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D-NICE POLYNOMIALS WITH FOUR ROOTS OVER INTEGRAL DOMAINS D OF ANY CHARACTERISTIC

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Abstract. Let D be an integral domain of any characteristic. We say that $p(x) \in D[x]$ is *D-nice* if $p(x)$ and its derivative $p'(x)$ split in $D[x]$. We begin by presenting a new equivalence relation for D-nice polynomials over integral domains D of characteristic $p > 0$, which leads to an important modification of our definition of equivalence classes of D-nice polynomials. We then present a partial solution to the unsolved problem of constructing and counting equivalence classes of D-nice polynomials $p(x)$ with four distinct roots. We consider the following three cases separately: (1) D has characteristic 0, (2) D has characteristic $p > 0$ and the degree of $p(x)$ is not a multiple of p, and (3) D has characteristic $p > 0$ and the degree of $p(x)$ is a multiple of p. In all these cases we give formulas for constructing some examples. In the final case we also count equivalence classes of D-nice polynomials for certain choices of the multiplicities of the roots of $p(x)$. To conclude, we state several problems about D-nice polynomials with four roots that remain unsolved.

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1. Introduction

Let D be any integral domain of any characteristic. We say that a polynomial $p(x) \in D[x]$ *splits* in $D[x]$ if, for some $n \geq 0$, $p(x)$ can be written in the form $p(x) = a(x - r_1)(x - r_2) \cdots (x - r_n)$ where $a \neq 0$ and $a, r_1, \ldots, r_n \in D$. We say that $p(x) \in D[x]$ is *D-nice* (or *nice in D*) if $p(x)$ and its derivative $p'(x)$ split in $D[x]$. By our definition of splitting, if $p(x)$ is D-nice, then $p'(x) \neq 0$. Most mathematicians who had researched $\mathbb{Z}$-nice polynomials since about 1960 were interested in constructing polynomials with integer coefficients, roots, and critical points—polynomials that are "nice" for calculus students to sketch (see [1] and [4], for example). Since many earlier papers use the term *nice* instead of *$\mathbb{Z}$-nice* (see [1] and [3], for example), we too will often use the term *nice* instead of *$\mathbb{Z}$-nice*.

The problem of constructing, describing, and classifying nice polynomials is now worthy in its own right (see [11]-[12]). We find it worthwhile to extend these earlier results in $\mathbb{Z}$ to all integral domains D of any characteristic.

The papers [5] and [9] present a new approach to D-nice polynomials by considering the relations between the roots and critical points of polynomials in $D[x]$. This new approach has led to many important new results on nice and D-nice polynomials (see [6]-[10]), including the results in this paper. Before these papers were written, [2] was considered the most important paper on nice polynomials. This same paper gives an extensive list of many of the papers on nice polynomials written before 2000.

In the next section, we present a new equivalence relation for D-nice polynomials $p(x)$ over integral domains D of characteristic $p > 0$. This leads to an important modification of our definition of equivalence classes of D-nice polynomials. As a major consequence of this new equivalence relation, we may assume none of the multiplicities of the roots of $p(x)$ are multiples of p. This assumption greatly simplifies the study of D-nice polynomials over integral domains of characteristic $p > 0$. In Section 3, we present a partial solution to the open problem of constructing and counting equivalence classes of D-nice polynomials $p(x)$ with four distinct roots. We consider two cases separately: the case where D has characteristic 0 and the case where D has characteristic $p > 0$ and the degree of $p(x)$ is not a multiple of p. In Section 4, we present a partial solution to the analogous problem concerning D-nice polynomials $p(x)$ with four distinct roots where D has characteristic $p > 0$ and where the degree of $p(x)$ is a multiple of p. In our partial solution we give formulas for constructing all such D-nice polynomials over integral domains D of characteristics 2 or 3. In Section 5, we conclude by stating several problems about D-nice polynomials with four roots that remain unsolved.

2. Preliminaries

The *type* of a polynomial is a list of the multiplicities of its distinct roots. For example, all polynomials of the type (6,5,5,3) are of the form $p(x) = a(x-r_1)^6(x-r_2)^5(x-r_3)^5(x-r_4)^3$ where r_1, r_2, r_3, and r_4 are all distinct and $a \neq 0$.

Most of the earlier papers on nice polynomials note that horizontal translations by integers, horizontal or vertical stretches by integer factors, and reflections over the coordinate axes transform a nice polynomial $p_1(x)$ into another nice polynomial

$p_2(x)$. In other words, these transformations preserve nicety. Each of these transformations has an inverse transformation that transforms $p_2(x)$ into $p_1(x)$. The paper [5] extends these transformations and their inverses to all integral domains D of characteristic 0 [5, Proposition 2.1 and Corollary 2.2]. But it is easy to extend these to integral domains D of characteristic $p > 0$. For convenience, we will refer to these transformations by using the same geometric descriptions we use in $\mathbb{Z}$ and in $\mathbb{Q}$, even if D is not ordered (i.e., even if D has no true geometric meaning). As in earlier papers, we define the horizontal translation of $p(x)$ by $a \in D$ units to be $p(x-a)$. The horizontal stretch and compression of $p(x)$ by a nonzero factor of $a \in D$ are defined by $p(x/a)$ and $p(ax)$, respectively. If necessary, the division occurs in the field of fractions of D. The vertical stretch and compression of $p(x)$ by a nonzero factor of $a \in D$ are defined by $ap(x)$ and $\frac{1}{a}p(x)$, respectively. Reflections of $p(x)$ over the x- and y-axes are defined by $-p(x)$ and $p(-x)$, respectively. We note that all these transformations preserve the type of a polynomial.

A recently discovered transformation that preserves nicety is the power transformation; its inverse is the root transformation [6, Theorem 2.1]: For any natural number n, a polynomial $p(x)$ is nice iff $[p(x)]^n$ is nice. This result clearly holds in any integral domain D of characteristic 0 and holds in integral domains D of characteristic $p > 0$ as long as n is not a multiple of p. If n is a multiple of p, then $\frac{d}{dx}[p(x)]^n = 0$; so, by our definition, $[p(x)]^n$ is not D-nice. It is obvious that the root transformation transforms a D-nice polynomial $p_1(x)$ into another D-nice polynomial $p_2(x)$ iff $p_1(x) = [p_2(x)]^n$ for some natural number n and some D-nice polynomial $p_2(x)$. The power transformation and the root transformation do not preserve the type of a polynomial. More precisely, if $p(x)$ is of the type $(m_1, \ldots, m_s)$, then $[p(x)]^n$ is of the type $(nm_1, \ldots, nm_s)$.

The most recently discovered transformation that preserves nicety of polynomials $p(x) = a(x-r_1)^{m_1}(x-r_2)^{m_2} \cdots (x-r_n)^{m_n}$ over integral domains D of characteristic $p > 0$ is the transformation that replaces one or more of the m_i's with positive integers m_i''s so that $m_i = m_i' \bmod p$. We shall call this transformation a *replacement of multiplicities mod p*. In particular, with this transformation, we may replace multiplicities m_i that are multiples of p with 0 (i.e., remove the factors $(x-r_i)^{m_i}$ for such m_i). Thus, if $m_1, \ldots, m_s$ are not multiples of p but $m_1', \ldots, m_t'$ are, then a D-nice polynomial of the type $(m_1, \ldots, m_s, m_1', \ldots, m_t')$ becomes a D-nice polynomial of the type $(m_1, \ldots, m_s)$ under this transformation. Since this transformation is important for the study of D-nice polynomials over integral domains D of characteristic $p > 0$ and since, as of now, this paper is the only one that mentions this transformation, we state this transformation as a theorem and prove below that it preserves nicety.

Theorem 2.1. *Suppose D is an integral domain of characteristic $p > 0$. Suppose that $m_1, \ldots, m_s$ are not multiples of p but that $m'_1, \ldots, m'_t$ are. Then $p_1(x) = a(x-r_1)^{m_1} \cdots (x-r_s)^{m_s}(x-r'_1)^{m'_1} \cdots (x-r'_t)^{m'_t}$ is D-nice iff $p_2(x) = a(x-r_1)^{m_1} \cdots (x-r_s)^{m_s}$ is. Furthermore, if $m_i = m'_i$ mod p for all i, then $p_1(x) = a(x-r_1)^{m_1} \cdots (x-r_s)^{m_s}$ is D-nice iff $p_2(x) = a(x-r_1)^{m'_1} \cdots (x-r_s)^{m'_s}$ is.*

Proof. To see this, first note that if $p_1(x)$ is a polynomial of the type $(m_1, \ldots, m_s)$ and $p_2(x)$ is a polynomial of the type $(m'_1, \ldots, m'_t)$ where $p_1(x)$ and $p_2(x)$ split in $D[x]$ and have no roots in common, then $p(x) = p_1(x)p_2(x)$ is a polynomial of the type $(m_1, \ldots, m_s, m'_1, \ldots, m'_t)$. Since $p'(x) = p'_1(x)p_2(x)$, $p(x)$ is D-nice iff $p_1(x)$ is. A slight variant of this argument proves the second part of the theorem. □

Translating horizontally $a \in D$ units, stretching horizontally or vertically by factors of $a \in D$, reflecting over the coordinate axes, taking powers, and replacing multiplicities mod p (m_i with m'_i) are transformations we call *equivalence transformations*. The corresponding inverse transformations are translating horizontally $-a \in D$ units, compressing horizontally or vertically by factors of $a \in D$, reflecting over the coordinate axes, taking roots, and replacing multiplicities mod p (m'_i with m_i). Although equivalence transformations do transform a D-nice polynomial into a D-nice polynomial, these inverse transformations do not necessarily transform a D-nice polynomial into another D-nice polynomial. For example, a horizontal or vertical compression may transform a D-nice polynomial $p_1(x)$ into a polynomial $p_2(x)$ where $p_2(x)$ is nice in the field of fractions of D rather than in D. The root transformation applied to arbitrary D-nice polynomials may result in nonpolynomials.

