequence

$$0 \to K_m \oplus G' \to P_{m-1} \oplus Q_{m-1} \to \cdots \to P_0 \oplus Q_0 \to M \oplus G \to 0.$$

This means that the m^{th} syzygy $K_m \oplus G'$ of $M \oplus G$ is $(1, 0)$-SG-projective. Therefore, from Theorem 2.7, $M \oplus G$ is $(1, m)$-SG-projective. □

The second corollary investigates the relation between (n, m)-SG-projective modules and the usual projective dimension. It is known, for a module M, that $\mathrm{Gpd}(M) \leq \mathrm{pd}(M)$ with equality if $\mathrm{pd}(M) < \infty$. For (n, m)-SG-projective modules we have the following result, which is an extension of [4, Proposition 2.2]:

Corollary 2.9. *Let M be an (n, m)-SG-projective module for some integers $n \geq 1$ and $m \geq 0$. Then,* $\mathrm{pd}(M) < \infty$ *if and only if* $\mathrm{fd}(M) < \infty$.

Proof. We only need to proof the converse implication. Assume that $\mathrm{fd}(M) < \infty$, then so every syzygy of M has finite flat dimension. From Theorem 2.4, an m^{th} syzygy of M is $(n, 0)$-SG-projective, and so it is projective from [4, Proposition 2.2]. This implies that $\mathrm{pd}(M) < \infty$, as desired. □

The above result leads us to conjecture that every module of finite Gorenstein projective dimension has finite projective dimension if it has finite flat dimension. From [4, Corollary 2.3], we have an affirmative answer over rings with finite weak global dimension. In the following result, we give an affirmative answer in a more general context. Recall that the left finitistic flat dimension of R is the quantity $l.\mathrm{FFD}(R) = \sup\{\mathrm{fd}_R(M) \mid M \textit{ is an } R\textit{-module with } \ \mathrm{fd}_R(M) < \infty\}$.

Proposition 2.10. *If $l.\mathrm{FFD}(R) < \infty$, then every module with both finite Gorenstein projective dimension and finite flat dimension has finite projective dimension.*

Proof. Assume that $l.\mathrm{FFD}(R) = n$ for some positive integer n. Let M be a module such that $\mathrm{fd}(M) < \infty$ and $\mathrm{Gpd}(M) = k < \infty$. To see that $\mathrm{pd}(M) < \infty$, it is sufficient, from Corollary 2.8 and its proof, to show that $K_m \oplus G'$ is projective (we use the notation of Corollary 2.8 and its proof). From the proof of [3, Theorem

2.7], $K_m \oplus G'$ can be considered as the direct sum of all the images of a complete projective resolution of K_m. Now, since $\mathrm{fd}(K_m) \leq n$ (since $\mathrm{fd}(M) < \infty$), all the images of this complete projective resolution have finite flat dimension, which is at most n (since $l.\mathrm{FFD}(R) = n$). This implies that $\mathrm{fd}(K_m \oplus G') \leq n$. Therefore, from [4, Proposition 2.2], $\mathrm{pd}(K_m \oplus G') < \infty$, as desired. □

Finally, it is convenient to note that one could define and study (n, m)-SG-injective modules as a dual notion to the current one of (n, m)-SG-projective modules. Then, every result established here for (n, m)-SG-projective modules, except Corollary 2.9 and Proposition 2.10, has a dual version for (n, m)-SG-injective modules.

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