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GENERALIZED COFINITELY SEMIPERFECT MODULES

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Abstract. In the present paper, we define generalized (amply) cofinitely supplemented modules, and generalized $\oplus$-cofinitely supplemented modules are defined as a generalization of (amply) cofinitely supplemented modules and $\oplus$-cofinitely supplemented modules, respectively, and show, among others, the following results:

(1) The class of generalized cofinitely supplemented modules is closed under taking homomorphic images, generalized covers and arbitrary direct sums.

(2) Any finite direct sum of generalized $\oplus$-cofinitely supplemented modules is a generalized $\oplus$-cofinitely supplemented module.

(3) M is a generalized cofinitely semiperfect module if and only if M is a generalized cofinitely supplemented -module by supplements which have generalized projective covers.

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1. Introduction

Supplemented modules have been discussed by several authors (see [9], [14]) in the literature, and these modules are useful in characterizing semiperfect modules and rings. Azumaya [3] introduced the notion of generalized projective covers to characterize semiperfect modules. Recently, in [12], Wang and Ding studied on generalized (amply) supplemented modules to characterize semiperfect modules and rings.

Alizade et al. studied certain modules whose maximal submodules have supplements, and introduced cofinitely supplemented modules [1]. In [5], Çalışıcı and Pancar studied cofinitely semiperfect modules as a generalization of semiperfect modules.

In this study, generalized (amply)cofinitely supplemented modules are defined as analogues of (amply) cofinitely supplemented modules. In Section 3, we obtain some basic properties of generalized (amply) cofinitely supplemented modules and generalized $\oplus$-cofinitely supplemented modules.

In Section 4, we characterize generalized cofinitely semiperfect modules via generalized projective covers of the generalized supplement submodules. For more detailed discussion on generalized supplements and generalized covers, we refer to [13]. Here, we also prove that;

Theorem. *For any module M the following statements are equivalent:*

(1) *M is a generalized cofinitely semiperfect module.*

(2) *M is a generalized amply cofinitely supplemented module by supplements which have generalized projective covers.*

(3) *M is a generalized cofinitely supplemented module by supplements which have generalized projective covers.*

2. Conventions and Notations

Unless otherwise stated, we use the following conventions and notations.

All rings are associative with unity and all R-modules are unital right R-modules.

A submodule N of a module M is called *small*, written $N \ll M$, if $M \neq N+L$ for every proper submodule L of M. $Rad(M)$ denotes the radical of M. The symbols, "$\leq$" will denote a submodule, "$\leq_d$" a module direct summand. For submodules A and B of M with $M = A + B$, B is called a *supplement* of A if it is minimal with respect to this property, equivalently if $A \cap B$ is small in B. A submodule N of M has *ample supplements* in M if every submodule L such that $M = N + L$ contains a supplement of N in M. The module M is called *(amply) supplemented* if every submodule of M has a (an ample) supplement submodule. M is called a *$\oplus$-supplemented module* if every submodule of M has a supplement that is a direct summand of M.

Following Wisbauer [14], M is said to be an *(amply) f-supplemented* module if every finitely generated submodule of M has an/a (amply) supplement in M. A submodule N of M is called *cofinite* (in M) if the factor module M/N is finitely generated. The module M is called *(amply)cofinitely supplemented* if every cofinite submodule of M has a (an ample) supplement in M ([1] and [11]).

Definition 2.1. Let M be a module. For submodules A and B of M with $M = A + B$, B is called a *generalized supplement* of A in M in case $A \cap B \subseteq Rad(B)$. Clearly, each supplement is a generalized supplement submodule. Adapting this concept, M is called *generalized (amply) cofinitely supplemented* if every cofinite submodule of M has (ample) generalized supplements in M and denoted by *gcs* (*gacs*)-module, respectively (see [4]).

Following [7] , M is called *$\oplus$-cofinitely supplemented* or briefly *$\oplus$-cof-supplemented* if every cofinite submodule of M has a supplement that is a direct summand of M. We call M *a generalized $\oplus$-cofinitely supplemented module* if every cofinite submodule of M has a generalized supplement that is a direct summand of M, denoted by $g-\oplus-cs$-module. It is easy to see that, $\oplus$-supplemented modules are generalized $\oplus$-cofinitely supplemented modules and converse is true if the modules is finitely generated. Note that hollow modules are generalized $\oplus$-supplemented so that local modules are also generalized $\oplus$-supplemented.