Since any finite composition of equivalence transformations transform a D-nice polynomial $p_1(x)$ into another D-nice polynomial $p_2(x)$ (the corresponding inverse transformations transform $p_2(x)$ into $p_1(x)$), we say that the two D-nice polynomials $p_1(x)$ and $p_2(x)$ are *equivalent* whenever $p_1(x)$ can be transformed into $p_2(x)$ and vice-versa by a finite composition of equivalence transformations or their inverse transformations. Since, by the power transformation and its inverse the root transformation, any D-nice polynomial of the type $(nm_1, \ldots, nm_s)$ is equivalent to a D-nice polynomial of the type $(m_1, \ldots, m_s)$ for any $n > 1$ if D has characteristic 0 and any $n > 1$ that is not a multiple of p if D has characteristic $p > 0$, we consider these types to be equivalent. By Theorem 2.1, if D has characteristic $p > 0$, any

D-nice polynomial of the type $(m_1, \ldots, m_s)$ is equivalent to a D-nice polynomial of the type $(m'_1, \ldots, m'_s)$ where $m_i = m'_i \bmod p$, so we consider these types to be equivalent. Likewise, if $m_1, \ldots, m_s$ are not multiples of p but $m'_1, \ldots, m'_t$ are, then we consider the types $(m_1, \ldots, m_s, m'_1, \ldots, m'_t)$ and $(m_1, \ldots, m_s)$ equivalent. When we count equivalence classes of D-nice polynomials, we count equivalence classes of D-nice polynomials of equivalent types.

Using the horizontal translation, we may assume $p(x)$ has a root at 0. Multiplying $p(x)$ by a nonzero element in D (which is a vertical stretch or compression) results in an equivalent D-nice polynomial, so we may assume $p(x)$ is monic. Using the power transformation, we may assume the multiplicities m_i have no common factor. These three assumptions greatly simplify the problem of constructing and counting equivalence classes of D-nice polynomials $p(x)$ with four distinct roots. By our assumptions, such polynomials have the form $p(x) = x^{m_0}(x - r_1)^{m_1}(x - r_2)^{m_2}(x - r_3)^{m_3}$ with m_0, m_1, m_2, and m_3 having no common factor. Furthermore, if D has characteristic $p > 0$ and one of the multiplicities is a multiple of p, then, by Theorem 2.1, such a D-nice polynomial is equivalent to a D-nice polynomial with three roots. A paper on constructing and counting equivalence classes of all types of D-nice polynomials with three roots is currently in progress and will be submitted for publication soon. Then the methods and results in this paper can be used to construct and count equivalence classes of these types of D-nice polynomials. If two or three of the multiplicities are multiples of p, then such D-nice polynomials are equivalent to D-nice polynomials with two roots or one root, respectively. It is easy to check that all types of D-nice polynomials with one or two roots exist and that any two of the same type are equivalent: Any D-nice polynomial with one root of multiplicity d is equivalent to the D-nice polynomial $p(x) = x^d$, and any D-nice polynomial of the type (m_0, m_1) is equivalent to the D-nice polynomial $p(x) = x^{m_0}(x-d)^{m_1}$ where $d = m_0 + m_1$ is not a multiple of the characteristic of D and to the D-nice polynomial $p(x) = x^{m_0}(x - 1)^{m_1}$ if d is. In general, if a D-nice polynomial $p(x)$ has N roots and one or more of the multiplicities are multiples of p, then $p(x)$ is equivalent to a D-nice polynomial with fewer with N roots. Hence, we may assume that none of the multiplicities of the roots of $p(x)$ are multiples of p. With this assumption, we significantly reduce the number of cases we need to consider when we study D-nice polynomials with a specified degree or a specified number of roots since we need not consider any of the possible cases where one or more of the multiplicities of the roots are multiples of p. Indeed Theorem 2.1 is an important new result for the study of D-nice polynomials over integral domains D of characteristic $p > 0$.

3. D-Nice Polynomials $p(x)$ with Four Roots Where the Degree of $p(x)$ Is Not a Multiple of p

We now consider the problem of constructing and counting equivalence classes of D-nice polynomials $p(x)$ with four distinct roots. In this section we consider the cases where D has characteristic 0 and where D has characteristic $p > 0$ and the degree of $p(x)$ is not a multiple of p. Some of our results will apply to both cases simultaneously (for example, Lemma 3.1 below does). To simplify wording for these results, we will describe both cases simultaneously by saying that $p(x)$ is a D-nice polynomial over an integral domain D of characteristic $p \geq 0$ where the degree of $p(x)$ is not a multiple of p. For such a $p(x) = x^{m_0}(x-r_1)^{m_1}(x-r_2)^{m_2}(x-r_3)^{m_3}$, its derivative is $p'(x) = dx^{m_0-1}(x-r_1)^{m_1-1}(x-r_2)^{m_2-1}(x-r_3)^{m_3-1} \cdot (x-c_1)(x-c_2)(x-c_3)$. We now state in Lemma 3.1 below the relations between the roots and critical points of such polynomials $p(x) \in D[x]$. We will use this lemma to derive a formula for constructing some (but not all) of the D-nice polynomials of these types.

Lemma 3.1. *Let D be any integral domain of characteristic $p \geq 0$. Let $p(x) = x^{m_0}(x-r_1)^{m_1}(x-r_2)^{m_2}(x-r_3)^{m_3} \in D[x]$ be a polynomial of degree $d = m_0 + m_1 + m_2 + m_3$ with four roots in D where d is not a multiple of p. Let $p'(x) = dx^{m_0-1}(x-r_1)^{m_1-1}(x-r_2)^{m_2-1}(x-r_3)^{m_3-1}(x-c_1)(x-c_2)(x-c_3)$. Then $p(x)$ is D-nice iff there exist c_1, c_2, and c_3 in D such that*

$$\sum_{i=1}^{3}(d-m_i)r_i = d(c_1+c_2+c_3), \tag{3.1}$$

$$\sum_{1\leq i<j\leq 3}(d-m_i-m_j)r_ir_j = d(c_1c_2+c_1c_3+c_2c_3), \tag{3.2}$$

$$m_0r_1r_2r_3 = dc_1c_2c_3. \tag{3.3}$$

Proof. These relations follow from [9, Corollary 3.3]. □

Remark. The proof of Lemma 4.2, which appears later in this paper, can be modified appropriately to prove Lemma 3.1 directly without using [9, Corollary 3.3].

By Lemma 3.1, there is a one-to-one correspondence between all solutions to (3.1)-(3.3) in D and all D-nice polynomials $p(x)$ of the type (m_0, m_1, m_2, m_3) as described above. Thus, to derive a formula for constructing these types of D-nice polynomials, we may solve (3.1)-(3.3) in D. The problem of deriving such a formula is only partially solved since no formula that gives all solutions to (3.1)-(3.3) in D

has been derived (see Problems 5.1-5.2). But the formula we derive does allow us to construct some of the solutions to (3.1)-(3.3) in D. In particular, our formula will allow us to construct D-nice polynomials of any given type (m_0, m_1, m_2, m_3) over integral domains of characteristic 0. We begin with the case $D = \mathbb{Z}$ since nice polynomials (i.e., $\mathbb{Z}$-nice polynomials) are of special interest. We can then easily generalize our work to derive a formula for constructing D-nice polynomials over other integral domains D of characteristic $p \geq 0$.

Since it is easier to construct solutions to (3.1)-(3.3) in $\mathbb{Q}$ than in $\mathbb{Z}$, we construct these solutions in $\mathbb{Q}$ instead. Whenever we construct a $\mathbb{Q}$-nice polynomial $p(x)$ with our formula, we may construct an equivalent nice polynomial by stretching $p(x)$ horizontally by an appropriate factor. We note that it is easy to extend Lemma 3.1 from polynomials over D to polynomials over the field of fractions of D. The papers [7], [8] and [10] present constructions of nice and D-nice polynomials by using a similar approach.

Using the horizontal compression for $\mathbb{Q}$-nice polynomials, we may assume $r_1 = 1$. In this case, (3.1)-(3.3) become

$$d - m_1 + (d - m_2)r_2 + (d - m_3)r_3 = d(c_1 + c_2 + c_3), \tag{3.4}$$

$$(d - m_1 - m_2)r_2 + (d - m_1 - m_3)r_3 + (d - m_2 - m_3)r_2r_3 = d(c_1c_2 + c_1c_3 + c_2c_3), \tag{3.5}$$

$$m_0r_2r_3 = dc_1c_2c_3. \tag{3.6}$$

We now solve each of the equations (3.4)-(3.6) for r_2 in terms of c_1. Solving (3.4) for r_2 in terms of c_1, we have

$$r_2 = \frac{d}{d - m_2}c_1 + A \tag{3.7}$$

where $A = \frac{d(c_2+c_3)+(m_3-d)r_3+m_1-d}{d-m_2}$. Solving (3.5) for r_2 in terms of c_1, we have

$$r_2 = Bc_1 + C \tag{3.8}$$

where $B = \frac{d(c_2+c_3)}{d-m_1-m_2+(d-m_2-m_3)r_3}$ and $C = \frac{dc_2c_3+(m_1+m_3-d)r_3}{d-m_1-m_2+(d-m_2-m_3)r_3}$. Finally, solving (3.6) for r_2 in terms of c_1, we have

$$r_2 = Dc_1 \tag{3.9}$$

where $D = \frac{dc_2c_3}{m_0r_3}$. If we find a value of c_1 (in terms of the other variables) so that (3.7) and (3.8) are equal and so that (3.9) and (3.7) are equal as well, then, for that particular c_1, all three expressions for r_2 are equal. It will then follow that this value of c_1 gives us a solution to (3.4)-(3.6). We now find a value of c_1

(in terms of the other variables) so that the expressions for r_2 in (3.7) and (3.8) are equal. Setting these expressions equal and assuming $\frac{d}{d-m_2} \neq B$, we see that $c_1 = (C-A)(\frac{d}{d-m_2} - B)^{-1}$ is a solution to this equation. For this value of c_1, the expressions for r_2 in (3.7) and (3.8) are equal.

