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ROTA-BAXTER CATEGORIES

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Abstract. We introduce Rota-Baxter categories and construct examples of such structures.

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1. Introduction

This work takes part in the efforts to understand the categorification of rings and other related algebraic structures. The idea of categorification of algebraic structures has been around for several decades and has gradually become better appreciated and understood. The expanding scope and applications of the notion of categorification has been greatly influenced by the works of Baez-Dolan [2,3], Crane-Frenkel [10], Crane-Yetter [11], Khovanov [24], among others. The basic idea is that it is worthwhile to look at the categorical foundations of set theoretical structures. Often sets arise as the equivalences classes of objects in a category. Going from a category to the set of equivalences classes of its objects is the process of decategorification. Categorification goes in the reverse direction, uncovering categories whose set of equivalences classes of objects reproduces a given set. Categorifications always exist but are no unique. Thus two general problems arise: the classification of categorifications and the extraction of information regarding a given set theoretical construction from its categorical counterpart.

Our approach to the categorification of rings, reviewed in Section 2, was first discussed in [17] with a view towards the categorification of the ring of functions on non-commutative spaces and the categorification of the algebra of annihilation and creation operators. Further developments aimed at the elaboration of a general setting for the study of the combinatorial properties of rational numbers were reported in [6,7], where the combinatorics of Bernoulli numbers and hypergeometric functions, respectively, are discussed. Our aim in this work is to study the categorification of Rota-Baxter rings [26], an algebraic structure under current active research because of its capability to unify notions coming from probability theory, combinatorics, symmetric functions, renormalization of Feynman integrals, among others. Applications of Rota-Baxter categories in the context of renormalization of Feynman integrals will be study in the forthcoming works [8,14].

This paper is organized as follows. In Section 2 we introduce Rota-Baxter categories and provide a couple of basic examples. In Section 3 we provided the simplest and most ubiquitous example of Rota-Baxter category. This sort of Rota-Baxter category are constructed from any distributive category and may be thought as categorifications of the Rota-Baxter ring of formal Laurent series. In Section 4 we construct Rota-Baxter categories associated with an arbitrary comonoidal category and a given Rota-Baxter category. This construction should be thought as the categorification of the ring of formal power series with coefficients in a Rota-Baxter ring. In Sections 5 and 6 we construct Rota-Baxter categories from idempotent and arbitrary bimonoidal functors, respectively. In Sections 7 and 8 we define categorical integration and show in three different contexts, categorical Riemannian integration, discrete analogues of integration and categorical Jackson integrals from q-calculus, that functorial integration provides examples of (twisted) Rota-Baxter categories of various weights. In Section 9 and 10 we construct Rota-Baxter categories naturally arising from classical and quantum field theory, respectively.

2. Categorification of rings

We assume the reader to be familiar with basic notions of category theory [25]. Let us begin recalling the notion of categorification of rings and semi-rings from [7,17].

Definition 2.1. A category C is *distributive* if it is equipped with functors $\oplus\colon C\times C\rightarrow C$ and $\otimes\colon C\times C\rightarrow C$ called sum and product, respectively; There are distinguished objects 0 and 1 in C; $(C,\oplus,0)$ is a symmetric monoidal category with unit 0; $(C,\otimes,1)$ is a monoidal category with unit 1; There are natural isomorphisms

$$x\otimes(y\oplus z)\simeq(x\otimes y)\oplus(x\otimes z)\quad\text{and}\quad(x\oplus y)\otimes z\simeq(x\otimes z)\oplus(y\otimes z),$$

for x,y,z objects of C. A distributive category have negative objects if it comes with a functor $-\colon C\rightarrow C$ and for x,y objects of C there are natural isomorphisms

$$-(x\oplus y)\simeq -x\oplus -y,\quad -0\simeq 0,\quad\text{and}\quad -(-x)\simeq x.$$

Coherence theorems for distributive categories were studied by Laplaza [23]. An interesting research problem is to find coherence theorems for distributive categories with negative objects.

Definition 2.2. Let C be a distributive category. A functor $P : C \to C$ is *additive* if for x, y objects of C there are natural isomorphisms $P(x \oplus y) \simeq P(x) \oplus P(y)$. If C has a negative functor we also demand the existence of natural isomorphisms $P(-x) \simeq -P(x)$. P is *bimonoidal* if it is additive and in addition there are natural isomorphisms $P(x \otimes y) \simeq P(x) \otimes P(y)$.

Definition 2.3. A *categorification* of a ring R is a distributive category C with negative functor together with a valuation map $|\ \ |: Ob(C) \to R$ such that:

$$|x| = |y| \text{ if } \quad x \simeq y, \quad |x \oplus y| = |x| + |y|, \quad |x \otimes y| = |x||y|,$$

$$|1| = 1, \quad |0| = 0, \quad \text{and} \quad |-x| = -|x|.$$

If we omit the existence of the negative functor in the definition above we arrive to the notion of categorification of semi-rings, which will be used quite often in this work. Next we introduce the main concept of this work, the notion of Rota-Baxter categories. A Rota-Baxter ring, see [26] and the references therein, is a triple (R, λ, p) where R is a ring, $\lambda \in \{-1, 0, 1\}$, and $p\colon R \longrightarrow R$ is a morphism of abelian groups satisfying:

$$p(x)p(y) = p(xp(y)) + p(p(x)y) + \lambda p(xy).$$

R may or may not have a unit, and may or may not be commutative. Notice that the notion of Rota-Baxter semi-ring makes perfect sense; for $\lambda = -1$, the required identity is

$$p(x)p(y) + p(xy) = p(xp(y)) + p(p(x)y).$$

Definition 2.4. A *Rota-Baxter category* of weight $\lambda \in \{-1, 0, 1\}$ is a distributive category C together with an additive functor $P : C \longrightarrow C$ and natural isomorphisms

$$\begin{aligned} P(x) \otimes P(y) \oplus P(x \otimes y) &\simeq P(P(x) \otimes y) \oplus P(x \otimes P(y)) \\ P(x) \otimes P(y) &\simeq P(P(x) \otimes y) \oplus P(x \otimes P(y)) \\ P(x) \otimes P(y) &\simeq P(P(x) \otimes y) \oplus P(x \otimes P(y)) \oplus P(x \otimes y). \end{aligned}$$

for x, y objects of C and $\lambda = -1, 0, 1$, respectively.

Definition 2.5. *A categorification of a Rota-Baxter ring* (R,p) is a Rota-Baxter category (C,P) together with a valuation map $|\ \ |: Ob(C) \to R$ such that $|P(x)| = p(|x|)$ for x object of C.

A Rota-Baxter ring may be regarded as a Rota-Baxter category and, as such, it is a categorification of itself. In the next sections the reader will find interesting examples of Rota-Baxter categories making this notion worth studying; we begin pointing out a couple of simple but useful examples. Any ring may be regarded as a Rota-Baxter ring with vanishing p. Any abelian group R provided with a group morphisms $P : R \to R$ may be regarded as a Rota-Baxter ring with multiplication constantly equal to zero. The analogues of these simple facts hold in the categorical context as well.

Proposition 2.6. *A distributive category may be regarded as a Rota-Baxter category with functor P constantly equal to zero. A distributive category may be regarded as a Rota-Baxter category of weight -1 with P equal to the identity functor. A symmetric monoidal category C together with an additive functor $P : C \to C$ may be regarded as a Rota-Baxter category with $\otimes$ constantly equal to zero.*

Rota-Baxter rings with vanishing p play a fundamental role in the theory of renormalization as formalized by Connes and Kreimer [13]. Rota-Baxter rings with vanishing product, though less studied, should not be overlooked. If (C,P) is a Rota-Baxter category then we let $Ker(P)$ be the full subcategory of C such that c is an object of $Ker(P)$ iff $P(c) \simeq 0$. Similarly, let $Im(P)$ be the full subcategory of C whose objects are isomorphic to objects of the form $P(c)$ for some $c \in Ob(C)$. The axioms for Rota-Baxter categories imply the following result.

Proposition 2.7. $(Ker(P),0)$ *is a Rota-Baxter category.* $(Im(P),I)$ *is a Rota-Baxter category.*

3. Main examples

The reason why the examples considered in this section are Rota-Baxter categories is succinctly encoded in the identity between sets with multiplicities shown in Figure 1. Let us start with the simplest and most prominent example. Let $\mathbb{Z}$-vect$_b$ be the category of $\mathbb{Z}$-graded vector spaces $V = \bigoplus V_n$ such that: V_n is finite dimensional and there exists $k \leq 0$ such that $V_n = 0$ for $n \leq k$. $\mathbb{Z}$-vect$_b$ is a distributive category with direct sums and tensor products given as usual by

$$(V \oplus W)_n = (V_n \oplus W_n) \quad \text{and} \quad (V \otimes W)_n = \bigoplus_{k+l=n} (V_k \otimes W_l).$$

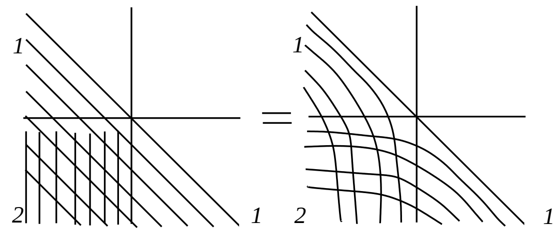

FIGURE 1. Geometric meaning of the Rota-Baxter identity.

Let $\mathbb{N}[z^{-1}, z]]$ be the semi-ring of formal Laurent series with integral coefficients, it is known that the map $p : \mathbb{N}[z^{-1}, z]] \to \mathbb{N}[z^{-1}]$ turns $\mathbb{N}[z^{-1}, z]]$ into a Rota-Baxter semi-ring of weight -1.