Definition 2.2. If P and M are modules, we call an epimorphism $p : P \to M$ a *(generalized) small cover* in case $(Ker(p) \subseteq Rad(P))$ $Ker(p) \ll P$, respectively. Since $Rad(P)$ is the sum of all small submodules of P, every small cover is a generalized cover. If P is a projective module, it is called *(generalized) projective cover*. As observed in [3], every projective cover is a generalized projective cover. We have the following basic properties of generalized covers.

Lemma 2.3. ([13, Lemmas 1.1 and 1.2]) (1) *If $f : P \to M$ and $g : M \to N$ are generalized covers for M and N, respectively, then $gf : P \to N$ is a generalized cover for N.*
(2) *Let $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ be such that all $p_i : P_i \to M_i$ are generalized covers. Let $P = P_1 \oplus P_2 \oplus \cdots \oplus P_n$. Then $p = \oplus p_i : P \to M$ is a generalized cover.*

An R-module M is called *semiperfect* if every factor module of M has a projective cover. If R_R is semiperfect, then R is called a *semiperfect* ring. Following [5], an R-module M is called *cofinitely semiperfect* if every finitely generated factor module of M has a projective cover. Clearly, semiperfect modules are cofinitely semiperfect and finitely generated cofinitely semiperfect modules are semiperfect.

Hence, we call an R-module M *generalized cofinitely semiperfect* if every finitely generated factor module of M has a generalized projective cover.

An account of these concepts can be found in the texts by Mohammed and Müller and Wisbauer, referenced in the manuscript as [9] and [14], respectively.

3. Generalized Cofinitely Supplemented Modules

In this section, we give some characterizations of *gcs* (*gacs*)-modules.

Lemma 3.1. *If $f : M \to N$ is a homomorphism and L is a generalized supplement in M with $Kerf \leq L$, then $f(L)$ is a generalized supplement in $f(M)$.*

Proof. Assume that L is a generalized supplement of K in M. Then $M = L + K$ and $L \cap K \subseteq Rad(L)$. Then $f(M) = f(L+K) = f(L)+f(K)$. Since $L \cap K \subseteq Rad(L)$ and $Ker(f) \leq L$, we have $f(L) \cap f(K) = f(L \cap K) \leq Radf(L)$ by [2, Proposition 9.14]. So $f(L)$ is a generalized supplement of $f(K)$ in $f(M)$. □

Proposition 3.2. *Any homomorphic image of a gcs (gacs)-module is a gcs (gacs)-module, respectively.*

Proof. Let M be an R-module. By [4, Theorem 3.5], any homomorphic image of M is a *gcs*-module. By adapting this argument, we can show similarly that any homomorphic image of M is a *gacs*-module. □

Lemma 3.3. *Let M be an R-module and $K, L, N \leq M$. Then;*
(1) If K is a generalized supplement of N in M and $T \leq Rad(M)$ then K is a generalized supplement of $N + T$ in M.
(2) If $f : M \to N$ is a generalized cover epimorphism, then, the submodule L of M is a generalized supplement in M if and only if $f(L)$ is a generalized supplement in N.
(3) If K is a gcs-module, L is cofinite and $K + L$ has a generalized supplement U in M, then $K \cap (L + U)$ has a generalized supplement V in K and $U + V$ is a generalized supplement of L in M.

Proof. (1) Let K be a generalized supplement of N in M. Then $M = N + K$ and $N \cap K \leq Rad(K)$. We consider the homomorphism $f : M \to (M/N) \oplus (M/K)$, $g : (M/N) \oplus (M/K) \to (M/(N + T)) \oplus (M/K)$ and canonical epimorphism $\pi : M \to M/N$. Since $Ker(f) = N \cap K \leq Rad(K)$, the homomorphism f is a generalized cover epimorphism . Note that $Ker(g) = (N + T)/N \oplus (0)$. Since $(N + T)/N = \pi(T) =\leq Rad(M/N)$ by [2, Proposition 9.14]. Hence g is a generalized cover epimorphism. By Lemma 2.3, fg is a generalized cover epimorphism, i.e., $Ker(fg) \leq Rad(M)$. Since $(N+T) \cap K = Ker(fg)$, K is a generalized supplement of $N + T$ in M.
(2) Let L be a generalized supplement of K in M. Then K is a generalized supplement of $L + Ker(f)$ by (1). By Lemma 3.1, $f(L) = f(L + Kerf)$ is also a generalized supplement in N. Conversely, let $f(L)$ be a generalized supplement of a submodule T in N, i.e., $N = f(L) + T$ and $f(L) \cap T \leq Rad(f(L))$. Then $M = L + f^{-1}(T)$. Because $L \cap f^{-1}(T) \leq f^{-1}(f(L) \cap T) \leq Rad(f^{-1}(T))$ by [6, Corollary 9.1.5], $f^{-1}(T)$ is a generalized supplement of L in M.
(3) Let U be generalized supplement of $K + L$ in M. Then $M = (K + L) + U$