We note that (3.7) and (3.9) are equal if $D = \frac{d}{d-m_2}$ and $A = 0$. Since these conditions are not necessary (we give an example later to show this), the formula we derive will not allow us to construct all rational solutions to (3.4)-(3.6). It is not clear right now how to find necessary conditions and how to use them to construct all rational solutions to (3.4)-(3.6). However, with our formula, we can still construct nice polynomials of any given type with four distinct roots, which we will prove later. The equations $D = \frac{d}{d-m_2}$ and $A = 0$ lead to the system of equations

$$\begin{aligned}(d-m_3)r_3 + d - m_1 &= d(c_2 + c_3),\\ m_0 r_3 &= (d-m_2)c_2 c_3.\end{aligned}$$

Solutions to this system of equations tell us which values of c_2, c_3, and r_3 give us a value of c_1 that leads to a solution to (3.4)-(3.6). To solve this system, we regard r_3 and c_3 as variables and treat c_2 as a free variable. Then the system becomes a linear system of equations in the two variables r_3 and c_3. We then rewrite the system of equations as

$$\begin{aligned}(d-m_3)r_3 - dc_3 &= dc_2 + m_1 - d,\\ m_0 r_3 - (d-m_2)c_2 c_3 &= 0.\end{aligned}$$

If we assume there are values of c_2 so that the solution to this system is unique (which we will prove shortly), then Cramer's rule gives us the following solution:

$$r_3 = \frac{c_2(m_2-d)(m_1+dc_2-d)}{m_0 d - d^2 c_2 + dm_2 c_2 + dm_3 c_2 - m_2 m_3 c_2}, \tag{3.10}$$

$$c_3 = \frac{m_0(d - dc_2 - m_1)}{m_0 d - d^2 c_2 + dm_2 c_2 + dm_3 c_2 - m_2 m_3 c_2}. \tag{3.11}$$

Summary of Construction. In short, to construct a $\mathbb{Q}$-nice polynomial $p(x)$ with four distinct roots, we choose a nonzero rational value for c_2, and we use (3.10) and (3.11) to find c_3 and r_3. Then we let $c_1 = C(\frac{d}{d-m_2} - B)^{-1}$ (since $A = 0$ with these choices of c_2, c_3, and r_3). Finally, we let $r_2 = \frac{d}{d-m_2}c_1$ (actually, we may use any one of the values (3.7)-(3.9) since all three are equal in this case). And $r_1 = 1$, as we assumed from the beginning. Stretching $p(x)$ horizontally by an appropriate factor will give us a nice polynomial with four distinct roots.

The following example shows that the sufficient conditions $A = 0$ and $D = \frac{d}{d-m_2}$ for finding a solution to (3.4)-(3.6) are not necessary (we found this example with a computer search). It easy to check that $m_0 = 2$, $m_1 = m_2 = m_3 = 1$, $r_1 = 1$, $r_2 = 4/3$, $r_3 = 7/4$, $c_1 = 1/2$, $c_2 = 8/5$, and $c_3 = 7/6$ is a solution to (3.4)-(3.6). However, $A = 17/24 \neq 0$ and $D = 8/3 \neq d/(d - m_2)(= 5/4)$, which proves that these conditions are not necessary for finding a solution to (3.4)-(3.6).

As an illustration, we now construct several examples of nice polynomials using our formulas. For the first example, let $m_0 = 2$ and $m_1 = m_2 = m_3 = 1$. Choose $c_2 = 2$. Then $r_3 = 24/11$, $c_3 = 6/11$, $r_2 = 12/7$, and $c_1 = 48/35$. Stretching this example horizontally by 385, we obtain the nice polynomial

$$\begin{aligned} p(x) &= x^2(x-385)(x-660)(x-840), \\ p'(x) &= 5x(x-210)(x-528)(x-770). \end{aligned} \tag{3.12}$$

For the second example, let $m_0 = m_1 = 2$ and $m_2 = m_3 = 1$. Choose $c_2 = 3/2$. Then $r_3 = 25/17$, $c_3 = 20/51$, $c_1 = 75/59$, and $r_2 = 90/59$. Stretching this example horizontally by 6018, we obtain the nice polynomial

$$\begin{aligned} p(x) &= x^2(x-6018)^2(x-9180)(x-8850), \\ p'(x) &= 6x(x-6018)(x-2360)(x-7650)(x-9027). \end{aligned} \tag{3.13}$$

Note that this sextic is not symmetric. A polynomial $p(x) \in D[x]$ is called *symmetric* if there exists a c, called a *center*, such that $p(c+x) = p(c-x)$ and *antisymmetric* if there exists a center c such that $p(c+x) = -p(c-x)$. It is easy to check that if $p(x)$ is symmetric with a center c, then $p'(x)$ is antisymmetric with a center c. Furthermore, if D has any characteristic but 2, then $p(c) = -p(c)$ for any antisymmetric $p(x)$, which says that any center of an antisymmetric polynomial is a root. Using these facts, we see that, if example (3.13) were symmetric, then the only possible choices for a center are its critical points. But it is easy to check that none of its critical points are centers, so it follows that this sextic is not symmetric.

For the last example, let $m_0 = 2$, $m_1 = m_2 = 1$, and $m_3 = 3$. Choose $c_2 = 4$. Then $r_3 = 264/41$, $c_3 = 22/41$, $c_1 = 1056/833$, and $r_2 = 176/119$. Stretching this example horizontally by 34153, we obtain the nice polynomial

$$\begin{aligned} p(x) &= x^2(x-34153)(x-50512)(x-219912)^3, \\ p'(x) &= 7x(x-219912)^2(x-18326)(x-43296)(x-136612). \end{aligned} \tag{3.14}$$

Note that when we derived our formulas above, we used only those properties that hold for any field, not any properties specific to $\mathbb{Q}$, so our formulas allow us to construct D-nice polynomials over any integral domain D of characteristic 0. Since Lemma 3.1 holds whenever D is an integral domain of characteristic $p > 0$ and none of the numbers d, m_0, m_1, m_2, and m_3 are multiples of p, we may use our formulas to construct D-nice polynomials of these types over such integral domains D. However, our formulas fail if D is a finite field that is "too small." And there are choices of the characteristic p and the type (m_0, m_1, m_2, m_3) where we cannot guarantee that our formulas work. We will explain these problems in more detail later.

We now construct two examples of $\mathbb{Z}[i]$-nice polynomials using our formulas. For the first example, let $m_0 = 2$ and $m_1 = m_2 = m_3 = 1$, and choose $c_2 = 1 + i$. Then $r_3 = 72/73 + (100/73)i$, $c_3 = 43/73 + (7/73)i$, $c_1 = 112/85 + (8/17)i$, and $r_2 = 28/17 + (10/17)i$. Stretching this example horizontally by $6205 + 6205i$, we obtain the $\mathbb{Z}[i]$-nice polynomial

$$\begin{aligned} p(x) &= x^2[x - (6205 + 6205i)][x - (6570 + 13870i)][x - (-2380 + 14620i)], \\ p'(x) &= 5x[x - (5256 + 11096i)][x - 12410i][x - (3060 + 4250i)]. \end{aligned} \tag{3.15}$$

This example is not equivalent to a nice polynomial. Any two such polynomials of this type with the double root at 0 are equivalent under only horizontal stretches and compressions and reflections over the y-axis. Thus, if example (3.15) were equivalent to a nice polynomial, then the ratio of any two nonzero roots would be a rational number, which gives us a contradiction.

For the second example, let $m_0 = m_1 = 2$ and $m_2 = m_3 = 1$, and choose $c_2 = i$. Then $r_3 = -140/769 + (990/769)i$, $c_3 = 396/769 + (56/769)i$, $c_1 = 85/106 + (165/106)i$, and $r_2 = 51/53 + (99/53)i$. Stretching this example horizontally by $81514 + 81514i$, we obtain the $\mathbb{Z}[i]$-nice polynomial

$$\begin{aligned} p(x) &= x^2[x - (81514 + 81514i)]^2[x - (-73824 + 230700i)] \\ &\quad \cdot [x - (-119780 + 90100i)], \\ p'(x) &= 6x[x - (81514 + 81514i)][x - (-61520 + 192250i)] \\ &\quad \cdot [x - (-81514 + 81514i)][x - (36040 + 47912i)]. \end{aligned} \tag{3.16}$$

Note that this sextic is not symmetric and is not equivalent to a nice polynomial. Checking that this sextic is not symmetric is similar to checking that example (3.13) is not symmetric. Since all nice sextics of this type are equivalent to $\mathbb{Q}$-nice sextics with double roots at 0 and 1 under only horizontal stretches and compressions and reflections over the y-axis, if the polynomial (3.16) were equivalent to a nice or

$\mathbb{Q}$-nice polynomial, it would be equivalent to one with double roots at 0 and 1. But the ratio of any two nonzero roots of (3.16) is nonreal, which is a contradiction.

Let D be an integral domain of characteristic 0. Given the type (m_0, m_1, m_2, m_3), we now prove that there are only finitely many values of c_2 where our formulas fail to produce an example of a D-nice polynomial of this type. To find all such c_2, we check when r_3 and c_3 are 0, when r_3 and c_3 are undefined, when B and C are undefined, when $C = 0$, when $\frac{d}{d-m_2} - B = 0$, when $r_3 = 1$, when r_2 is undefined, when $r_2 = 0$, when $r_2 = 1$, and when $r_2 = r_3$. Since the solutions to many of these equations are too complicated to write effectively, we decide instead in these particular cases to argue that the equations do not reduce to $0 = 0$. If all such equations never reduce to $0 = 0$, regardless of the choice of the multiplicities of the roots, then it will follow that we may choose c_2 so that all these cases where the formulas fail to produce examples of D-nice polynomials with four distinct roots can be avoided. Once we check these details, then the following result will follow.