Proposition 3.1. *The functor* $P\colon \mathbb{Z}\text{-vect}_b \to \mathbb{Z}\text{-vect}_b$ *given for* $k < 0$ *by*

$$P\Big(\bigoplus_{k\leq n} V_n\Big) = \bigoplus_{k\leq n<0} V_n,$$

turns $\mathbb{Z}\text{-vect}_b$ *into a Rota-Baxter category of weight* -1. $\mathbb{Z}\text{-vect}_b$ *is a categorification of* $\mathbb{N}[z^{-1}, z]]$ *with valuation map* $|\ \ | : \mathbb{Z}\text{-vect}_b \to \mathbb{N}[z^{-1}, z]]$ *given by* $|V| = \sum_n dim(V_n) z^n$.

Proposition 3.1 is an instance of a general construction of Rota-Baxter categories to be developed presently. For a distributive category C let $C_b^{\mathbb{Z}}$ be the category whose objects are maps $f : \mathbb{Z} \to C$, such that there exists $k \leq 0$ with $f(n) \simeq 0$ for $n < k$. Morphisms in $C_b^{\mathbb{Z}}$ are given by

$$C_b^{\mathbb{Z}}(f, g) = \prod_{n\in\mathbb{Z}} C(f(n), g(n)).$$

The category $C_b^{\mathbb{Z}}$ is distributive category with $\oplus$, $\otimes$, and negative functor given by:

$$(f \oplus g)(n) = f(n) \oplus g(n), \quad (f \otimes g)(n) = \bigoplus_{k+l=n} f(k) \otimes g(l), \quad \text{and} \quad (-f)(n) = -f(n),$$

where f, g belong to $C_b^{\mathbb{Z}}$ and $k, l, n \in \mathbb{Z}$.

Theorem 3.2. *The category* $C_b^{\mathbb{Z}}$ *together with the functor* $P : C_b^{\mathbb{Z}} \to C_b^{\mathbb{Z}}$ *given by* $P(f) = f_{<0}$ *where*

$$f_{<0}(n) = \begin{cases} 0 & \text{if } n \geq 0 \\ f(n) & \text{if } n < 0 \end{cases}$$

is a Rota-Baxter category of weight -1*. If* C *is a categorification of* R*, then* $C_b^{\mathbb{Z}}$ *is a categorification of* $R[z^{-1}, z]]$.

Proof. For f and g objects of $C_b^{\mathbb{Z}}$, we have that:

$$\begin{aligned} P(f\otimes g) &= (f\otimes g)_{<0} \\ P(f)\otimes P(g) &= f_{<0}\otimes g_{<0} \\ P(f\otimes P(g)) &= (f\otimes g_{<0})_{<0} \\ P(P(f)\otimes g) &= (f_{<0}\otimes g)_{<0}. \end{aligned}$$

We have to check that there are canonical isomorphisms

$$(f\otimes g)_{<0}\oplus f_{<0}\otimes g_{<0}\simeq (f_{<0}\otimes g)_{<0}\oplus (f\otimes g_{<0})_{<0},$$

which we do evaluating both sides at $n\in\mathbb{Z}$. If $n\geq 0$ we obtain the identity $0\oplus 0=0\oplus 0$. If $n<0$, then we have to show that there are canonical isomorphisms

$$\bigoplus_{k+l=n} f(k)\otimes g(l) \quad \bigoplus \bigoplus_{k+l=n,k<0,l<0} f(k)\otimes g(l)\simeq$$

$$\bigoplus_{k+l=n,k<0} f(k)\otimes g(l) \quad \bigoplus \bigoplus_{k+l=n,l<0} f(k)\otimes g(l),$$

which is clear. This proves the first statement of the theorem. For the second statement consider the valuation map $|\ \ |: C_b^{\mathbb{Z}}\rightarrow R[z^{-1},z]]$ given by

$$|f|=\sum_{n\in\mathbb{Z}}|f(n)|z^n$$

satisfies all required axioms. □

Proposition 3.1 is obtained from Theorem 3.2 letting C be *vect* the category of finite dimensional vector spaces with valuation map $|V|=dim(V)$.

Let $\mathbb{Z}$-*set* be the category of $\mathbb{Z}$-graded finite sets, i.e. an object of $\mathbb{Z}$-*set* is a pair (x,f) where x is a finite set and $f:x\rightarrow\mathbb{Z}$ is a map. Morphisms in $\mathbb{Z}$-*set* from (x,f) to (y,g) are maps $\alpha:x\rightarrow y$ such that $g\circ\alpha=f$. Disjoint union and Cartesian product are given, respectively, by

$$(x,f)\sqcup(y,g)=(x\sqcup y,f\sqcup g) \text{ and } (x,f)\times(y,g)=(x\times y,f\circ\pi_x+g\circ\pi_y),$$

where π_x and π_y are the canonical projections of $x\times y$ onto x and y, respectively. Consider the functor $P:\mathbb{Z}\text{-}set\longrightarrow\mathbb{Z}\text{-}set$ given by

$$P(x,f)=(f^{-1}(-\infty,0),f|_{f^{-1}(-\infty,0)}).$$

Proposition 3.3. *$(\mathbb{Z}$-set$,P)$ is a Rota-Baxter category of weight -1. $\mathbb{Z}$-set is a categorification of* $\mathbb{N}[z^{-1},z]$.

Proof. The result follows from Theorem 3.2 since there is a natural functor $i:\mathbb{Z}$-$set \to set_b^{\mathbb{Z}}$ which exhibits $\mathbb{Z}$-set as a full subcategory of $set_b^{\mathbb{Z}}$ closed under sum, product and P. The valuation map on $set_b^{\mathbb{Z}}$, induced from the valuation map on set sending x into its cardinality $|x|$, restricts to a valuation map on $\mathbb{Z}$-set. □

4. Comonoidal categories

If C is a co-ring, i.e., an abelian group provided with a co-product, and R is a Rota-Baxter ring, then the set $Hom(C,R)$ of morphisms of abelian groups from C to R is Rota-Baxter ring with product $fg(x) = (f \otimes g)\Delta(x)$ and operator p given by $p(f)(c) = p(f(c))$ for $f \in Hom(C,R)$ and $c \in C$. We proceed to state the corresponding facts for Rota-Baxter categories.

Let Cat be the category whose objects are essentially small categories, morphisms in Cat are functors. We a define a functor $\otimes : Cat \times Cat \longrightarrow Cat$ as follows. The tensor product category $C \otimes D$ of the categories C and D has as objects triples (x,f,g) where x is a finite set, $f : x \to Ob(C)$ and $g : x \to Ob(D)$ are maps. Morphisms from (x_1,f_1,g_1) to (x_2,f_2,g_2), objects of $C \otimes D$, are given by

$$C \otimes D((x_1,f_1,g_1),(x_2,f_2,g_2)) = \bigsqcup_{\alpha:x_1\to x_2} \prod_{i\in x} C(f_1(i), f_2(\alpha(i))) \times D(g_1(i), g_2(\alpha(i))),$$

where $\alpha : x_1 \to x_2$ is an arbitrary bijection.

The following definition formalizes the categorical analogue of the notion of a co-ring without co-unit.

Definition 4.1. A category D is *comonoidal* if it comes equipped with a functor $\delta : D \longrightarrow D \otimes D$ such that there is a natural isomorphisms $(\delta \otimes 1_D)\delta \longrightarrow (1_D \otimes \delta)\delta$ satisfying Mac Lane's pentagon axiom.

Theorem 4.2. *Suppose D with a functor $\delta : D \longrightarrow D\otimes D$ is a comonoidal category and C a Rota-Baxter category. Then C^D, the category of functors from D to C, is a Rota-Baxter category with functor P given by $P(F)(x) = P(F(x))$.*

Proof. First we show that C^D is distributive. Define sum and product by

$$(F+G)(x) = F(x) \oplus G(x)$$

$$(FG)(x) = \sum_{d\delta(x)} F(\delta_1(x)) \otimes G(\delta_2(x)),$$

where $\delta(x) : d\delta(x) \longrightarrow Ob(D) \otimes Ob(D)$, where $d\delta(x)$ is the domain of $\delta(x)$. The negative functor is $(-F)(x) = -F(x)$. Next assume C is a Rota-Baxter category of weight -1, the other cases being similar. The desired result follows from the natural isomorphisms

$$P(F)\otimes P(G)(x)\oplus P(F\otimes G)(x) \simeq \bigoplus_{d\delta(x)} P(F(\delta_1(x))\otimes P(G(\delta_2(x)))\oplus P(F(\delta_1(x))\otimes G(\delta_2(x))),$$

$$P(P(F)\otimes G)(x)\oplus P(F\otimes P(G)) \simeq \bigoplus_{d\delta(x)} P(F(\delta_1(x)))\otimes G(\delta_2(x))\oplus F(\delta_1(x))\otimes P(G(\delta_2(x))).$$

□

Given a positive integer n we use the notation $[n] = \{1, 2, ..., n\}$. A category D may be regarded as a comonoidal category with functor $\delta : D \longrightarrow D \otimes D$ sending an object x in D to the map $\delta(x) : [1] \to D \times D$ such that $\delta(x)(1) = (x, x)$. This canonical comonoidal structure induces the monoidal structure on C^D given by $FG(x) = F(x)G(x)$.

Corollary 4.3. *Assume C is a Rota-Baxter category. Then C^D is a Rota-Baxter category with functor $P : C^D \to C^D$ given by $P(F)(x) = P(F(x))$, and the product of functors given by $FG(x) = F(x)G(x)$.*

Let us consider a rather simple example of comonoidal category. Recall that a set x may be regarded as the category with objects x and identity morphisms only. The category $[n] \times [n]$ is comonoidal with functor $\delta : [n] \times [n] \longrightarrow ([n] \times [n]) \otimes ([n] \times [n])$ such that $\delta(i, j)$ is the map

$$\delta(i, j) : [n] \longrightarrow ([n] \times [n]) \otimes ([n] \times [n])$$

given by

$$\delta(i, j)(k) = ((i, k), (k, j)).$$

If C is a distributive category then $M_n(C) = C^{[n]\times[n]}$, the category of $n\times n$ matrices with values in C, is also a distributive category. Concretely, an object A in $M_n(C)$ is a family A_{ij} of objects in C, for $1 \leq i, j \leq n$. Sum and product of objects in $M_n(C)$ are given, respectively, by

$$(A \oplus B)_{ij} = A_{ij} \oplus B_{ij} \quad \text{and} \quad (A \otimes B)_{ij} = \bigoplus_k (A_{ik} \otimes B_{kj}),$$

for $A, B \in M_n(C)$.