and $(K+L)\cap U \leq Rad(U)$. Note that $M/(L+U) \cong K/(K\cap(L+U))$. We consider the homomorphism $f: M/L \to M/(L+U)$. Since M/L is finitely generated, $M/(L+U)$ is finitely generated and so $K\cap(L+U)$ a cofinite submodule of K. Because K is a *gcs*-module, $K\cap(L+U)$ has a generalized supplement V in K. That is $K=[K\cap(L+U)]+V$ and $[K\cap(L+U)]\cap V \leq Rad(V)$. Since V is a submodule of K, we have $[K\cap(L+U)]\cap V = V\cap(L+U) \leq Rad(V)$. Then $M = L+(U+V)$. By [14, Lemma 19.3], $L\cap(U+V) \leq [U\cap(K+L)]+[V\cap(L+U)]$. Since $U\cap(K+L) \leq Rad(U)$ and $V\cap(L+U) \leq Rad(V)$, then $L\cap(U+V) \leq Rad(U)+Rad(V) \leq Rad(U+V)$. □

Theorem 3.4. *The class of gcs-modules is closed under taking generalized cover and arbitrary direct sums.*

Proof. By Lemma 3.3(2), a generalized cover of a *gcs*-module is a *gcs*-module. By [4, Theorem 3.5], the class of *gcs*-modules is closed under taking arbitrary direct sums. □

Theorem 3.5. *Let M be a gcs-module and let A be a cofinite submodule of M. If A is a generalized supplement in M, then $Rad(A) = A$.*

Proof. Let K be a generalized supplement of M with M/K finitely generated. Then $M = N+K$ and $N\cap K \leq Rad(K)$, where N is a cofinite submodule of M. Since $M/N \overset{f}{\cong} K/(N\cap K)$ and M/N is finitely generated, we have $K/(N\cap K) = Rad(K/(N\cap K))$. By [2, Proposition 9.15], $Rad(K/(N\cap K)) = Rad(K)/(N\cap K)$ because $Ker(f) = N\cap K \leq Rad(K)$. Hence $K/(N\cap K) = Rad(K/(N\cap K)) = Rad(K)/(N\cap K)$. This implies that $K = Rad(K)$. □

We consider the following condition for a module M.

$(*)$ Every cofinite submodule of the module $M/Rad(M)$ is a direct summand.

Proposition 3.6. *Every gcs-module satisfies the $(*)$-condition.*

Proof. Let M be a *gcs*-module and $Rad(M) \leq N \leq M$ with N a cofinite submodule of M. Since M is a *gcs*-module, there exist $X \leq M$ such that $M = N+X$ and $N\cap X \leq Rad(X)$ and so $N\cap X \leq Rad(M)$. Then

$$\begin{aligned} M/Rad(M)) &= N/Rad(M) + ((X+Rad(M))/Rad(M)) \\ &= N/Rad(M) \oplus ((X+Rad(M))/Rad(M)) \end{aligned}$$

because $N\cap(X+Rad(M)) = (N\cap X)+Rad(M) = Rad(M)$. □

A module M is called *local* if the sum of all proper submodules of M is a proper submodule of M. Note that 0 is a local submodule and also a cofinitely supplemented submodule of M, and so is a generalized cofinitely supplemented submodule of M.

For any module M, we shall denote the socle of M by $Soc(M)$.

Lemma 3.7. (See [1, Lemma 2.7]) *Let R be a ring. The following statements are equivalent for an R-module M.*
1. *Every cofinite submodule of M is a direct summand of M.*
2. *Every maximal submodule of M is a direct summand of M.*
3. *$M/Soc(M)$ does not contain a maximal submodule.*

For any module M, $Cof(M)$ will denote the sum of all cofinitely supplemented submodules of M, $Loc(M)$ will denote the sum of all local submodules of M (see [1]) and $g-Cof(M)$ will denote the sum of all generalized cofinitely supplemented submodules of M, respectively. In case M does not contain a local submodule, $Loc(M)$ is the zero submodule. Thus $Loc(M) \leq Cof \leq g - Cof$.