Theorem 3.2. *Let D be an integral domain of characteristic 0. Let m_0, m_1, m_2, and m_3 be any four positive integers. Then there exists a D-nice polynomial of the type (m_0, m_1, m_2, m_3) with four distinct roots.*

Proof. We now check all the equations that we mentioned above. In checking most of these equations, we often assume one or more of the variables are nonzero or that the denominators in some of our expressions are nonzero. We may do so since these separate cases where these assumptions do not hold do not lead us to cases where the equation we are checking has any additional solutions. Finally, we may assume that $c_2 \neq 0$ throughout the proof since the value $c_2 = 0$ is always excluded. To begin, first note that r_3 and c_3 are 0 iff $c_2 = \frac{d-m_1}{d}$.

Next note that the denominators in (3.10) and (3.11) are 0 iff $m_0 d - d^2 c_2 + d m_2 c_2 + d m_3 c_2 - m_2 m_3 c_2 = 0$ iff $c_2 = \frac{m_0 d}{d^2 - d m_2 - d m_3 + m_2 m_3}$. Furthermore, the denominators in B and C are 0 iff $d - m_1 - m_2 + (d - m_2 - m_3) r_3 = 0$. We may assume the denominator in r_3 is nonzero, so we substitute (3.10) for r_3 and clear denominators to rewrite the equation as $(d - m_1 - m_2)(m_0 d - d^2 c_2 + d m_2 c_2 + d m_3 c_2 - m_2 m_3 c_2) + c_2(d - m_2 - m_3)(m_2 - d)(m_1 + d c_2 - d) = 0$. This equation is of the form $A_2 c_2^2 + A_1 c_2 + A_0 = 0$ with $A_0 = m_0 d^2 - m_0 m_1 d - m_0 m_2 d = d m_0 (m_0 + m_3) \neq 0$, so this equation does not reduce to $0 = 0$.

We now check when $C = 0$. The equation $C = 0$ is $d c_2 c_3 + (m_1 + m_3 - d) r_3 = 0$. Substitute (3.10) and (3.11) for r_3 and c_3 and clear denominators (we may assume the denominators are nonzero) to rewrite the equation as $-m_0 d c_2 (m_1 + d c_2 - d) +$

$c_2(m_1+m_3-d)(m_2-d)(m_1+dc_2-d)=0$. Since we may assume $r_3, c_3 \neq 0$ (so we assume $m_1+dc_2-d \neq 0$) and since we may assume $c_2 \neq 0$, we may cancel $c_2(m_1+dc_2-d)$, so our equation is now $-m_0d+(m_1+m_3-d)(m_2-d)=0$. We now use $d=m_0+m_1+m_2+m_3$ to rewrite this equation as $-m_0d-(m_0+m_2)(m_2-d)=0$, which we can further rewrite as $m_2(m_1+m_3)=0$ by expanding the left-hand side and using the definition of d again. But this equation is a contradiction since D has characteristic 0, so $C \neq 0$ iff $c_2 \neq 0$ and c_2 is chosen so that r_3 and c_3 are nonzero and defined.

We now check the equation $\frac{d}{d-m_2}-B=0$. Substitute the value for B into this equation. Assuming B has a nonzero denominator (which we may do so) and noting that $d \neq 0$, we clear denominators and cancel d from both sides to rewrite the equation as $d-m_1-m_2+(d-m_2-m_3)r_3-(d-m_2)(c_2+c_3)=0$. Substituting (3.10) and (3.11) for r_3 and c_3 and assuming the denominator is nonzero (we may do so), we then clear denominators and cancel c_2 from both sides to write the equation as $(d-m_1-m_2)(m_0d-d^2c_2+dm_2c_2+dm_3c_2-m_2m_3c_2)+(d-m_2-m_3)(m_1-d)(m_1+dc_2-d)-(d-m_2)(m_0d-d^2c_2+dm_2c_2+dm_3c_2-m_2m_3c_2)+m_0(d-m_2)(dc_2+m_1-d)=0$. We now check that this equation does not reduce to $0=0$ by noting that this equation can be written in the form $B_2c_2^2+B_1c_2+B_0=0$ where $B_0=-m_0m_1m_2 \neq 0$.

The equation $r_3=1$ does not reduce to $0=0$. To see this, note that $r_3=1$ iff $c_2(m_2-d)(m_1+dc_2-d)=m_0d-d^2c_2+dm_2c_2+dm_3c_2-m_2m_3c_2$, which is of the form $D_2c_2^2+D_1c_2+D_0=0$ where $D_0=m_0d \neq 0$.

By (3.9), r_2 is undefined iff $r_3=0$ (with this equation considered earlier). And $r_2 \neq 0$ iff c_1, c_2, and c_3 are nonzero. But the equations $c_1 \neq 0$ and $c_3 \neq 0$ are equivalent to equations we checked earlier.

The last two equations require much more work to check than the previous equations had required. Before checking these equations, we write r_2 in terms of c_2, the degree of $p(x)$, and the multiplicities of the roots of $p(x)$. If we do so, then we have

$$r_2=\frac{c_2(dc_2-m_0-m_2-m_3)(m_1+m_3)}{m_0m_1c_2-m_3^2c_2-m_0m_3c_2+m_0m_3c_2^2+m_1m_3c_2^2+m_1^2c_2-m_0m_1+m_3^2c_2^2}. \tag{3.17}$$

We used Maple 10 to do this directly since the details are too complicated to present effectively and efficiently. We now start with the simpler equation, $r_2=1$. By (3.17), this equation is $c_2(dc_2-m_0-m_2-m_3)(m_1+m_3)=m_0m_1c_2-m_3^2c_2-m_0m_3c_2+m_0m_3c_2^2+m_1m_3c_2^2+m_1^2c_2-m_0m_1+m_3^2c_2^2$. This equation does not reduce to $0 = 0$ since this equation is of the form $E_2c_2^2 + E_1c_2 + E_0 = 0$ where $E_0 = m_0m_1 \neq 0$.

To complete the proof, we need to check that the equation $r_2 = r_3$ does not reduce to $0 = 0$. Setting (3.10) and (3.17) equal, cancelling c_2 from both sides, and cross-multiplying, we write the equation as $(m_0d - d^2c_2 + dm_2c_2 + dm_3c_2 - m_2m_3c_2)(dc_2 - m_0 - m_2 - m_3)(m_1 + m_3) = (m_2 - d)(m_1 + dc_2 - d)(m_0m_1c_2 - m_3^2c_2 - m_0m_3c_2 + m_0m_3c_2^2 + m_1m_3c_2^2 + m_1^2c_2 - m_0m_1 + m_3^2c_2^2)$. This equation is of the form $F_3c_2^3 + F_2c_2^2 + F_1c_2 + F_0 = 0$ where $F_3 = dm_3(d - m_2)^2 \neq 0$, so this equation does not reduce to $0 = 0$, as we wished to show. This check completes the proof of Theorem 3.2. □

Most of the proof of Theorem 3.2 extends to the case where D has characteristic $p > 0$. Note that all the equations we have checked cause no problems here except the equation $C = 0$, the equation where the denominators in (3.10) and (3.11) are 0, and the equation $r_2 = r_3$. When we had checked the equation $C = 0$, we had said this equation does not reduce to $0 = 0$ since $m_2(m_1 + m_3) \neq 0$. For such D, this is nonzero iff $m_1 \neq -m_3 \mod p$. When we checked the equation where the denominators in (3.10) and (3.11) are 0, we earlier had said that this equation does not reduce to $0 = 0$ since $dm_0(m_0 + m_3) \neq 0$. For such D, this is nonzero iff $m_0 \neq -m_3 \mod p$. Finally, when we checked the equation $r_2 = r_3$, we argued that the equation does not reduce to $0 = 0$ since $dm_3(d - m_2)^2 \neq 0$. But, for such D, this is nonzero iff $d \neq m_2 \mod p$. Thus, with these three additional assumptions, the argument above continues to hold if D is an infinite integral domain of characteristic $p > 0$ or if D is a finite field that is not "too small." More precisely, if the number of values of $c_2 \in D$ for which our formulas fail is less than $|D|$, then we can construct D-nice polynomials with four distinct roots using our formulas; otherwise, we cannot. For example, the total number of such values of $c_2 \in D$ is at most 13. Thus, if D has more than 13 elements, we can find such a D-nice polynomial with our formulas. If D has 13 or fewer elements, it is still possible in some cases to construct such D-nice polynomials with our formulas. The same can be said if one or more of our three assumptions do not hold. But stating all the conditions in which we may do so cannot be done effectively; therefore, for simplicity, we choose not to state all these conditions.

Our comments lead to the following theorem.

Theorem 3.3. *Let D be an integral domain of characteristic $p > 0$. Let m_0, m_1, m_2, m_3, and $d = m_0 + m_1 + m_2 + m_3$ be any positive integers that are not multiples* *of p and further assume that $m_1 \neq -m_3$ mod p, $m_0 \neq -m_3$ mod p, and $d \neq m_2$ mod p. If D has more than 13 elements, then there exists a D-nice polynomial of the type (m_0, m_1, m_2, m_3).*

Paper [7] proves that the number of equivalence classes of all types of nice polynomials with three roots is infinite and that papers [8] and [10] prove the analogous results for D-nice symmetric polynomials over infinite integral domains. Furthermore, in Section 4 of this paper, in all cases where we can count all equivalence classes of D-nice polynomials and D is infinite, the number of equivalence classes is infinite. These results suggest the following conjecture.