Corollary 4.4. *If C is a distributive category then $M_n(C)$ is a distributive category. If C is a Rota-Baxter category, then $M_n(C)$ is a Rota-Baxter category. If C is a categorification of R, then $M_n(C)$ is a categorification of $M_n(R)$.*

For our next constructions we need the theory of species introduced by Joyal [22] and further elaborated by Bergeron, Labelle, and Leroux [4]. Let $\mathbb{B}^n$ be the category whose objects are pairs (x, f) where x is a finite set and $f : x \to [n]$ is any map. Morphisms in $\mathbb{B}^n$ from (x, f) to (y, g) are maps $\alpha : x \to y$ such that $g \circ \alpha = f$. For a distributive category C we let $C^{\mathbb{B}^n}$, the category of C-species in n variables, be the category of functors from $\mathbb{B}^n$ to C.

The category $\mathbb{B}^n$ is comonoidal with functor $\delta : \mathbb{B}^n \to \mathbb{B}^n \otimes \mathbb{B}^n$ given on (x, f) in $\mathbb{B}^n$ by the map

$$\delta(x, f) : Par_2(x) \to Ob(\mathbb{B}^n) \otimes Ob(\mathbb{B}^n) \quad \text{such that}$$

$$\delta(x, f)(x_1, x_2) = ((x_1, f|_{x_1}), (x_2, f|_{x_2})),$$

where $Par_2(x)$ is the set of pairs (x_1, x_2) such that $x_1 \sqcup x_2 = x$. It follows that $C^{\mathbb{B}^n}$ is a distributive category with sum and product given by

$$(F+G)(x, f) = F(x, f) \sqcup G(x, f) \quad \text{and} \quad (FG)(x, f) = \bigoplus_{x_1 \sqcup x_2 = x} F(x_1, f|_{x_1}) \otimes G(x_2, f|_{x_2}).$$

If R is a ring then we let $R[[x_1, ..., x_n]]$ be the ring of formal divided power series in variables $x_1, ..., x_n$. The latter algebra is the free R-module generated by symbols:

$$\frac{x^k}{k!} \quad \text{where} \quad k \in \mathbb{N}^n, \quad x^k = x_1^{k_1} ... x_n^{k_n}, \quad \text{and} \quad k! = k_1! ... k_n!.$$

The product is defined on generators via the identity

$$\frac{x^k}{k!} \frac{x^s}{s!} = \binom{k+s}{s} \frac{x^{k+s}}{(k+s)!}, \quad \text{where} \quad \binom{k+s}{s} = \prod_{i=1}^{n} \binom{k_i + s_i}{s_i}.$$

Corollary 4.5. *If (C, P) is Rota-Baxter category of weight λ, then $C^{\mathbb{B}^n}$ is a Rota-Baxter category of weight λ with functor P given by $P(F)(x, f) = P(F(x, f))$, for F in $C^{\mathbb{B}^n}$ and (x, f) in $\mathbb{B}^n$. If C is a categorification of R, then $C^{\mathbb{B}^n}$ is a categorification of $R[[x_1, ..., x_n]]$.*

Proof. Follows from Theorem 4.2. The valuation map $|\ \ | : C^{\mathbb{B}^n} \to R[[x_1, ..., x_n]]$ is defined by

$$F = \sum_{k \in \mathbb{N}^n} F([k]) \frac{x^k}{k!} \quad \text{where} \quad [k] = ([k_1], ..., [k_n]).$$

$\square$

Next we consider non-commutative species introduced in [17]. Let $R\langle\langle x_1, \ldots, x_n\rangle\rangle$ be the ring of formal power series in non-commutative variables $x_1, \ldots, x_n$ and coefficients in R. We construct a categorification C^{L_n} of $R\langle\langle x_1, \ldots, x_n\rangle\rangle$ with the property that each valuation map $| \ |\colon Ob(C) \to R$, induces a valuation map

$$| \ |\colon Ob(C^{L_n}) \to R\langle\langle x_1, \cdots, x_n\rangle\rangle.$$

Let L_n be the category whose objects are triples $(x, <, f)$ where x is a finite set, $<$ is a linear order on x, $f\colon x \to [n]$ is a map. Morphisms from $(x, <, f)$ to $(y, <, g)$ are given by

$$L_m((x,<,f),(y,<,g)) = \{\varphi\colon x \to y \mid g \circ \varphi = f, \text{ and } \varphi(i) < \varphi(j) \ \text{ for all } \ i < j\}.$$

The disjoint union $(x_1, <_1) \sqcup (x_2, <_2)$ of linearly order sets is $(x_1 \sqcup x_2, <)$, where the order on $x_1 \sqcup x_2$ extends the order on x_1 and the order on x_2, and $i < j$ for $i \in x_1$, $j \in x_2$. An order partition in n-blocks of $(x, <)$ is a n-tuple $(x_1, <_1), \cdots, (x_n, <_n)$ of posets such that $(x_1, <) \sqcup \cdots \sqcup (x_n, <) = (x, <)$. Let $OPar_n(x, <)$ be the set of order partitions of $(x, <)$ in n blocks. L_n is comonoidal category with $\delta : L_n \to L_n \otimes L_n$ sending $(x, <, f)$ into the map

$$\delta(x,<) : OPar_2(x,<) \longrightarrow Ob(L_n) \times Ob(L_n)$$

such that

$$\delta(x,<)((x_1,<_1),(x_1,<_2)) = (x_1,<_1,f|_{x_1}),(x_1,<_2,f|_{x_2}).$$

It follows that C^{L_n} is distributive with sum

$$(F+G)(x,<,f) = F(x,<,f) \oplus G(x,<,f)$$

and product

$$(FG)(x,<,f) = \bigoplus F(x_1,<_1,f|_{x_1}) \otimes G(x_2,<_2,f|_{x_2}),$$

where the sum runs over all pairs $((x_1, <), (x_2, <_1)) \in OPar_2(x, <_2)$.

Corollary 4.6. *If C is a categorification of a Rota-Baxter ring R, then C^{L_n} is a categorification of the Rota-Baxter ring $R\langle\langle x_1, \ldots, x_n\rangle\rangle$.*

Proof. By Theorem 4.2 we only need to define the valuation map $| \ | : C^{L_n} \to R\langle\langle x_1, \ldots, x_n\rangle\rangle$ which is given by $|F| = \sum_{f:[m]\to[n]} |F([m], <, f)|x^f$, where $x_f = x_{f(1)} \ldots x_{f(m)}$. $\square$

5. Idempotent bimonoidal functors

In this section we provided a general construction which generates a wide variety of examples of Rota-Baxter categories.

Theorem 5.1. *Let C be a distributive category and $P : C \to C$ a bimonoidal functor such that there is a natural isomorphisms $P^2 \simeq P$. Then (C, P) is a Rota-Baxter category of weight -1.*

Proof. Since P is bimonoidal and $P^2 = P$ we have natural isomorphisms

$$P(P(x) \otimes y) \simeq P(x \otimes P(y)) \simeq P(x \otimes y) \simeq P(x) \otimes P(y). \qquad \square$$

Theorem 5.2. Let C_1 and C_2 be distributive categories. The product $C_1 \times C_2$ is a distributive category with sum, product, and negative functor given by $(c_1, c_2) \oplus (d_1, d_2) = (c_1 \oplus c_2, d_1 \oplus d_2)$, $(c_1, c_2) \otimes (d_1, d_2) = (c_1 \otimes c_2, d_1 \otimes d_2)$ and $-(c_1, c_2) = (-c_1, -c_2)$. The functor $\pi_1 : C_1 \times C_2 \to C_1 \times C_2$ sending (c_1, c_2) into $(c_1, 0)$ turns $C_1 \times C_2$ into a Rota-Baxter category of weight -1.

Proof. Follows from Theorem 5.1 and the fact that π_1 is bimonoidal and $\pi_1^2 = \pi_1$. Equivalently, one can check the identities

$P((c_1, c_2) \otimes (d_1, d_2)) = P(c_1, c_2) \otimes P(d_1, d_2) = P(P(c_1, c_2) \otimes (d_1, d_2)) = P((c_1, c_2) \otimes P(d_1, d_2)) = (c_1 \otimes d_1, 0)$. $\square$

Next we consider examples of Rota-Baxter categories all of which arise from Theorem 5.1. The examples will gradually become closer to geometric and topological matters. In particular, we will see that the intersection homology for stratified manifolds can be naturally recast within our settings. Given f a morphism in a category C, we let df and cf be the domain and codomain of f, respectively. If x is an object of C we let 1_x be the identity morphisms from x to itself, thus $d1_x = c1_x = x$. We let $C^{(1)}$ be the category whose objects are morphisms in C. A morphisms in $C^{(1)}$ from f to g is a pair of morphisms (α_1, α_2) in C such that $\alpha_2 \circ f = g \circ \alpha_1$. If C is a distributive category, then we have naturally induced functors $\oplus : C^{(1)} \times C^{(1)} \to C^{(1)}$ and $\otimes : C^{(1)} \times C^{(1)} \to C^{(1)}$ turning $C^{(1)}$ into a distributive category.

Theorem 5.3. *If C is a distributive category, then $C^{(1)}$ is a Rota-Baxter category of weight -1 with functor $P : C^{(1)} \to C^{(1)}$ given by $P(f) = 1_{df}$.*

Proof. Follows from the natural isomorphisms

$$P(f \otimes g) \simeq P(f) \otimes P(g) \simeq P(P(f) \otimes g) \simeq P(f \otimes P(g)) \simeq 1_{df} \otimes 1_{dg}. \qquad \square$$

Recall that a morphisms f in a category C is called injective or monic if $g \circ f = h \circ f$ implies $f = g$ for all morphisms f, g in $C^{(1)}$. Let $IC^{(1)}$ be the full subcategory of $C^{(1)}$ whose objects are injective morphisms. Assume that C is a distributive category and that the functors $\oplus$ and $\otimes$ on $C^{(1)}$ induce by restriction functors $\oplus$ and $\otimes$ on $IC^{(1)}$. The functor P on $C^{(1)}$ induces, by restriction, a functor P on $IC^{(1)}$.