Theorem 3.8. *Let R be a ring and M be a right R-module. Then the following statements are equivalent:*
(1) *M is gcs-module.*
(2) *Every maximal submodule of M has a generalized supplement in M.*
(3) *The module $M/Loc(M)$ does not contain a maximal submodule.*
(4) *The module $M/g - Cof(M)$ does not contain a maximal submodule.*

Proof. Similar to [1, Theorem 2.8]. □

Through the rest of this section, we focus on the notion of generalized ⊕-cofinitely supplemented modules.

Lemma 3.9. *For any ring R, finite direct sum of g-⊕-cs R-modules is a g-⊕-cs-module.*

Proof. Let M_i be a g-⊕-cs-module for each $1 \leq i \leq n$ and $M = \oplus_{i=1}^{n} M_i$. To prove that M is a g-⊕-cs-module, it is sufficient to prove that this is the case when $n = 2$. Let L be any cofinite submodule of M. Then $M = M_1 + M_2 + L$ so that $M_1 + M_2 + L$ has a generalized supplement 0 in M. Since L is cofinite submodule of M, then M/L is finitely generated and so $M/(M_1 + L)$ is finitely generated. Note that

$$\frac{M}{M_1 + L} = \frac{M_1 + M_2 + L}{M_1 + L} = \frac{M_2 + (M_1 + L)}{M_1 + L} \cong \frac{M_2}{M_2 \cap (M_1 + L)}$$

so that $M_2/(M_2 \cap (M_1 + L))$ is finitely generated. Then $M_2 \cap (M_1 + L)$ is a cofinite submodule of M_2. Since M_2 is a g-$\oplus$-cs-module, there exist a direct summand generalized supplement H of $M_2 \cap (M_1 + L)$ in M_2, namely $M_2 = [M_2 \cap (M_1 + L)] + H = H \oplus H^{'}$ for some $H' \leq M_2$ and $[M_2 \cap (M_1 + L)] \cap H \leq Rad(H)$. By Lemma 3.4, H is a generalized supplement of $M_1 + L$ in M. That is $M = (M_1 + L) + H$ and $(M_1 + L) \cap H \leq Rad(H)$. Note that

$$\frac{M}{L + H} = \frac{M_1 + (L + H)}{L + H} \cong \frac{M_1}{M_1 \cap (L + H)}$$

so that $M_1/(M_1 \cap (L + H))$ is finitely generated. Then $M_1 \cap (L + H)$ is a cofinite submodule of M_1. Since M_1 is a g-$\oplus$-cs-module, there exist a direct summand generalized supplement K of $M_1 \cap (L+H)$ in M_1, namely $M_1 = [M_1 \cap (L+H)] + K = K \oplus K^{'}$ for some $K' \leq M_1$ and $[M_1 \cap (L+H)] \cap K \leq Rad(K)$. Since K is a submodule of M_1, $(L + H) \cap K \leq Rad(K)$. By Lemma 3.4, L is a generalized supplement of $H + K$ in M. Since H is a direct summand of M_2 and K is a direct summand of M_1, then $H + K = H \oplus K$ is a direct summand of M. Clearly, $M = L + (H + K)$ and $L \cap (H + K) \leq Rad(H + K)$. Hence $M = M_1 \oplus M_2$ is a g-$\oplus$-cs-module. □

Theorem 3.10. *Any finite direct sum of g-$\oplus$-cs-modules is a g-$\oplus$-cs-module.*

Proof. By Lemma 3.9. □

Let M be a module. A submodule X of M is called *fully invariant* if for every $h \in End_R(M)$, $h(X) \subseteq X$. The module M is called *duo* , if every submodule of M is fully invariant.

Lemma 3.11. *Let M be a duo module. If $M = M_1 \oplus M_2$, then $A = (A \cap M_1) \oplus (A \cap M_2)$ for A is submodule of M.*

Proof. See [8]. □

One of the our aims in this section is to investigate conditions which ensure that a factor submodule of a g-$\oplus$-cs-module will be a g-$\oplus$-cs-module.