Conjecture 3.4. Suppose D is an infinite integral domain of characteristic $p \geq 0$. Then the number of equivalence classes of all types of D-nice polynomials $p(x)$ with four roots where the degree of $p(x)$ is not a multiple of p is infinite.

Application. Although (3.1)-(3.3) has not been completely solved, these relations can help us program a computer to search for examples of such D-nice polynomials, especially for examples in $D = \mathbb{Z}$ or $D = \mathbb{Z}/(p)$ for a prime p. Computer searches for such $\mathbb{Z}/(p)$-nice polynomials with four roots have revealed that no $\mathbb{Z}/(p)$-nice nonsymmetric quartics with four distinct roots exist when $p < 23$. Searching for examples with a computer and counting equivalence classes has revealed that all $\mathbb{Z}/(23)$-nice nonsymmetric quartics are equivalent to

$$p(x) = x(x-1)(x-4)(x-6),$$
$$p'(x) = 4(x-17)(x-13)(x-7).$$

Searching for examples with a computer and counting equivalence classes has revealed that all $\mathbb{Z}/(29)$-nice nonsymmetric quartics are equivalent to

$$p(x) = x(x-1)(x-2)(x-6),$$
$$p'(x) = 4(x-27)(x-23)(x-22).$$

The details of these facts and the details of other computer searches for $\mathbb{Z}/(p)$-nice polynomials with four roots will later be written as a separate paper and submitted for publication.

4. D-Nice Polynomials $p(x)$ with Four Roots Where the Degree of $p(x)$ Is a Multiple of p

Let D be an integral domain of characteristic $p > 0$. We now consider the case where the degree d of $p(x) = x^{m_0}(x-r_1)^{m_1}(x-r_2)^{m_2}(x-r_3)^{m_3}$ is a multiple of

p. In this case, the degree of $p'(x)$ is $d-2$ or less, so Lemma 3.1 does not apply to such $p(x)$. For such $p(x)$, $p'(x) = x^{m_0-1}(x-r_1)^{m_1-1}(x-r_2)^{m_2-1}(x-r_3)^{m_3-1}q(x)$ where $q(x) = (m_1r_1 + m_2r_2 + m_3r_3)x^2 + [(m_0+m_3)r_1r_2 + (m_0+m_2)r_1r_3 + (m_0 + m_1)r_2r_3]x - m_0r_1r_2r_3$. The degree of $q(x)$ determines the degree of $p'(x)$. We summarize these important details in the following proposition. Since the proof is easy, we omit it.

Proposition 4.1. *Let D be an integral domain of characteristic $p > 0$ with field of fractions $QF(D)$. Suppose (m_0, m_1, m_2, m_3) is a type where none of the multiplicities are multiples of p but that $d = m_0 + m_1 + m_2 + m_3$ is. Let $p(x) = x^{m_0}(x-r_1)^{m_1}(x-r_2)^{m_2}(x-r_3)^{m_3}$. Then the following results hold:*

(a) *$p'(x)$ has degree $d-2$ iff $m_1r_1 + m_2r_2 + m_3r_3 \neq 0$. This nonzero number is the leading coefficient of $p'(x)$.*
(b) *$p'(x)$ has degree $d-3$ iff $m_1r_1 + m_2r_2 + m_3r_3 = 0$ and $(m_0+m_3)r_1r_2 + (m_0+m_2)r_1r_3 + (m_0+m_1)r_2r_3 \neq 0$. This nonzero number is the leading coefficient of $p'(x)$. Furthermore, in this case, $p(x)$ is $QF(D)$-nice iff $p(x)$ splits in $QF(D)[x]$.*
(c) *$p'(x)$ has degree $d-4$ iff $m_1r_1 + m_2r_2 + m_3r_3 = 0$ and $(m_0+m_3)r_1r_2 + (m_0+m_2)r_1r_3 + (m_0+m_1)r_2r_3 = 0$. The leading coefficient of $p'(x)$ is $-m_0r_1r_2r_3$. In this case, $p(x)$ is D-nice iff $p(x)$ splits in $D[x]$.*

If $p(x)$ is a polynomial as described by Proposition 4.1 but $m_1r_1+m_2r_2+m_3r_3 \neq 0$ and $p(x)$ splits in $\mathrm{QF}(D)[x]$, then $p(x)$ need not be $\mathrm{QF}(D)$-nice since $q(x)$ has degree 2. In this case, we derive the relations between the roots and critical points of such polynomials $p(x)$. Though these relations follow directly from [9, Theorem 3.4], we derive these relations directly without using this theorem since [9] has not yet been accepted for publication and so that the reader may see how these relations are derived. We will use Lemma 4.2 later to derive formulas for constructing some examples of these types of D-nice polynomials.

Lemma 4.2. *Let D be an integral domain of characteristic $p > 0$. Suppose m_0, m_1, m_2, and m_3 are not multiples of p but that $d = m_0 + m_1 + m_2 + m_3$ is, and let $p(x) = x^{m_0}(x-r_1)^{m_1}(x-r_2)^{m_2}(x-r_3)^{m_3} \in D[x]$. Assume $m_1r_1+m_2r_2+m_3r_3 \neq 0$ so that $p'(x) = (m_1r_1+m_2r_2+m_3r_3)x^{m_0-1}(x-r_1)^{m_1-1}(x-r_2)^{m_2-1}(x-r_3)^{m_3-1}(x-c_1)(x-c_2)$ has degree $d-2$. Then $p(x)$ with four roots in D is D-nice iff there exist*

c_1 and c_2 in D so that

$$(m_0+m_3)r_1r_2+(m_0+m_2)r_1r_3+(m_0+m_1)r_2r_3 = \\ -(m_1r_1+m_2r_2+m_3r_3)(c_1+c_2), \qquad (4.1)$$

$$m_0r_1r_2r_3 = -(m_1r_1+m_2r_2+m_3r_3)c_1c_2. \qquad (4.2)$$

Proof. If $p(x)$ is as above, then $p'(x) = x^{m_0-1}(x-r_1)^{m_1-1}(x-r_2)^{m_2-1}\cdot (x-r_3)^{m_3-1}q(x)$ where $q(x) = (m_1r_1+m_2r_2+m_3r_3)(x-c_1)(x-c_2)$ since $q(x)$ is quadratic. As mentioned above, this $q(x)$ is equal to $(m_1r_1+m_2r_2+m_3r_3)x^2 + [(m_0+m_3)r_1r_2+(m_0+m_2)r_1r_3+(m_0+m_1)r_2r_3]x - m_0r_1r_2r_3$ as well. Equating coefficients of both forms of $q(x)$ gives us (4.1)-(4.2). □

In Section 4.1, we count equivalence classes of D-nice polynomials $p(x)$ of degree d with four distinct roots where the degree of $p'(x)$ is $d-3$ or $d-4$ and where the characteristic of D is $p>2$. In Section 4.2, we derive a formula for constructing D-nice polynomials $p(x)$ with four distinct roots over integral domains D of characteristic 2. We also count equivalence classes of certain types of such $p(x)$. In Section 4.3, we derive a formula for constructing D-nice polynomials of degree d with four distinct roots over integral domains D of characteristic 3 where the degree of $p'(x)$ is $d-2$. We also count equivalence classes of these types of D-nice polynomials. The cases where $p'(x)$ has degree $d-2$ and D has characteristic $p>3$ are unsolved since solutions to (4.1)-(4.2) in integral domains D of characteristic $p>3$ are unknown (see Problem 5.4).

4.1. Case 1: $p'(x)$ Has Degree $d-3$ or $d-4$. For $p(x)$ as described above, $p'(x)$ has degree $d-3$ or $d-4$ iff $m_1r_1+m_2r_2+m_3r_3=0$. By Proposition 4.1 this $p(x)$ is QF(D)-nice iff $p(x)$ splits in QF(D)[x]. Thus, we can construct all examples of such D-nice polynomials and count equivalence classes without deriving formulas. To count equivalence classes of D-nice polynomials $p(x)$ of degree d with four roots where the degree of $p'(x)$ is either $d-3$ or $d-4$, we construct a set S of representatives of all the equivalence classes. Although S may contain some QF(D)-nice polynomials, S can be used to count equivalence classes of D-nice polynomials since any two equivalent D-nice polynomials are equivalent to the same QF(D)-nice polynomial, and any two equivalent QF(D)-nice polynomials are equivalent to the same D-nice polynomial.

We may assume $r_1=1$ since, by horizontal compressions, all D-nice polynomials are equivalent to QF(D)-nice polynomials with $r_1=1$. For such $p(x)=x^{m_0}(x-$

$1)^{m_1}(x-r_2)^{m_2}(x-r_3)^{m_3}$, the degree of $p'(x)$ is $d-3$ or $d-4$ iff $m_1+m_2r_2+m_3r_3=0$ iff $r_3=\frac{-m_1-m_2r_2}{m_3}$ where r_2 is any element in $\mathrm{QF}(D)$. We determine when one or both of r_2 and r_3 are 0, 1, or equal to each other since in these cases (and only these) $p(x)$ does not have four distinct roots. Note that r_3 is 0 iff $-m_1-m_2r_2=0$ iff $r_2=\frac{-m_1}{m_2}(\neq 0)$. This choice of r_2 is not 1 iff $m_1\neq -m_2$ in D. And r_3 is 1 iff $-m_1-m_2r_2=m_3$ iff $r_2=\frac{-(m_1+m_3)}{m_2}(\neq 1)$. This value of r_2 is nonzero iff $m_1\neq -m_3$ in D. Finally, note that $r_3=r_2$ iff $r_2=\frac{-m_1-m_2r_2}{m_3}$ iff $r_2=\frac{-m_1}{m_2+m_3}$ and $m_2\neq -m_3$ in D (if $m_2=-m_3$ in D, then it is easy to check that r_3 is never equal to r_2). This particular value of r_2 is neither 0 nor 1. These results are summarized below.