Corollary 5.4. *$(IC^{(1)}, P)$ is a Rota-Baxter category of weight -1.*

Consider the category $Top_{\subseteq}$ whose objects are pairs of topological spaces (X_1, X_2) with $X_1 \subseteq X_2$. A morphism from (X_1, X_2) to (Y_1, Y_2) is a continuous map $f : X_2 \to Y_2$ such that $f(X_1) \subseteq Y_1$. Componentwise disjoint union and Cartesian product give $Top_{\subseteq}$ the structure of a distributive category. The functor $P : Top_{\subseteq} \to Top_{\subseteq}$ given by $P(X_1, X_2) = (X_1, X_1)$ is bimonoidal and idempotent $P^2 = P$.

Corollary 5.5. *$(Top_{\subseteq}, P)$ is a Rota-Baxter category of weight -1.*

Consider $Vect_{\subseteq}$ the category whose objects are pairs (V, W) where W is a vector space and V a subspace of W. Morphisms are linear transformations between the bigger spaces that preserve the given subspaces. $Vect_{\subseteq}$ is a distributive category with sum and product given by componentwise direct sum and tensor product. The functor $P : Vect_{\subseteq} \to Vect_{\subseteq}$ given by $P(V, W) = (V, V)$ is bimonoidal and satisfies $P^2 = P$.

Corollary 5.6. *$(Vect_{\subseteq}, P)$ is a Rota-Baxter category of weight -1.*

Let $vect_{\subseteq}$ be the full subcategory of $Vect_{\subseteq}$ whose objects are pairs of finite dimensional vector spaces. Let $P : vect_{\subseteq} \to vect_{\subseteq}$ be the functor given by $P(V, W) = (0, W/V)$, where W/V is the quotient vector space and 0 the vector space with one element. Again it is easy to check that P is bimonoidal and that $P^2 = P$.

Corollary 5.7. *$(vect_{\subseteq}, P)$ is a Rota-Baxter category of weight -1.*

Let $\mathbb{T}$ be the category whose objects are triples (x, y, f) where $x \subseteq y$ are sets and $f : y \to y$ is a map. Morphisms in $\mathbb{T}$ from (x_1, y_1, f_1) to (x_2, y_2, f_2) are maps $\alpha : y \to y$ such that $\alpha(x_1) \subseteq x_2$ and $\alpha \circ f_1 = f_2 \circ \alpha$. Sum and product are given by

$$(x_1, y_1, f_1) \sqcup (x_2, y_2, f_2) = (x_1 \sqcup x_2, y_1 \sqcup y_2, f_1 \sqcup f_2),$$

$$(x_1, y_1, f_1) \times (x_2, y_2, f_2) = (x_1 \times x_2, y_1 \times y_2, f_1 \times f_2).$$

The functor $P : \mathbb{T} \to \mathbb{T}$ given by $P(x, y, f) = (a, a, g)$, where $a = \{i \in x \mid f(i) \in x\}$ and $g = f|_a$, is bimonoidal and $P^2 = P$.

Corollary 5.8. *$(\mathbb{T}, P)$ is a Rota-Baxter category of weight -1.*

An interesting refinement of $\mathbb{T}$ is the category $\mathbb{T}_G$ where G is a group. Objects in $\mathbb{T}_G$ are triples (x, y, ρ) where $x \subseteq y$ are sets and $\rho : G \times x \to x$ is a group action of G on x. The distributive structure on $\mathbb{T}_G$ is defined just as for $\mathbb{T}$. The functor $P : \mathbb{T}_G \to \mathbb{T}_G$ given by $P(x, y, \rho) = (a, a, \rho|_a)$, where $a = \{i \in x \mid gi \in x \text{ for all } g \in G\}$ and $\rho|_a$ is the restriction of the action of ρ on x to a, is bimonoidal and $P^2 = P$.

Corollary 5.9. *$(\mathbb{T}_G, P)$ is a Rota-Baxter category of weight -1.*

Consider the category $Comp_{\subseteq}$ whose objects are triples (V, W, d), where $V \subseteq W$ are $\mathbb{Z}$-graded vector spaces and $\partial : W \to W$ is a degree -1 linear map such that $\partial \circ \partial = 0$. Componentwise direct sum and tensor product of $\mathbb{Z}$-graded vector spaces give $Comp_{\subseteq}$ the structure of a distributive category. The differential on the sum and tensor product is given by $d_{V \oplus W} = d_V \oplus d_W$ and $d_{V \otimes W} = d_V \otimes 1_W \oplus 1_V \otimes d_W$. The homology of (V, W, d) in $Comp_{\subseteq}$ is by definition given by

$$H((V, W, d)) = H(W, d)).$$

The functor $P : Comp_{\subseteq} \to Comp_{\subseteq}$ given by $P(V, W, d) = (Z, Z, d_Z)$, where for $i \in \mathbb{Z}$ we set

$$Z_i = \{v \in V_i \mid dv \in V_{i-1}\}$$

and d_Z is the restriction of d to Z, is bimonoidal and idempotent $P^2 = P$.

Corollary 5.10. *$(Comp_{\subseteq}, P)$ is a Rota-Baxter category of weight -1.*

Let us point out the relation between the Corollary 5.10 above and the notion of intersection homology introduced by Goresky and McPhearson [21]. In a nutshell the construction of intersection homology may be summarized as follows. Fix a map $p : \mathbb{Z}^{\geq 1} \to \mathbb{Z}$, called the perversity, satisfying

$$p(k) \leq p(k+1) \leq p(k) + 1 \quad \text{and} \quad p(1) = p(2) = 0.$$

Construct functor $C^p : sman \to Comp_{\subseteq}$. An object in *sman* the groupoid of stratified manifolds of dimension n is a topological space X together with a filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq ... \subseteq X_n = X$$

such that $X_j \setminus X_{j-1}$ with the induced topology is a smooth manifold of dimension j. Morphisms are homeomorphisms which are smooth when restricted to the smooth pieces $X_j \setminus X_{j-1}$.

Given a stratified manifold and a perversity p, let $(C^p(X), C(X))$ be the object of $Comp_{\subseteq}$ where $C(X)$ is the complex of chains on X, i.e. $C_i(X)$ is the space of continuous maps from Δ_i into X, and $C_i^p(X)$ is the subspace of $C_i(X)$ generated by the "allowed chains" of dimension i, i.e. the space generated by chains $c : \Delta_i \longrightarrow X$ such that $c^{-1}(X_j \setminus X_{j-1})$, for $j < n$, is included in the union of sub-simplices of Δ_i of dimension less or equal to $i+j-n+p(j)$. The intersection homology $IH^p(X)$ with perversity p of a stratified manifold X is defined to be the homology of $P(C^p(X))$, i.e. $IH^p(X) = H(P(C^p(X)))$. The discovery of the intersection homology for stratified manifolds has been regarded as one of the greatest achievements of twenty century mathematics. One may only wonder at the fact that a Rota-Baxter functor was already lurking around its very definition.

6. Bimonoidal functors

In this section we assume that C is a distributive category with infinite sums and infinite distributivity.

Theorem 6.1. *Let $F\colon C \to C$ be a bimonoidal functor. The functor $P : C \to C$ given by*
$$P(x) = \bigoplus_{n=0}^{\infty} F^n(x)$$
makes C a Rota-Baxter category of weight -1.

Proof. The desired result follows from the following natural isomorphisms
$$P(x) \otimes P(y) \simeq \bigoplus_{n,m=0}^{\infty} (F^n(x) \otimes F^m(y)), \quad P(x \otimes y) \simeq \bigoplus_{n=0}^{\infty} (F^n(x) \otimes F^n(y)),$$
$$P(x \otimes P(y)) \simeq \bigoplus_{0=n\leq m}^{\infty} (F^n(x) \otimes F^m(y)), \quad P(P(x) \otimes y) \simeq \bigoplus_{0=m\leq n}^{\infty} (F^n(x) \otimes F^m(y)).$$
□

Corollary 6.2. *Fix species F_i in $C_0^{\mathbb{B}^n}$ for $1 \leq i \leq n$. The functor $(F_1, ..., F_n) : C_0^{\mathbb{B}^n} \longrightarrow C_0^{\mathbb{B}^n}$ given by $(F_1, ..., F_n)(F) = F \circ (F_1, ..., F_n)$ is bimonoidal. The functor $P = P(F_1, ..., F_n) : C_0^{\mathbb{B}^n} \longrightarrow C_0^{\mathbb{B}^n}$ given by*
$$P(F) = \sum_{m=0}^{\infty} F \circ (F_1, \cdots, F_m)^{\circ m}$$
gives $C_0^{\mathbb{B}^n}$ the structure of a Rota-Baxter category of weight -1.

Let $C_0^{L_n}$ be the full subcategory of C^{L_n} whose objects are functors F such that $F(\emptyset) = 0 \in Ob(C)$. Let $F, F_1, \cdots, F_n$ be non-commutative species in $C_0^{L_n}$ and $(x, <, f)$ an object of L_n. The composition or substitution of non-commutative species is given by

$$F \circ (F_1, \cdots, F_n)(x, <, f) = \bigoplus_{p,g} F(p, <_p, g) \otimes \bigotimes_{B \in p} F_{p(B)}(B, <_B, f|_B),$$

where the sum runs over $p \in \mathrm{OPar}(x, <)$ and $g \colon p \to [n]$.