Proposition 3.12. *Assume that M is a g-$\oplus$-cs-duo module and $N \leq M$. Then M/N is a g-$\oplus$-cs-module.*

Proof. Let $N \leq K \leq M$ with K/N a cofinite submodule of M/N. Then $M/K \cong (M/N)/(K/N)$ is finitely generated. Since M is a g-$\oplus$-cs-module, there exists a submodule L and $L^{'}$ of M such that $M = K + L = L \oplus L^{'}$, and $K \cap L \leq Rad(K)$. Note that $M/N = K/N + (L + N)/N$, by modularity, $K \cap (L + N) = (K \cap L) + N$. Since $K \cap L \leq Rad(L)$, we have $K/N \cap (L + N)/N = [(K \cap L) + N]/N \leq Rad((L+$

$N)/N)$. This implies that $(L+N)/N$ is a generalized supplemented of K/N in M/N. Now $N=(N\cap L)\oplus(N\cap L^{'})$ by Lemma 3.11, implies that

$$(L+N)\cap(L^{'}+N)\leq N+(L+N\cap L+N\cap L^{'})\cap L^{'}.$$

It follows that $(L+N)\cap(L^{'}+N)\leq N$ and $M/N=((L+N)/N)\oplus((L^{'}+N)/N)$. Then $(L+N)/N$ is a direct summand of M/N. Consequently, M/N is a g-$\oplus$-cs-module. □

A module M is called *distributive* if its lattice of submodules is a distributive lattice, equivalently for submodules K,L,N of M, $N+(K\cap L)=(N+K)\cap(N+L)$ or $N\cap(K+L)=(N\cap K)+(N\cap L)$.

A module M is said to have *the Summand Sum Property* (SSP) if the sum of any pair of direct summands of M is a direct summand of M, i.e., if N and K are direct summands of M then $N+K$ is also a direct summand of M.

Theorem 3.13. (1) *Let M be a g-$\oplus$-cs-module and N a submodule of M. If for every direct summand K of M, $(N+K)/N$ is a direct summand of M/N then M/N is a g-$\oplus$-cs-module.*
(2) *Let M be a g-$\oplus$-cs-module with the SSP. Then every direct summand of M is a g-$\oplus$-cs-module.*
(3) *Let M be a g-$\oplus$-cs-distributive module. Then M/N is a g-$\oplus$-cs-module for every submodule N of M.*

Proof. (1) Any cofinite submodule of M/N has the form T/N where T is a cofinite submodule of M and $N\subseteq T$. Since M is a g-$\oplus$-cs-module, there exists a direct summand D of M such that $M=T+D=D\oplus D^{'}$ and $T\cap D\leq Rad(D)$ for some submodule D' of M. Now $M/N=T/N+(D+N)/N$. By hypothesis, $(D+N)/N$ is a direct summand of M/N. Note that $(T/N)\cap((D+N)/N)=[T\cap(D+N)]/N=[N+(D\cap T)]/N$. Since $T\cap D\leq Rad(D)$, we have $[(D\cap T)+N]/N\leq Rad((D+N)/N)$. This implies that $(D+N)/N$ is a generalized supplement submodule of T/N in M/N. Hence M/N is a g-$\oplus$-cs-module.
(2) Let N be a direct summand of M. Let $M=N\oplus N'$ for some $N'\leq M$. We want to show that M/N' is a g-$\oplus$-cs-module. Assume that L is a direct summand of M. Since M has the SSP, $L+N^{'}$ is a direct summand of M. Let $M=(L+N')\oplus K$ for some $K\leq M$. Then $M/N'=(L+N')/N'\oplus(K+N')/N'$. Therefore M/N' is a g-$\oplus$-cs-module by (1).
(3) Let D be a direct summand of M. Then $M=D\oplus D'$ for some submodule D' of M. Now $M/N=[(D+N)/N]+[(D'+N)/N]$ and $N=N+(D\cap D')=$

$(N+D)\cap(N+D')$ by distributivity of M. This implies that $M/N = [(D+N)/N]\oplus[(D'+N)/N]$. By (1), M/N is a g-$\oplus$-cs-module. □

A module M is said to have *the Summand Intersection Property* (SIP) if the intersection of any pair of direct summands of M is a direct summand of M, i.e., if N and K are direct summands of M then $N \cap K$ is also a direct summand of M.

Lemma 3.14. ([8, Corollary 18]) *Let M be a duo module. Then M has the SIP and the SSP.*

As a result of Theorem 3.13 and Lemma 3.14, we can obtain the following corollary.

Corollary 3.15. *Let M be a g-$\oplus$-cs-duo module. Then every direct summand of M is a g-$\oplus$-cs-module.*

4. Generalized Cofinitely Semiperfect Modules

We start with a connection between generalized projective covers and cofinitely generalized supplements.