Lemma 4.3. *Suppose $m_1+m_2r_2+m_3r_3=0$.*

(a) *If $r_2=\frac{-m_1}{m_2}$, then $r_3=0$. This choice of r_2 is never 0 and is not equal to 1 iff $m_1\neq -m_2$ mod p.*
(b) *If $r_2=\frac{-(m_1+m_3)}{m_2}$, then $r_3=1$. This choice of r_2 is never equal to 1 and is not equal to 0 iff $m_1\neq -m_3$ mod p.*
(c) *If $r_2=\frac{-m_1}{m_2+m_3}$, then $r_2=r_3$. This choice of r_2 is never equal to 0 or 1. If $m_2=-m_3$ mod p, then r_2 and r_3 are never equal.*

In all other cases, r_2 and r_3 are neither 0 nor 1 and are distinct.

Remark. To simplify the following results and proofs, we shall not consider all the different cases where the set $A=\{0,1,\frac{-m_1}{m_2},\frac{-(m_1+m_3)}{m_2},\frac{-m_1}{m_2+m_3}\}$ consists of 5, 4, or 3 elements. Instead, we combine all these cases and then use $|A|$ instead.

We look first at the case where the four multiplicities are all distinct mod p since this case is the simplest. Note that in this case the characteristic of D is $p>3$.

Corollary 4.4. *Let D be an integral domain of characteristic $p>3$. Let m_0, m_1, m_2, and m_3 be distinct mod p and $d=m_0+m_1+m_2+m_3$ a multiple of p. Let $A=\{0,1,\frac{-m_1}{m_2},\frac{-(m_1+m_3)}{m_2},\frac{-m_1}{m_2+m_3}\}$. Then the number of equivalence classes of D-nice polynomials of the type (m_0,m_1,m_2,m_3) whose derivatives are of degree $d-3$ or $d-4$ is $p^n-|A|$ if D has p^n elements and is infinite if D is infinite.*

Proof. Let $S=\{p_{a,b}(x)=x^{m_0}(x-1)^{m_1}(x-a)^{m_2}(x-b)^{m_3}: m_1+m_2a+m_3b=0,\ p_{a,b}(x)$ has four distinct roots$\}$. It is easy to see that any D-nice polynomial of these types is equivalent to a polynomial in S. Furthermore, any two polynomials in S are equivalent iff they are equal since all the multiplicities are distinct mod p. Thus, S is a set of representatives of the equivalence classes of D-nice polynomials of these types. By Lemma 4.3, if $r_2 \notin A$, there exists a unique r_3 so that $p_{r_2,r_3}(x) \in S$; and if $r_2 \in A$, then there is no such r_3. Since $p_{a,b}(x) \neq p_{b,a}(x)$, it then follows that the number of equivalence classes is as stated. □

We now consider the case where the set of multiplicities consists of three distinct numbers mod p. Without loss of generality, we may assume $m_0 \neq m_1$ mod p and that both of these numbers are different from $m_2 = m_3 = m$ mod p. Again the characteristic of D is $p > 3$.

Corollary 4.5. *Let D be an integral domain of characteristic $p > 3$. Let $m_0 \neq m_1$ mod p and both be different mod p from $m_2 = m_3 = m$ mod p, and let $d = m_0 + m_1 + m_2 + m_3$ be a multiple of p. Let $A = \{0, 1, \frac{-m_1}{m_2}, \frac{-(m_1+m_3)}{m_2}, \frac{-m_1}{m_2+m_3}\}$. Then the number of equivalence classes of D-nice polynomials of the type (m_0, m_1, m_2, m_3) whose derivatives are of degree $d-3$ or $d-4$ is $\frac{1}{2}(p^n - |A|)$ if D has p^n elements and is infinite if D is infinite.*

Proof. We may assume that $m_2 = m_3 = m$, and we use the same argument used in the proof of Corollary 4.4 but now noting that, since $p_{a,b}(x) = p_{b,a}(x) \in S$, every polynomial in S is counted twice. □

We now consider the two cases where the set of multiplicities consists of two distinct numbers mod p. In one case, we may assume $m_0 = m_1 = m$ mod p, $m_2 = m_3 = m'$ mod p, and $m \neq m'$ mod p. In this case, the characteristic of D is $p > 2$. In the other case, we may assume m_0 is different from $m_1 = m_2 = m_3 = m$ mod p. In this case, the characteristic of D is $p > 3$.

Corollary 4.6. *Let D be an integral domain of characteristic $p > 2$. Let $m_0 = m_1 = m$ mod p and $m_2 = m_3 = m'$ mod p where $m \neq m'$ mod p. Suppose $d = m_0 + m_1 + m_2 + m_3$ is a multiple of p. Let $A = \{0, 1, \frac{-m_1}{m_2}, \frac{-(m_1+m_3)}{m_2}, \frac{-m_1}{m_2+m_3}\}$. Then the number of equivalence classes of D-nice polynomials of the type (m_0, m_1, m_2, m_3) whose derivatives are of degree $d-3$ or $d-4$ is $\frac{1}{2}(p^n - |A|)$ if D has p^n elements and is infinite if D is infinite.*

Proof. We may assume $m_0 = m_1 = m$, $m_2 = m_3 = m'$, and $m \neq m'$. Let S be the same set S as stated above. Since $p_{a,b}(x) = p_{b,a}(x) \in S$, S has $\frac{1}{2}(p^n - |A|)$ polynomials if D has p^n elements. However, under a horizontal translation by 1 and a reflection over the y-axis, $p_{a,b}(x)$ is equivalent to $p_{1-a,1-b}(x) \in S$ but to no other polynomials in S. We claim that these two polynomials are actually equal, so our claim implies that the number of equivalence classes equals the number of polynomials in S. To see the claim, we note that the roots of any polynomial in S satisfy $m + m'(a+b) = 0$, which holds iff $a + b = -m/m' = 1$ since $m + m' = 0$ in D. For all such pairs of roots a and b, $a+b = 1$; therefore, $a = 1-b$ and $b = 1-a$. This proves our result. □

Before we consider the other case $m_1 = m_2 = m_3 = m$ mod p and $m_0 \neq m$ mod p, we prove the following lemma we will use in counting equivalence classes. In both the lemma below and in Corollary 4.8, we call the cube roots of unity different from 1 the *nontrivial* cube roots of unity.

Lemma 4.7. *Let D be an integral domain of characteristic $p > 3$. Suppose $m_1 = m_2 = m_3 = m$ mod p and $m_0 \neq m$ mod p. Let $S^* = \{p_{a,b}(x) = x^{m_0}(x-a)^m(x-b)^m(x-1)^m : a+b+1 = 0, p_{a,b}(x)$ is D-nice$\}$. Let a_1, a_2 denote the nontrivial cube roots of unity.*

(a) *If $a_1, a_2 \in D$, then $p_{a_1,a_2}(x) \in S^*$. The polynomial $p_{a_1,a_2}(x) \in S^*$ is not equivalent to any other polynomial in S^*.*
(b) *Every polynomial $p_{a,b}(x) \in S^*$ different from $p_{a_1,a_2}(x)$ is equivalent to exactly two other polynomials in S^*.*

Proof. (a). To prove this, we need to prove that $a_1+a_2+1 = 0$. Both a_1 and a_2 are roots of the polynomial x^3-1, which factors as $(x-1)(x^2+x+1)$. Neither a_1 nor a_2 is 1, so both a_1 and a_2 are roots of x^2+x+1. Thus, $a_1^2+a_2^2+a_1+a_2+2 = 0$. But $a_1^2 = a_2$ and $a_2^2 = a_1$ since both a_1 and a_2 are generators of the unique subgroup of order 3 of D^*, the multiplicative group of D. This equation then becomes $2a_1 + 2a_2 + 2 = 0$, so $a_1 + a_2 + 1 = 0$, as needed.

Since $p_{a,b}(x)$ is monic and $m_0 \neq m$, $p_{a,b}(x)$ is equivalent to both $p_{1/a,b/a}(x)$ and $p_{1/b,a/b}(x)$ under horizontal compressions and to no other polynomials in S^*. Since a_1 and a_2 are the only elements of order 3 in D^* and since $1/a_1$ has order 3, $1/a_1 = a_2$. Thus, a_1 and a_2 are multiplicative inverses. Then it follows that $a_1^3 = 1 = a_1a_2$, so $a_1^2 = a_2$ and $a_1a_2 = 1$. Hence, $a_1 = a_2/a_1$ and $a_2 = 1/a_1$, which shows that $p_{a_1,a_2}(x) = p_{1/a_1,a_2/a_1}(x)$. By similar reasoning, $p_{a_1,a_2}(x) = p_{a_1/a_2,1/a_2}(x)$. We then conclude that $p_{a_1,a_2}(x) \in S^*$ is not equivalent to any other polynomials in S^*.

(b). We may reverse the steps above to prove that $p_{a,b}(x)$, $p_{1/a,b/a}(x)$, and $p_{1/b,a/b}(x)$ are all distinct in this case. This completes the proof. □

Corollary 4.8. *Let D be an integral domain of characteristic $p > 3$. Let m_0 be different mod p from $m_1 = m_2 = m_3 = m$ mod p, and suppose $d = m_0 + m_1 +$*

$m_2 + m_3$ is a multiple of p. Let $A = \{0, 1, \frac{-m_1}{m_2}, \frac{-(m_1+m_3)}{m_2}, \frac{-m_1}{m_2+m_3}\}$. Then the number of equivalence classes of D-nice polynomials of the type (m_0, m_1, m_2, m_3) whose derivatives are of degree $d-3$ or $d-4$ is $\frac{1}{3}[\frac{1}{2}(p^n - |A|) - 1] + 1$ if D has p^n elements and contains both nontrivial cube roots of unity and is $\frac{1}{6}(p^n - |A|)$ if D has p^n elements and contains no nontrivial cube roots of unity and is infinite if D is infinite.