Corollary 6.3. *Fix species F_i in $C_0^{L^n}$ for $1 \le i \le n$. The functor $(F_1, ..., F_n)$: $C_0^{L^n} \longrightarrow C_0^{L^n}$ given by*

$$(F_1, ..., F_n)(F) = F \circ (F_1, ..., F_n)$$

is bimonoidal. The functor $P = P(F_1, ..., F_n) : C_0^{L^n} \longrightarrow C_0^{L^n}$ given by

$$P(F) = \sum_{m=0}^{\infty} F \circ (F_1, \cdots, F_m)^{\circ m}$$

gives $C_0^{L^n}$ the structure of a Rota-Baxter category of weight -1.

Let us close this section with an example of Rota-Baxter category related with q-calculus. For a nice introduction to q-calculus the reader may consult [13]. Recent results on q-calculus related with Gaussian and Feynman integration are given in [15,16,19]. In Section 8 we discuss further applications to q-calculus. Consider the ring $R[[x, q]]$ of formal power series in variables x, q with coefficients in R. A fundamental role in q-calculus is play by the shift operator

$$s : R[[x, q]] \longrightarrow R[[x, q]]$$

given by

$$s(f)(x, q) = f(qx, q)$$

for $f \in R[[x, q]]$. Suppose C is a categorification of R, then $C^{\mathbb{B}^2}$ is a categorification of $R[[x, q]]$. Our next goal is to find a categorification of the shift operator, namely, we define functor $S : C^{\mathbb{B}^2} \longrightarrow C^{\mathbb{B}^2}$ such that $|S(F)| = s(|F|)$ for any F in $C^{\mathbb{B}^2}$. Let $Inj : \mathbb{B}^2 \longrightarrow \mathbb{B}$ be the species such that

$$Inj(x, y) = \{\alpha : x \to y \mid \alpha \text{ is injective } \},$$

and define $S : C^{\mathbb{B}^2} \longrightarrow C^{\mathbb{B}^2}$ by the rule

$$S(F)(x, y) = \bigoplus_{\alpha \in Inj(x,y)} F(x, y \setminus \alpha(x)).$$

Theorem 6.4. *The functor $S : C^{\mathbb{B}^2} \longrightarrow C^{\mathbb{B}^2}$ given by the formula above is bimonoidal and satisfies $|S(F)| = s(|F|)$. The functor $P_S : C^{\mathbb{B}^2} \longrightarrow C^{\mathbb{B}^2}$ given by*

$$P_S(F)(x,y) = \bigoplus_{k\geq 0} \bigoplus_{\alpha \in Inj(x,y)^k} F(x, y \setminus \cup_{i=1}^{k} \alpha_i(x)),$$

where $\alpha = (\alpha_1, ...\alpha_k)$ and $\alpha_i(x) \cap \alpha_j(x) = \emptyset$, turns $C^{\mathbb{B}^2}$ into a Rota-Baxter category of weight -1.

Proof. The first part follows from the identities

$$|S(F)| = \sum_{k\leq n} |S(F)([k],[n])| \frac{x^k}{k!}\frac{x^n}{n!} = \sum_{k\leq n} k!\binom{n}{k} f_{k,n-k}\frac{x^k}{k!}\frac{x^n}{n!} = s(|F|).$$

From Theorem 6.1 we know that the functor $P_S(F) = \bigoplus_{n=0}^{\infty} S^n(F)$ is Rota-Baxter of weight -1. It is easy to check that P_S is given by the formula above. □

From the expression above for P_S we can easily compute that

$$p_s(f) = |P_S(F)| = \sum_{k\leq n} (\sum_{pk \leq n} \frac{n! F_{n,n-pk}}{(n-pk)!}) \frac{x^k}{k!}\frac{x^n}{n!}.$$

7. Functorial integration

The most prominent example of a Rota-Baxter ring of weight 0 is the ring of continuous functions on the real line. The Rota-Baxter operator is just integration

$$P(f) = \int_0^x f dt.$$

The Rota-Baxter identity in this case is equivalent to the integration by parts formula. We consider categorical analogues of this construction, what is needed are categorifications of continuous functions such that it is possible to define categorical analogues of the notion of integration. Since we do not have at our disposal a surjective categorification of the ring of continuous or smooth functions on the real line, we restrict our attention to the sub-ring of polynomial functions. Indeed, we work with formal power series instead of polynomial functions. Thus, we are looking for categorifications $|\ \ | : Ob(C) \rightarrow R[[x]]$ of the ring of formal power series with coefficients in a commutative ring R, with the property that there exists a Rota-Baxter functor $P : C \rightarrow C$ such that $|P(c)| = \int_0^x |c| dt$. The categories with these properties we know of are categories of functors, and thus P provides a notion of functorial integration.

Let L be the category of finite linearly order sets. Morphisms in L are order preserving bijections. If x is a linearly order set and $x_1 \sqcup x_2 = x$, then x_1 and x_2 are linearly order with the induced orders. If x is a linearly order finite set, then for $k \leq |x|$, we let $m_k(x)$ be the maximal interval of length k of x. For example $m(x) = m_1(x)$ is the maximal element of x. Suppose that C is a categorification of R and let C^L be the category of functors from L to C. Objects of C^L are called non-symmetric or linear C-species. Sum and product of linear species are given by $F + G(x) = F(x) \oplus G(x)$ and

$$FG(x) = \bigoplus_{x_1 \sqcup x_2 = x} F(x_1) \otimes F(x_2),$$

for F, G in C^L and x in L. Define functor $P : C^L \to C^L$ by the rule $P(F)(x) = F(x \setminus m(x))$.

Theorem 7.1. *(C^L, P) is a Rota-Baxter category of weight 0. The valuation map $|\ \ | : C^L \to R[[x]]$ given by*

$$|F| = \sum_{n \in \mathbb{N}} |F(n)| \frac{x^n}{n!}$$

satisfies $|P(F)| = \int_0^x |F| dt$ for any linear C-species F.

There is a forgetful functor $f : L \to \mathbb{B}$ which sends an ordered set $(x, <)$ into x. The functor f induces by pullback the bimonoidal functor $f^* : C^{\mathbb{B}} \to C^L$ and the commutative diagram

$$\begin{tikzcd} C^{\mathbb{B}} \arrow[r, "|\ |"] \arrow[d, "f^*"'] & R[[x]] \arrow[d, "I"] \\ C^L \arrow[r, "|\ |"] & R[[x]] \end{tikzcd}$$

Unfortunately the functor $P : C^L \to C^L$ is not well defined on $C^{\mathbb{B}}$ since there is no canonical way to choose an element from an unordered set. We see that in order to define P we should break the symmetries of $\mathbb{B}$, i.e. reduce the isotropy groups to the identity and work with L. The proof that C^L is a distributive category may be found in [17]. We show that P is a Rota-Baxter functor using graphical notation, the reader should bare in mind that the grammatical codification of the pictorial proof is purely mechanical. For example the evaluation of the species F on the set $x = \{1, 2, 3, 4, 5, 6, 7\}$ is shown in Figure 2. If we do not need, and this is usually the case, to specify the elements of x, then we prefer the use the abstract representation shown in Figure 3.

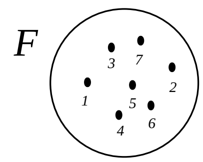

Figure 2. Evaluation of a species.

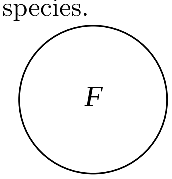

Figure 3. Abstract representation.

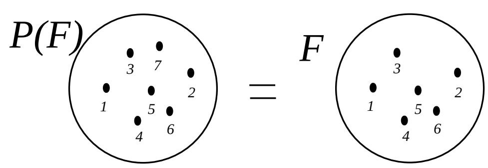

Figure 4. Graphical representation of P.

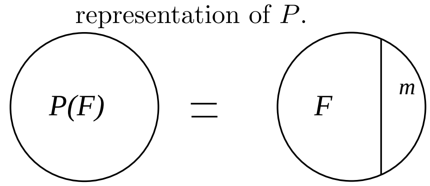

Figure 5. Abstract representation.

The action of P on a species F is graphically represented in Figures 4 and 5. With this conventions the Rota-Baxter isomorphism for P is represented in Figure 6. Both sides are isomorphic because either $n < m$ and then the graph on the left agrees with the first graph on the right, or $m < n$ and in that case it agrees with the second graph on the right hand side.

As in [18] one can show that in any Rota-Baxter category of weight zero there are natural isomorphisms

$$P^a(x)P^b(y) \simeq \bigoplus_{i=1}^{a} \binom{b-1+a-i}{b-1} P^{a+b-i}(P^i(x)\otimes y) \oplus \bigoplus_{i=1}^{b} \binom{a-1+b-i}{a-1} P^{a+b-i}(F\otimes P^i(G)).$$

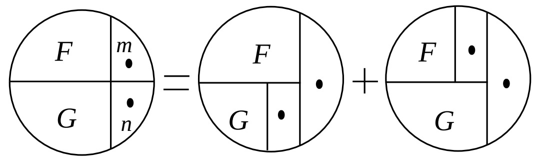

FIGURE 6. Graphical representation of the Rota-Baxter identity.