Lemma 4.1. *Let N be a submodule of the module M and $f : M \to M/N$ be the canonical epimorphism. Also let P be any module, $g : P \to M/N$ and $h : P \to M$ such that g is h composed with f. Then the map g is a generalized cover epimorphism if and only if $Im(h)$ is a generalized supplement of N and $Ker(h) \leq Rad(P)$.*

Proof. If g is a generalized cover epimorphism then $N \cap Im(h) = h(Ker(g)) \leq Rad(Im(h)) = Rad(h(P))$. This implies that $Im(h) = h(P)$ is a generalized supplement of N since g is an epimorphism. Note that $Ker(h) \subseteq Ker(g)$. Therefore, we can obtain $Ker(h) \leq Rad(P)$. The converse is clear by Lemma 2.3. □

We come now to our main result.

Theorem 4.2. *For any module M the following statements are equivalent:*
(1) *M is a generalized cofinitely semiperfect module.*
(2) *M is a gacs-module by supplements which have generalized projective covers.*
(3) *M is a gcs-module by supplements which have generalized projective covers.*

Proof. (1) ⇒ (2) Let $M = A + B$ with M/A finitely generated and let $f : P \to M/A$ be a generalized projective cover. Since P is projective and M/A is isomorphic to $B/(A \cap B)$, the map f lifts to a map $g : P \to B$. Since f is a generalized cover, then $Im(g)$ is a generalized supplement of $A \cap B$ in B and g is a generalized cover by Lemma 4.1. Hence P is a generalized projective cover of $Im(g)$ which is clearly contained in B.

(2) ⇒ (3) Clear.

(3) ⇒ (1) Let M/N be a finitely generated factor module of M. Then N is a cofinite submodule. Let K denote a generalized supplement of N and let $f : P \to K$ be a generalized projective cover. K is a generalized cover of $K/(N \cap K)$, i.e., the natural epimorphism $g : K \to K/(N \cap K) \overset{h}{\cong} (N + K)/N = M/N$ is a generalized cover. Hence $hgf : P \to M/N$ is a generalized projective cover by Lemma 2.3. □

Theorem 4.3. *For any finitely generated module M the following statements are equivalent:*

(1) *M is a generalized semiperfect module.*

(2) *M is a generalized cofinitely semiperfect module.*

(3) *M is an amply f-supplemented module by finitely generated supplements which have generalized projective covers.*

(4) *M is an f-supplemented module by finitely generated supplements which have generalized projective covers.*

Proof. The proof is an easy modification of the proof of Theorem 4.2. □

Proposition 4.4. (1) *Every homomorphic image of a generalized cofinitely semiperfect module is generalized cofinitely semiperfect.*

(2) *Every generalized cover of generalized cofinitely semiperfect module is generalized cofinitely semiperfect.*

Proof. (1) Follows from Theorem 4.2 and Proposition 3.2.

(2) Follows from Theorems 4.2 and 3.5. □

A generalized cover $f : P \to M$ is called a *generalized M-projective cover* in case P is an M-projective module. We give a theorem without the proof which characterizes a generalized M-projective cover under conditions because its proof is similar to the proof of Theorem 4.2.

Theorem 4.5. *For any module M the following statements are equivalent:*

(1) *Every finitely generated factor module of M has a generalized M-projective cover.*

(2) *M is a gacs-module by supplements which have generalized M-projective covers.*

(3) *M is a gcs-module by supplements which have generalized M-projective covers.*

Proposition 4.6. *Let M be an R-module.*
(1) *If M is a projective g-$\oplus$-cs-module then M is a generalized cofinitely semiperfect module.*
(2) *Assume that $f : P \to M$ is a generalized projective cover of M. If P is a g-$\oplus$-cs-module, then the following statements are equivalent.*
  (a) *M is a generalized cofinitely semiperfect module.*
  (b) *P is a generalized cofinitely semiperfect module.*

Proof. (1) Let M/N be a finitely generated factor module of M. Then N is cofinite. Since M is a g-$\oplus$-cs-module, there exist submodules K and $K^{'}$ of M such that $M = N + K$, $N \cap K \leq Rad(K)$, and $M = K \oplus K^{'}$ for some $K' \leq M$. Clearly, K is projective. For the inclusion homomorphism $i : K \to M$ and the canonical epimorphism $\pi : M \to M/N$, we have $\pi i : K \to M/N$ is an epimorphism and $Ker(\pi i) = N \cap K \leq Rad(K)$.
(2) Clear from Theorem 4.5 and (1). □

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