Proof. We may assume $m_1 = m_2 = m_3 = m$ and $m_0 \neq m$. Let S^* be the same set as described by Lemma 4.7. Let a_1 and a_2 be the nontrivial cube roots of unity. By Lemma 4.7, $p_{a,b}(x)$ is equivalent to exactly two other polynomials in S^* if $p_{a,b}(x) \neq p_{a_1,a_2}(x)$. But $p_{a_1,a_2}(x)$ is not equivalent to any other polynomial in S^*. If $a_1, a_2 \in D$, we form the set S of representatives of the equivalence classes by keeping 1/3 of all the polynomials in $S^* - \{p_{a_1,a_2}(x)\}$ then adding $p_{a_1,a_2}(x)$ to S to complete the set of representatives. If $a_1, a_2 \notin D$, then to form S, we keep 1/3 of the polynomials in S^*. Then it follows by this construction that the number of equivalence classes is the number stated above. $\square$

4.2. Case 2: D Has Characteristic 2. We now look at the case where D has characteristic 2 and the degree of $p(x)$ is even. By our assumptions, all the multiplicities of the roots of $p(x)$ are odd. If D is a finite field of characteristic 2, then we do not need a formula for constructing such D-nice polynomials $p(x)$, regardless of what the degree of $p'(x)$ is. We now prove this below.

Proposition 4.9. *Suppose D is a finite field of characteristic 2. Let m_0, m_1, m_2, and m_3 all be odd. Then $p(x) = x^{m_0}(x - r_1)^{m_1}(x - r_2)^{m_2}(x - r_3)^{m_3} \in D[x]$ is D-nice iff $p(x)$ splits in $D[x]$.*

Proof. Note that $p'(x) = x^{m_0-1}(x - r_1)^{m_1-1}(x - r_2)^{m_2-1}(x - r_3)^{m_3-1} \cdot$
$[(r_1 + r_2 + r_3)x^2 + r_1r_2r_3]$, which splits in $D[x]$ since $p'(x) \neq 0$ and the Frobenius homomorphism $y \mapsto y^2$ is an automorphism. $\square$

By the proof of Proposition 4.9, $p'(x)$ has degree $d-2$ or $d-4$, and this degree is $d-4$ iff $r_1+r_2+r_3 = 0$. Furthermore, the conclusion of Proposition 4.9 holds if $p'(x)$ has degree $d-4$ and D is infinite but does not hold if $p'(x)$ has degree $d-2$ and D is infinite. To see that the former holds, note that $p'(x) = r_1r_2r_3x^{m_0-1}(x-r_1)^{m_1-1} \cdot$ $(x - r_2)^{m_2-1}(x - r_3)^{m_3-1}$ in this case. For convenience, we state this result in Proposition 4.10 below. To see that the latter holds, take $D = \mathbb{Z}/(2)(y)$, the field of rational functions over $\mathbb{Z}/(2)$. Then $p(x) = x(x-1)[x-(y^2+y)][x-(y^3+y+1)]$

splits in $D[x]$, but $p'(x) = (y^3 + y^2)x^2 + (y^5 + y^4 + y^3 + y)$ does not split in $D[x]$. It is easy to see that the only elements in D that that have square roots in D are those rational functions in reduced form where the only powers of y are even. Since the rational function $\frac{y^5+y^4+y^3+y}{y^3+y^2}$ in reduced form is $\frac{y^3+1}{y}$, it does not have a square root in D. Thus, $p'(x)$ does not split in $D[x]$.

Proposition 4.10. *Suppose D is an infinite integral domain of characteristic 2. Let m_0, m_1, m_2, and m_3 all be odd. Then $p(x) = x^{m_0}(x - r_1)^{m_1}(x - r_2)^{m_2}(x - r_3)^{m_3} \in D[x]$ with $r_1 + r_2 + r_3 = 0$ is D-nice iff $p(x)$ splits in $D[x]$.*

Remark. Proposition 4.9 implies Proposition 4.10 whenever D is finite; so, to avoid redundancy, we choose to assume D is infinite in Proposition 4.10.

We now derive a formula for constructing D-nice polynomials $p(x)$ of degree d with four roots over integral domains D of characteristic 2 where all four roots have odd multiplicities and the degree of $p'(x)$ is $d - 2$. In this case, the relations (4.1)-(4.2) simplify as follows:

$$r_1r_2r_3 = (r_1 + r_2 + r_3)c^2. \tag{4.3}$$

To see this, note that (4.1) becomes $(r_1 + r_2 + r_3)(c_1 + c_2) = 0$, so $c_1 + c_2 = 0$ since $r_1 + r_2 + r_3 \neq 0$. Thus, $c_1 = c_2 = c$. Substitution into (4.2) gives (4.3) above.

We now solve (4.3) in QF(D) to derive a formula for constructing D-nice polynomials $p(x)$ of degree d over (infinite) integral domains D of characteristic 2 where the multiplicities of the four roots of $p(x)$ are odd and where $p'(x)$ has degree $d-2$. We must assume $r_1 + r_2 + r_3 \neq 0$; otherwise, (4.3) does not hold. By Theorem 2.1, we may assume these multiplicities are all 1 and that $p(x)$ has degree 4.

Theorem 4.11. *Let D be an infinite integral domain of characteristic 2. Then the polynomial $p(x) = x(x - r_1)(x - r_2)(x - r_3)$ with derivative $p'(x) = (r_1 + r_2 + r_3)(x - c)^2 \neq 0$ is QF(D)-nice and has four distinct roots iff*

$$r_1 = \frac{(r_2 + r_3)c^2}{r_2r_3 + c^2} \tag{4.4}$$

for some r_2, r_3, and c that are nonzero and all distinct and where $r_2r_3 \neq c^2$.

Proof. Rewrite (4.3) as $r_1r_2r_3 + r_1c^2 = (r_2 + r_3)c^2$ and solve for r_1. We check when $r_1 + r_2 + r_3 \neq 0$. By (4.3), if c, r_1, r_2, and r_3 are nonzero, then $r_1 + r_2 + r_3 \neq 0$. We check below that our choices of r_2, r_3, and c guarantee that all variables are nonzero; then, it will follow that $r_1 + r_2 + r_3 \neq 0$ for our choices of r_2, r_3, and c.

If $r_2r_3 = c^2$, then (4.3) becomes $r_1c^2 = (r_1 + r_2 + r_3)c^2$, so $r_1 = r_1 + r_2 + r_3$ (since $c \neq 0$), which implies that $r_2 = r_3$. Thus, in this case, $p(x)$ does not have four distinct roots. And $r_1 = r_2$ iff $r_2^2r_3 + r_2c^2 = r_2c^2 + r_3c^2$ (by the formula) iff $r_2^2r_3 = r_3c^2$ iff $r_2^2 = c^2$ iff $r_2 = c$. By similar reasoning, $r_1 = r_3$ iff $r_3 = c$. Combining these facts, we see that (4.3) has a solution where r_1, r_2, and r_3 are all nonzero and distinct iff $r_2r_3 \neq c^2$ and r_2, r_3, and c are all nonzero and distinct.

The converse of the theorem follows from Lemma 4.2. □

Remark. Because (4.3) holds if D is finite, Theorem 4.11 holds whenever D is finite, but we do not need to use formula (4.4) in this case. Since Theorem 4.11 is useful only when D is infinite, we assume D is infinite in this theorem.

We illustrate formula (4.4) with the following two examples of D-nice polynomials.

Let $D = \mathbb{Z}/(2)[y]$. Choose $r_2 = y + 1$, $r_3 = y^3 + y^2 + 1$, and $c = y^3 + y$. Then, by formula (4.4), $r_1 = \frac{y^9+y^8+y^7+y^5+y^4+y^3}{y^6+y^4+y+1}$. Stretching this polynomial horizontally by $y^6 + y^4 + y + 1$, we obtain the equivalent D-nice polynomial

$$\begin{aligned} p(x) &= x[x - (y^9 + y^8 + y^7 + y^5 + y^4 + y^3)][x - (y^7 + y^6 + y^5 + y^4 + y^2 + 1)] \\ &\qquad \cdot [x - (y^9 + y^8 + y^7 + y^2 + y + 1)], \qquad (4.5) \\ p'(x) &= (y^7 + y^6 + y^3 + y)[x - (y^9 + y^5 + y^4 + y^3 + y^2 + y)]^2. \end{aligned}$$

Let $D = \mathbb{Z}/(2)[y]$. Choose $r_2 = y$, $r_3 = y^2 + y + 1$, and $c = y^3$. Then, by formula (4.4), $r_1 = \frac{y^8+y^6}{y^6+y^3+y^2+y}$. Stretching this polynomial horizontally by $y^6 + y^3 + y^2 + y$, we obtain the equivalent D-nice polynomial

$$\begin{aligned} p(x) &= x[x - (y^8 + y^6)][x - (y^7 + y^4 + y^3 + y^2)] \\ &\qquad \cdot [x - (y^8 + y^7 + y^6 + y^5 + y^3 + y)], \qquad (4.6) \\ p'(x) &= (y^5 + y^4 + y^2 + y)[x - (y^9 + y^6 + y^5 + y^4)]^2. \end{aligned}$$

Counting equivalence classes of D-nice polynomials $p(x)$ over finite fields of characteristic 2 where all four multiplicities of the roots of $p(x)$ are odd is currently an unsolved problem (see Problem 5.3). Every such D-nice polynomial $p(x)$ is equivalent to a polynomial in $S^* = \{p_{a,b}(x) = x(x-1)(x-a)(x-b) : p_{a,b}(x)$ is D-nice and has four distinct roots$\}$. Counting these equivalence classes is difficult because determining when two polynomials in S^* are equivalent is a difficult problem. Even though every such D-nice polynomial is equivalent to a polynomial in S^*, two polynomials in S^* can be equivalent under a horizontal translation (by either 1, a, or b), under a horizontal stretch or compression (by either a or b), or under compositions of these transformations. However, if D is infinite, then it is easy to check that the number of equivalence classes is infinite.