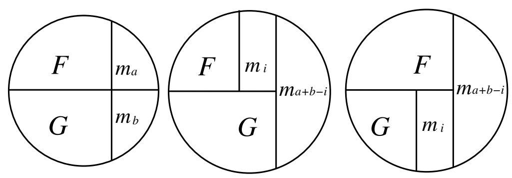

FIGURE 7. Graphical representation of the Rota-Baxter identity.

where $a, b > 1$ and $c \geq 0$ are integers, and by convention nx is the sum of n copies of x, for n a non-negative integer and x an object of a distributive category. It is interesting to elucidate the meaning of the natural isomorphisms above in the Rota-Baxter category (C^L, P). For species F and G, the species

$$P^a(F)P^b(G), P^{a+b-i}(P^i(F) \otimes G) \quad \text{and} \quad P^{a+b-i}(F \otimes P^i(G))$$

are represented graphically in Figure 7. Let us see how the desired isomorphisms arise. Each application of the Rota-Baxter isomorphisms to $P^a(F)P^b(G)$ yields a couple of the form

$$P^{a+b-1}(P(F) \otimes G) \quad \text{and} \quad P^{a+b-1}(F \otimes P(G)).$$

Thus it should be clear that we can apply recursively the Rota-Baxter isomorphisms until we reach elements of the form

$$P^{a+b-i}(P^i(F) \otimes G) \quad \text{and} \quad P^{a+b-i}(F \otimes P^i(G)),$$

where the process stop since then no further application of the Rota-Baxter isomorphisms is possible. Consider an application of the functor $P^a(F)P^b(G)$ on a finite set x. The object $P^a(F)P^b(G)(x)$ is sum of a family of objects of C constructed in several steps. First, x is partitioned in two blocks x_1 and x_2, then the top a, respectively b, elements are removed from x_1 and x_2, thus we obtain object

$$F(x_1 \setminus m_a(x_1)) \otimes F(x_2 \setminus m_b(x_2)).$$

This case corresponds with the left most picture in Figure 7. We need to count how many copies of

$$P^{a+b-i}(P^i(F)\otimes G)$$

arise in this process. The species $P^{a+b-i}(P^i(F)\otimes G)$ applied to a finite set x yields the sum of the following objects of C: first we remove the top $a+b-i$ elements of x, the resulting set is partitioned in two blocks x_1' and x_2', then we remove the top i elements from x_1', thus we obtain an object $F(x_1'\setminus m_i(x_1'))\otimes F(x_2')$. Assume now that the maximal element of x lies in x_2. The pairs x_1 and x_2 given rise to pairs x_1' and x_2' as above are constructed in the following way: from the $a+b-i$ top elements of x the maximal element should lie in x_2, thus we should choose a subsect s of cardinality $b-1$ from the $a+b-i-1$ top elements (excluding the maximal element), once this choice has been made x_1 and x_2 are uniquely determined from x_1' and x_2' via the identities $x_1 = x_1'\cup m_{a+b-i}(x)\setminus(s\cup\{m(x)\})$ and $x_2 = x_2'\cup s\cup\{m(x)\}$. Clearly there are as many as

$$\binom{b-1+a-i}{b-1}$$

different choices for s, thus justifying the claimed isomorphisms.

We closed this section describing a family of Rota-Baxter categories that may be regarded as categorical analogues of discrete integration. These examples are based on two techniques strongly promoted by Rota, the incidence algebra of posets free Rota-Baxter [26]. The reader will find other approaches to free Rota-Baxter algebras in [1,9,20]. Assume C is a distributive category and $(X,\leq)$ is a partially order set. Let C^X be the category whose objects are maps $f: X\to Ob(C)$. Morphisms in C^X from f to g are given by

$$C^X(f,g)=\prod_{i\in X} C(f(i),g(i)).$$

Sum and product on C^X are given by

$$(f\otimes g)(i)=f(i)\otimes g(i) \quad\text{and}\quad (f\otimes g)(i)=f(i)\otimes g(i),$$

respectively. Define the functors $P_<: C^X\to C^X$ and $P_\leq: C^X\to C^X$ by

$$P(f)(j)=\oplus_{i<j}f(i) \quad\text{and}\quad P(f)(j)=\oplus_{i\leq j}f(i).$$

Theorem 7.2. *$(C^X,P_<)$ is a Rota-Baxter category of weight 1. $(C^X,P_\leq)$ is a Rota-Baxter category of weight -1.*

Proof. The statements follow from the identities between set with multiplicities depicted in Figure 8 and Figure 9, respectively. □

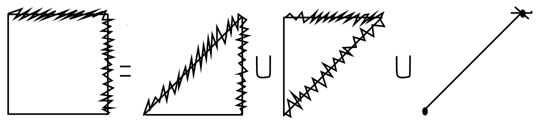

Figure 8. Rota-Baxter identity of weight 1.

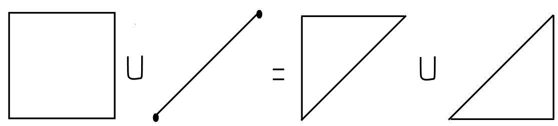

Figure 9. Rota-Baxter identity of weight -1.

For any distributive category C and finite poset X we define the incidence category $I(X,C)$ as the full subcategory of $C^{X\times X}$ whose objects $A : X \times X \longrightarrow C$ are such $A(i,j) = 0$ unless $i \leq j$. $I(X,C)$ is a distributive category with sum and product given by $(A+B)(i,j) = A(i,j) \oplus B(i,j)$ and $(AB)(i,k) = \bigoplus_{i\leq j\leq k} A(i,j) \otimes B(j,k)$. There is functor $P : I(X,C) \times C^X \longrightarrow C^X$ sending (A,f) into $P_A(f)$ given by $P_A(f)(j) = \bigoplus_{i\leq j} A(i,j)f(i)$. Letting ξ be given by $\xi(i,j) = 1$ for $i \leq j$ and 0 otherwise, then $P_\xi = P_\leq$ is a Rota-Baxter operator of weight -1. Similarly, letting $\overline{\xi}$ be given by $\overline{\xi}(i,j) = 1$ for $i < j$ and 0 otherwise, we obtain that $P_{\overline{\xi}} = P_<$ is a Rota-Baxter operator of weight 1. Unfortunately, P_A is not a Rota-Baxter functor for arbitrary A.

8. Categorification of q-calculus

Most applications of q-calculus assume that q is a real number in the interval $(0,1)$. As q approaches 1 one recovers computations in classical calculus. However, it is often the case that computations in q-calculus make sense for $q \geq 0$. In this section we adopt the convention that q is a non-negative integer. Setting $q = 1$ one recovers the theory of species from the theory of q-species developed in this section. Given a commutative ring R we let $R[[x]]_q$ be the ring of formal q-divided powers series defined as the free R-module generated by symbols

$$\frac{x^k}{[k]_q!} \quad \text{with} \quad k \in \mathbb{N},$$

with product
$$\frac{x^k}{[k]_q!}\frac{x^s}{[s]_q!} = \binom{k+s}{s}_q \frac{x^{k+s}}{[k+s]_q!},$$
where

$$\binom{k+s}{s}_q = \frac{[k+s]_q!}{[k]_q![s]_q!}, \quad [n]_q! = \prod_{k=1}^{n}[k], \quad \text{and} \quad [k] = 1+q+...+q^{k-1}.$$

We define operators $\partial_q : R[[x]]_q \longrightarrow R[[x]]_q$, $\int_q : R[[x]]_q \longrightarrow R[[x]]_q$ and $s_q : R[[x]]_q \longrightarrow R[[x]]_q$ by:
$$\partial_q\left(\sum_{n=0}^{\infty} f_n \frac{x^n}{[n]_q!}\right) = \sum_{n=0}^{\infty} f_{n+1}\frac{x^n}{[n]_q!},$$
$$\int_q\left(\sum_{n=0}^{\infty} f_n \frac{x^n}{[n]_q!}\right) = \sum_{n=1}^{\infty} f_{n-1}\frac{x^n}{[n]_q!},$$
$$s_q\left(\sum_{n=0}^{\infty} f_n \frac{x^n}{[n]_q!}\right) = \sum_{n=0}^{\infty} q^n f_n\frac{x^n}{[n]_q!}.$$

The operators ∂_q and $\int_q$ are the formal analogues of the notions of q-derivation and q-integration. The operator s_q is called the shift operator and plays a distinguished role in our next result. We are going to show that $\int_q$ is a twisted[1] Rota-Baxter operator.

Proposition 8.1. *For $f, g \in R[[x]]_q$ the following identity holds:*
$$\left(\int_q f\right)\left(\int_q g\right) = \int_q\left(\int_q f\right) g + \int_q\left(f s_q\left(\int_q g\right)\right).$$

Proof. Assume that
$$f = \sum_{n=0}^{\infty} f_n \frac{x^n}{[n]_q!} \quad \text{and} \quad g = \sum_{n=0}^{\infty} g_n \frac{x^n}{[n]_q!}.$$
The desired result follows from the identities:
$$(\int_q f)(\int_q g) = \sum_{n=2}^{\infty}\left(\sum_{k=1}^{n-1}\binom{n}{k}_q f_{k-1}g_{n-k-1}\right)\frac{x^n}{[n]_q!},$$
$$\int_q(\int_q f)g = \sum_{n=2}^{\infty}\left(\sum_{k=1}^{n-1}\binom{n-1}{k}_q f_{k-1}g_{n-k-1}\right)\frac{x^n}{[n]_q!},$$
$$\int_q(f s_q(\int_q g)) = \sum_{n=2}^{\infty}\left(\sum_{k=1}^{n-1}\binom{n-1}{k-1}_q f_{k-1}g_{n-k-1}\right)\frac{x^n}{[n]_q!},$$

[1]We thank Dominique Manchon for helping us clarify this subtle point.

$$\binom{n}{k}_q = \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q .$$

□

Let C be a distributive category, we are going to define a q-deformed distributive structure on C^{L_n} as follows: sum of species is given by $(F+G)(x) = F(x) \oplus G(x)$, and the q-product of species is given by

$$(FG)(x) = \bigoplus_{x_1 \sqcup x_2 = x} [q]^{c(x_1,x_2)} F(x_1) \otimes G(x_2), \quad \text{where}$$

$$c(x_1, x_2) = \{(i,j) \mid i \in x_1, j \in x_2 \text{ and } i > j\}.$$

In the definition above we used the following convention: if x is a finite set and c an object of a distributive category then we set

$$xc = \bigoplus_{i \in x} c.$$

We write $C_q^{L_n}$ instead of C^{L_n} to emphasize that we are using the q-deformed product. Notice that if $q = 1$ then $[q]^{c(x_1,x_2)}$ is a set with one element and plays no significant role, thus we recover the product of species of Section 7. We define functors $P_q : C_q^L \longrightarrow C_q^L$ and $S_q : C_q^L \longrightarrow C_q^L$ as follows. For species F, G in C_q^L and a linearly order set x we set:

$$\partial_q(F)(x) = F(x \sqcup \{x\}), \quad P_q(F)(x) = F(x'),$$

where $x' = x \setminus m(x)$ if x is non-empty and $P_q(F)(\emptyset) = \emptyset$. The functor S_q is given by

$$S_q(F)(x) = [q]^x F(x).$$

One can check that ∂_q and P_q are almost inverse of each other, indeed, we have that

$$\partial_q P_q F = F \quad \text{and} \quad P_q \partial_q F = F_+,$$

where $F_+(x) = F(x)$ if x in non-empty and $F_+(\emptyset) = \emptyset$.

Theorem 8.2. *(C_q^L, P_q, S_q) is a twisted Rota-Baxter category, i.e. there are natural isomorphisms*

$$P_q(F)P_q(G) \simeq P_q(P_q(F)G) \oplus P(FS_qP_q(G)).$$

If C is a categorification of R, then (C_q^L, P_q, S_q) is a categorification of $(R[[x]]_q, \int_q, s_q)$.

Proof. Let us show that P_q is a twisted Rota-Baxter functor. We have the identities

$$P_q(F)P_q(G)(x) = \bigoplus_{x_1 \sqcup x_2 = x} [q]^{c(x_1,x_2)} F(x_1') \otimes G(x_2'),$$

$$P_q(P_q(F)G)(x) = \bigoplus_{x_1 \sqcup x_2 = x'} [q]^{c(x_1,x_2)} F(x_1') \otimes G(x_2),$$

$$P_q(FS_qP(G))(x) = \bigoplus_{x_1 \sqcup x_2 = x'} [q]^{c(x_1,x_2)\sqcup x_2} F(x_1) \otimes G(x_2'),$$

The desired natural isomorphisms are constructed as follows. Consider the summand in $P_q(F)P_q(G)(x)$ corresponding with the partition $x_1 \sqcup x_2 = x$. Then $m(x)$ lies either in x_1 or in x_2. If $m(x) \in x_2$, then $x_1 \sqcup x_2' = x'$ and the summands corresponding to (x_1, x_2) and (x_1, x_2') in $P_q(F)P_q(G)(x)$ and $P_q(P_q(F)G)(x)$, respectively, agree since in this case $c(x_1, x_2) = c(x_1, x_2')$. On the other hand, if $m(x) \in x_1$, then $x_1' \sqcup x_2 = x'$ and the summands corresponding to (x_1, x_2) and (x_1, x_2') in $P_q(F)P_q(G)(x)$ and $P_q(FS_qP(G))(x)$, respectively, are naturally isomorphic since in this case $c(x_1, x_2) = c(x_1', x_2) \sqcup x_2$.

The valuation map $|\ \ | : C_q^L \longrightarrow R[[x]]_q$ is given by $|F| = \sum_{n=0}^{\infty} |F([n])| \frac{x^n}{[n]_q!}$. Let us check that it satisfies the multiplicative property:

$$\begin{aligned} |FG| &= \sum_{n=0}^{\infty} |(FG)([n])| \frac{x^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty} \sum_{x_1 \sqcup x_2 = [n]} q^{|c(x_1,x_2)|} |F(x_1)||G(x_2)| \frac{x^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(\sum_{x_1 \sqcup x_2 = [n], |x_1| = k} q^{|c(x_1,x_2)|} \right) |F(k)||G(n-k)| \frac{x^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k}_q |F(k)||G(n-k)| \frac{x^n}{[n]_q!} \\ &= |F||G|. \end{aligned}$$

□

9. Categorification of classical field theory

Let K be a set and J a subset of K. For a commutative ring R we let $R[[K]]$ be the ring of formal divided power series on variables $k \in K$, and $R[[J]]$ the subring of $R[[K]]$ consisting of formal divided power series with variables $k \in J$. Formally $R[[K]]$ is defined as follows. Let $M(K)$ be the set of maps

$$m : K \longrightarrow \mathbb{N}$$

with finite support $s(m) = \{i \in K \mid m(i) \neq 0\}$, then we set

$$R[[K]] = Maps(M(K), R).$$

The structural operations on $R[[K]]$ are given by

$$(f+g)(m) = f(m) + g(m)$$

and

$$(fg)(m) = \sum_{m_1+m_2=m} \binom{m}{m_1} f(m_1)g(m_2),$$

where $m, m_1, m_2 \in M(K)$ and

$$\binom{m}{m_1} = \prod_{i \in s(m)} \binom{m(i)}{m_1(i)}.$$

Consider the operator $p_J : R[[K]] \longrightarrow R[[J]] \subseteq R[[K]]$ given by

$$p_J(f)(m) = \begin{cases} f(m) & \text{if } s(m) \subseteq x \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that the operator p_J is Rota-Baxter of weight -1 since p_J is an idempotent ring morphism.

The construction above can be generalized to the categorical context without difficulties. Let $\mathbb{B}^{(K)}$ be the category whose objects are pairs (x, f) where x is a finite set and $f : x \longrightarrow K$ is a map. Morphisms in $\mathbb{B}^{(K)}$ from (x, f) to (y, g) are bijections $\alpha : x \to y$ such that $g \circ \alpha = f$. Let C be a distributive category with negative objects, see [5,6,7,17] for examples; and let $C^{\mathbb{B}^{(K)}}$ be the category of functors from $\mathbb{B}^{(K)}$ to C. We define the sum and product of functors as follows, given (x, f) in $\mathbb{B}^{(K)}$ and F, G in $C^{\mathbb{B}^{(K)}}$ then

$$(F+G)(x, f) = F(x, f) \oplus G(x, f)$$

and

$$FG(x, f) = \bigoplus_{x_1 \sqcup x_2 = x} F(x_1, f|_{x_1}) \otimes G(x_2, f|_{x_2}).$$

These structural functors turn $C^{\mathbb{B}^{(K)}}$ into a distributive category. Assume that C is a categorification of a ring R and let $R[[K]]$ be the ring of formal series in variables K with coefficients in R. Consider the functor $P_J : C^{\mathbb{B}^{(K)}} \longrightarrow C^{\mathbb{B}^{(K)}}$ given by

$$P_J(F)(x, f) = \begin{cases} F(x, f) & \text{if } f(x) \subseteq J \\ 0 & \text{otherwise} \end{cases}$$

The following result is easy to check.

Theorem 9.1. *$(C^{\mathbb{B}^{(K)}}, P_J)$ is a Rota-Baxter category of weight -1. Moreover $(C^{\mathbb{B}^{(K)}}, P_J)$ is a categorification of $(R[[K]], p_J)$ with valuation map given by*

$$|F|(m) = |F(x_m, f_m)|,$$

$$\textit{where} \quad x_m = \coprod_{k\in K}[m(k)] \quad \textit{and the map} \quad f_m : \coprod_{k\in K}[m(k)] \longrightarrow K$$

is such that $f_m(i) = k$ if $i \in [m(k)]$.

Let us now see how a Lagrangian field theory may be described from a categorical point. The basic objects appearing in a field theory, namely the fields, are locally identified with maps

$$\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}^k,$$

where d is the dimension of the space-time manifold and n is the number of scalar fields involved in the construction of φ. A Lagrangian theory is completely determined by a function l, the Lagrangian, depending on fields and its derivative. Fixing coordinates $x_1, ..., x_d$ on $\mathbb{R}^d$ and writing φ as $\varphi = (\varphi^1, ..., \varphi^k)$, then a translation invariant Lagrangian may be regarded as a polynomial or formal power series in variables $\partial_I\varphi^j$, where for $j \in [k]$ and $I \in \mathbb{N}^d$ we set

$$\partial_I\varphi^j = \partial_1^{I(1)}...\partial_d^{I(d)}\varphi^j.$$

Thus we see that a Lagrangian l may be regarded as an element of the ring of formal divided powers

$$R[[\mathbb{N}^d \times [k]]] = R[[\partial_I\varphi^j]],$$

where $(I, j) \in \mathbb{N}^d \times [k]$ and $\partial_I\varphi^j$ is regarded as a formal variable. Theorem 9.1 tell us that if C is a categorification of R, then $C^{\mathbb{B}^{(\mathbb{N}^d\times[k])}}$ is a categorification of $R[[\partial_I\varphi^j]]$. Objects in $C^{\mathbb{B}^{(\mathbb{N}^d\times[k])}}$ are triples (x, f, g) where x is a finite set, $f : x \longrightarrow \mathbb{N}^d$ and $g : x \longrightarrow [k]$. The valuation map

$$|\ \ | : C^{\mathbb{B}^{(\mathbb{N}^d\times[k])}} \longrightarrow R[[\partial_I\varphi^j]]$$

sends a species $F \in C^{\mathbb{B}^{(\mathbb{N}^d\times[k])}}$ into the formal divided power series

$$|F| = \sum_{f\in M(\mathbb{N}^d\times[k])} F(\bigsqcup_{(I,j)}[(f,g)^{-1}(I,j)])\frac{\varphi^f}{f!},$$

where

$$\frac{\varphi^f}{f!} = \prod_{(I,j)}\frac{(\partial_I\varphi^j)^{f(I,j)}}{f(I,j)!}.$$

For $(I, j) \in \mathbb{N}^d \times [k]$ we have the partial derivation functors

$$\partial_{(I,j)} : C^{\mathbb{B}^{(\mathbb{N}^d\times[k])}} \longrightarrow C^{\mathbb{B}^{(\mathbb{N}^d\times[k])}}$$

given by

$$\partial_{(I,j)}F(x,f) = F((x,f) \sqcup (*,(I,j))).$$

Notice that the ring $R[[\partial_I \varphi^j]]$ comes with additional natural vector fields

$$\partial_i : R[[\partial_I \varphi^j]] \longrightarrow R[[\partial_I \varphi^j]]$$

given by

$$\partial_i(\partial_I \varphi^j) = \partial_{I+\varepsilon_i}\varphi^j$$

where the vectors ε_i are the canonical generators of $\mathbb{N}^d$. In the categorical context we have functors

$$\partial_i : C^{\mathbb{B}(\mathbb{N}^d \times [k])} \longrightarrow C^{\mathbb{B}(\mathbb{N}^d \times [k])}$$

given for $1 \leq i \leq n$ by

$$\partial_i(F)(x,f,g) = \bigoplus_{a \in x} F(x, f + \delta_a \varepsilon_i, g)$$

where $\delta_a : x \longrightarrow \{0,1\}$ is the Kronecker delta function. More generally for $I \in \mathbb{N}^d$ we have differential functor

$$\partial_I : C^{\mathbb{B}(\mathbb{N}^d \times [k])} \longrightarrow C^{\mathbb{B}(\mathbb{N}^d \times [k])}$$

given by

$$\partial_I(F)(x,f,g) = \bigoplus_{a_i \in x^{I(i)}} F(x, f + \sum_{i=1}^{l} \delta_{a_{i,j}}\varepsilon_i, g).$$

The categorification of a Lagrangian l is thus obtained by finding a functor L in $C^{\mathbb{B}(\mathbb{N}^d \times [k])}$ such that $|L| = l$. Classical field theory main concern is understanding the solutions of a system of partial differential equations $e_j(l) = 0$, $j \in [k]$, called the Euler-Lagrange equations which are determined by the Lagrangian l. The the first step in the categorification of this system of partial differential equations is to find a categorification for each of the equations appearing in the Euler-Lagrange equations, namely, we need species $E_j(l)$ in $C^{\mathbb{B}(\mathbb{N}^d \times [k])}$ such that $|E_j(L)| = e_j(l)$.

Theorem 9.2. *The functors $E_j(L) : C^{\mathbb{B}(\mathbb{N}^d \times [k])} \longrightarrow C^{\mathbb{B}(\mathbb{N}^d \times [k])}$ given by*

$$E_j(L)(x,f,g) = \bigoplus_{I \in \mathbb{N}^d} \bigoplus_{a_i \in (x \sqcup \{*\})^{I(i)}} (-1)^{|I|} L(x \sqcup \{*\}, (f \sqcup (*,I)) + \sum_{i,m} \varepsilon_i \delta_{a_{i,m}}, g \sqcup (*,j))$$

are such that $|E_j(L)| = e_j(L)$.

Proof. Follows from the definitions given above and the well-know formula

$$e_j(l) = \sum_{I \in \mathbb{N}^d} (-1)^{|I|} \partial_I \frac{\partial l}{\partial(\partial_I \varphi^j)}.$$

$\square$

10. Categorification of quantum field theory

As discussed in the previous section the ring of functions on the configuration space of a field theory may be identified with $R[[K]]$ where $K = \mathbb{N}^d \times [k]$. Moreover $C^{\mathbb{B}^{(K)}}$ provides a categorification of the ring of functions on configuration space. In order to proceed to consider quantum structures in field theory, we consider functions in phase space which may be identified with the ring

$$\mathbb{R}[[K \sqcup \overline{K}]].$$

An element of the first copy of K is denoted by k, the corresponding element in the second copy is denoted $\overline{k}$. So we have an involution

$$K \sqcup \overline{K} \longrightarrow K \sqcup \overline{K}$$

sending k into $\overline{k}$ and $\overline{k}$ into k. Clearly $C^{\mathbb{B}^{(K\sqcup\overline{K})}}$ is a categorification of $\mathbb{R}[[K \sqcup \overline{K}]]$. The new algebraic structure present in the ring of functions on phase space is the Poisson bracket

$$\{\ \ ,\ \ \} : \mathbb{R}[[K \sqcup \overline{K}]] \times \mathbb{R}[[K \sqcup \overline{K}]] \longrightarrow \mathbb{R}[[K \sqcup \overline{K}]]$$

which is determined by the fact that it is antisymmetric, a derivation on each variable, and is given on generators by

$$\{k, \overline{k}\} = \delta_{k,\overline{k}}, \quad \{k, s\} = 0, \text{ and } \{\overline{s}, \overline{k}\} = 0.$$

So our first task is to define a bifunctor

$$\{\ \ ,\ \ \} : C^{\mathbb{B}^{(K\sqcup\overline{K})}} \times C^{\mathbb{B}^{(K\sqcup\overline{K})}} \longrightarrow C^{\mathbb{B}^{(K\sqcup\overline{K})}}$$

which plays the role of the Poisson bracket in the categorical context. The Poisson bracket of functors F and G in $C^{\mathbb{B}^{(K\sqcup\overline{K})}}$ turns out to be given by:

$$\begin{aligned}\{F,G\}(x,f) \quad &= \bigoplus_{x_1\sqcup x_2 = x, k\in K} F(x_1 \sqcup \{*\}, f \sqcup (*,k)) \otimes G(x_2 \sqcup \{*\}, f \sqcup (*,\overline{k})) \\ &- \bigoplus_{x_1\sqcup x_2 = x, k\in K} F(x_1 \sqcup \{*\}, f \sqcup (*,\overline{k})) \otimes G(x_2 \sqcup \{*\}, f \sqcup (*,k)).\end{aligned}$$

Theorem 10.1. *For F, G in $C^{\mathbb{B}^{(K\sqcup\overline{K})}}$ the following identity holds*

$$|\{F,G\}| = \{|F|, |G|\}.$$

Proof. The result follows from the fact that the Poisson bracket of functions on phase space is given by

$$\{f, g\} = \sum_{k\in K} \frac{\partial f}{\partial k}\frac{\partial g}{\partial \overline{k}} - \frac{\partial f}{\partial \overline{k}}\frac{\partial g}{\partial k},$$

and the fact that the formula for the Poisson bracket of species given above may be also be defined by

$$\{F, G\} = \textstyle\sum_{k\in K} \partial_k F \partial_{\overline{k}} G - \partial_{\overline{k}} F \partial_k G. \qquad \square$$

The commutative product of functions on phase space comes equipped with a natural deformation into a quantum product, often called the Moyal product, which is determined by the Poisson bracket. Our next goal is to describe the Moyal product at the categorical level. Recall that quantum phase space possesses an extra dimension parameterized by a formal variable $\hbar$. Thus a categorification of formal power series in quantum phase space is given by $C^{\mathbb{B}^{(K\sqcup\overline{K}\sqcup\hbar)}}$ with the natural valuation map into $R[[K \sqcup \overline{K} \sqcup \hbar]]$. Objects in $\mathbb{B}^{(K\sqcup\overline{K}\sqcup\hbar)}$ are triples (x, f, h) where x is a finite set, $f : x \longrightarrow K \sqcup \overline{K}$ is a map, and h is another finite set. Given a map $\alpha : h \to K \sqcup \overline{K}$, we define $s(\alpha)$ the sign of α as follows: ± 1 according to the parity of $\alpha^{-1}(\overline{K})$.

$$s(\alpha) = \begin{cases} 1 & \text{if } |\alpha^{-1}(\overline{K})| \text{ is even} \\ -1 & \text{if } |\alpha^{-1}(\overline{K})| \text{ is odd.} \end{cases}$$

We define the Moyal star product of species F and G in $C^{\mathbb{B}^{(K\sqcup\overline{K}\sqcup\hbar)}}$ is given by

$$F \star G(x, f, h) \quad = \quad \bigoplus s(\alpha) F(x_1 \sqcup h_3, f \sqcup \alpha, h_1) \otimes G(x_2 \sqcup h_3, f \sqcup \overline{\alpha}, h_2)),$$

where the sum runs over finite sets x_1, x_2, h_1, h_2, h_3 such $x_1 \sqcup x_2 = x$, $h_1 \sqcup h_2 \sqcup h_3 = h$ and $\alpha : h_3 \to K \sqcup \overline{K}$.

Theorem 10.2. *For F, G in $C^{\mathbb{B}^{(K\sqcup\overline{K}\sqcup\hbar)}}$ the following identity holds*

$$|F \star G| = |F| \star |G|.$$

Proof. The result is an instance of a general categorification theorem for the Kontsevich's star product proved in [17]. Alternatively, one notices that the expression given for the $\star$-product of species is the categorical version of the following expression for the $\star$-product of functions on phase space

$$f \star g = \sum_{n=0}^{\infty} \frac{h^n}{n!} m\Big(\sum_{k\in K} \frac{\partial}{\partial k} \otimes \frac{\partial}{\partial \overline{k}} - \frac{\partial}{\partial \overline{k}} \otimes \frac{\partial}{\partial k}\Big)^n (f \otimes g),$$

where for m denotes the product of functions on phase space. $\square$

We close the paper with an example of a quantum-like Rota-Baxter category. Fix a subset J of K. Consider the functor $P_J : C^{\mathbb{B}^{(K\sqcup\overline{K}\sqcup\hbar)}} \longrightarrow C^{\mathbb{B}^{(K\sqcup\overline{K}\sqcup\hbar)}}$ given by

$$P_J(F)(x, f, h) = \begin{cases} F(x, f, h) & \text{if } f(x) \subseteq J \sqcup \overline{J} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 10.3. *$(C^{\mathbb{B}^{(K\sqcup\overline{K}\sqcup\hbar)}}, \star, P_J)$ is Rota-Baxter category of weight -1. $(C^{\mathbb{B}^{(K\sqcup\overline{K}\sqcup\hbar)}}, \star, P_J)$ is a categorification of the Rota-Baxter algebra*

$$(R[[K \sqcup \overline{K} \sqcup \hbar]], \star, p_J)$$

with valuation map given by

$$|F|(m) = |F(x_m, f_m, [m(\hbar)])|,$$

where $x_m = \coprod_{k\in K}[m(k)] \sqcup [m(\overline{k})]$ and $f_m : \coprod_{k\in K}[m(k)] \sqcup [m(\overline{k})] \longrightarrow K$ is such that

$$f_m(i) = \begin{cases} k & \text{if } i \in [m(k)] \\ \overline{k} & \quad i \in [m(\overline{k})]. \end{cases}$$

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