Corollary 4.12. *Let D be an infinite integral domain of characteristic 2. Let m_0, m_1, m_2, and m_3 all be odd. Then the number of D-nice polynomials of the type (m_0, m_1, m_2, m_3) is infinite.*

Proof. Let S^* be the same set mentioned above. By Proposition 4.10 and Theorem 4.11, S^* is infinite. By the discussion above, every polynomial in S^* is equivalent to finitely many other polynomials in S^*, so the result follows. ☐

4.3. Case 3: $p'(x)$ Has Degree $d-2$ and D Has Characteristic 3. We now find a formula for constructing D-nice polynomials $p(x)$ with four roots over integral domains D of characteristic 3 where the degree d of $p(x)$ is a multiple of 3 and the degree of $p'(x)$ is $d-2$. The only possibility is that two of the multiplicities of the roots are 2 mod 3 and the other two are 1 mod 3. Without loss of generality, we may assume $m_0 = m_3 = 2$ and $m_1 = m_2 = 1$. Using the horizontal compression, we may assume $r_3 = 1$. In this case, the relations (4.1)-(4.2) become

$$2r_1r_2 = (r_1 + r_2 + 2)(c_1 + c_2), \tag{4.7}$$

$$r_1r_2 = (r_1 + r_2 + 2)c_1c_2. \tag{4.8}$$

To find our formula for constructing such D-nice polynomials, we solve this system of equations in QF(D) with the condition that $r_1+r_2+2 \neq 0$. If $r_1+r_2+2 = 0$, then (4.7)-(4.8) do not hold.

Theorem 4.13. *Let D be an integral domain of characteristic 3. The polynomial $p(x) = x^2(x-r_1)(x-r_2)(x-1)^2$ with derivative $p'(x) = (r_1+r_2+2)x(x-1)(x-c_1)(x-c_2) \neq 0$ is QF(D)-nice iff there exist $c_2 \neq 0$ and $r_2 \neq 0, 1$ in QF(D) so that*

$$c_1 = \frac{c_2}{2c_2 - 1}, \tag{4.9}$$

$$r_1 = \frac{r_2c_2^2 + 2c_2^2}{2c_2r_2 - r_2 - c_2^2} \tag{4.10}$$

where the denominators of (4.9) and (4.10) are nonzero.

Proof. We first proof that, for the given choices of r_2 and c_2 in the statement of the theorem and the choice of r_1 given by (4.10), $r_1 + r_2 + 2 \neq 0$. By (4.7)-(4.8), if r_1 and r_2 are nonzero, then $r_1 + r_2 + 2 \neq 0$. We choose $r_2 \neq 0, 1$ and $c_2 \neq 0$; thus, by (4.10), $r_1 \neq 0$. So $r_1 + r_2 + 2 \neq 0$.

By (4.7)-(4.8), if $p(x)$ has four distinct roots, neither c_1c_2 nor c_1+c_2 are 0 and, by (4.7), $\frac{2r_1r_2}{c_1+c_2} = \frac{r_1r_2}{c_1c_2}$. Furthermore, $r_1r_2 \neq 0$; so, by canceling, $\frac{2}{c_1+c_2} = \frac{1}{c_1c_2}$. Solving this equation for c_2 in terms of c_1, assuming $c_2 \neq -1$, we obtain (4.9). If $c_2 = -1$, then the equation we just solved has no solution. Now we solve (4.8) for r_1 in terms of r_2, c_1, and c_2. We rewrite the equation as $r_1(r_2 - c_1c_2) = r_2c_1c_2 + 2c_1c_2$ and divide, assuming $r_2 \neq c_1c_2$, which gives us $r_1 = \frac{r_2c_1c_2+2c_1c_2}{r_2-c_1c_2}$. Note that if $r_2 = c_1c_2$, then, by (4.8), $r_1+r_2+2 = r_1$, which implies that $r_2 = -2 = 1$, which is impossible if $p(x)$ has four distinct roots. Substituting (4.9) for c_1 into the expression for r_1 leads to (4.10) above.

The converse follows from Lemma 4.2. □

Remark. The method of solving (4.7)-(4.8) in the proof above does not extend to solving (4.1)-(4.2) in general. However, it is not difficult to see that this method can be generalized to the case where m_0+m_2 is a multiple of p. But we choose not to do that here since it is not clear right now if such a formula would be of interest.

We now illustrate formulas (4.9)-(4.10) with several examples of D-nice polynomials.

Let $D = \mathbb{Z}/(3)[y]$. Let $r_2 = 2y^2+y+1$ and $c_2 = y^3+2y+2$. Then, by the formulas, $r_1 = \frac{2y^8+y^7+2y^6+2y^3+y^2+y}{2y^6+y^5+y^4+2y+2}$ and $c_1 = \frac{y^3+2y+2}{2y^3+y}$. Stretching this horizontally by $(2y^3+y)(2y^6+y^5+y^4+2y+2)$, we obtain the equivalent D-nice polynomial

$$p(x) = x^2(x-r_1)(x-r_2)(x-r_3)^2, \tag{4.11}$$
$$p'(x) = (y^{10}+y^9+y^8+2y^7+y^6+2y^5+y^3)x(x-r_3)(x-c_1)(x-c_2)$$

where $r_1 = y^{11}+2y^{10}+y^8+2y^7+y^6+2y^5+y^4+y^3+y^2$,
$r_2 = 2y^{11}+2y^{10}+2y^9+2y^8+y^7+y^6+y^5+y^3+y^2+2y$,
$r_3 = y^9+2y^8+y^7+y^6+y^5+y^4+y^3+2y^2+2y$,
$c_1 = 2y^9+y^8+2y^7+y^5+y^4+2y^3+y^2+2y+1$, and
$c_2 = y^{12}+2y^{11}+y^9+y^8+2y^7+2y^6+2y^2+y$.

Let $D = \mathbb{Z}/(3)[y]/(f)$ where $f(y) = y^2+1$ is irreducible in $\mathbb{Z}/(3)[y]$. To simplify notation, we denote $g(y)+(f) \in D$ by $g(t)$. Choose $r_2 = 2t+1$ and $c_2 = t$. Then, by the formulas, $r_1 = 2t$ and $c_1 = t+1$. Thus, our D-nice polynomial is

$$p(x) = x^2(x-2t)[x-(2t+1)](x-1)^2, \tag{4.12}$$
$$p'(x) = tx(x-1)[x-(t+1)](x-t).$$

We now find all values of c_2 and r_2 where the formulas (4.9)-(4.10) give D-nice polynomials with four distinct roots.

Proposition 4.14. *Formulas* (4.9)-(4.10) *give D-nice polynomials with four distinct roots iff $c_2 \neq 0, \pm 1$ and $r_2 \neq c_2, 0, 1, \frac{c_2^2}{2c_2-1}, \frac{-c_2}{c_2-2}$. These five values are all distinct.*

Proof. To find all values of r_2 and c_2 where the formulas fail to give D-nice polynomials with four distinct roots, we consider the cases r_1 is undefined, $r_1 = 0$, $r_1 = 1$, and $r_1 = r_2$. The case where c_1 is undefined iff $c_2 = -1$ has been considered earlier. These cases are all the cases where the formulas fail to give D-nice polynomials with four distinct roots. Note that r_1 is undefined iff $2c_2r_2 - r_2 - c_2^2 = 0$ iff $r_2 = \frac{c_2^2}{2c_2-1}$. Next, note that $r_1 = 0$ iff $r_2c_2^2 + 2c_2^2 = 0$ iff $c_2^2(r_2+2) = 0$ iff $c_2 = 0$ or $r_2 = 1$. Next, $r_1 = 1$ iff $r_2c_2^2 + 2c_2^2 = 2c_2r_2 - r_2 - c_2^2$ iff $r_2(c_2^2 - 2c_2 + 1) = 0$ iff $r_2 = 0$ or $c_2 = 1$. Finally, $r_1 = r_2$ iff $2c_2r_2^2 - r_2^2 - c_2^2r_2 = r_2c_2^2 + 2c_2^2$ iff $(2c_2-1)r_2^2 + c_2^2r_2 + c_2^2 = 0$ iff $r_2 = \frac{-c_2}{c_2-2}$ or $r_2 = c_2$. Checking that these five values for r_2 are all distinct when $c_2 \neq 0, \pm 1$ is not difficult, and we omit the details. □

We will use the following result when we count equivalence classes of D-nice polynomials as described above. Since the result is important for studying D-nice polynomials with a specified number of roots, we present this result for polynomials with any number of roots over any integral domain D. In this result r being a root of multiplicity 0 means that r is not a root.

Proposition 4.15. *Suppose $p(x) = a\prod(x-r_i)^{m_i} \in D[x]$ and $r_i \neq r_j$ for $i \neq j$. If D has characteristic 0, then r_i is a root of multiplicity $m_i - 1$ of $p'(x)$. If D has characteristic $p > 0$, then r_i is a root of multiplicity $m_i - 1$ of $p'(x)$ if m_i is not a multiple of p. Otherwise, the multiplicity of the root r_i of $p'(x)$ is m_i or greater.*

Proof. Let $p(x) = a(x-r_1)^{m_1}(x-r_2)^{m_2}\cdots(x-r_n)^{m_n}$. Differentiating $p(x)$ by the product rule, we see that $p'(x) = a(x-r_1)^{m_1-1}(x-r_2)^{m_2-1}(x-r_n)^{m_n-1}q(x)$ where $q(x) = \sum_{i=1}^n m_i\prod_{j\neq i}(x-r_j)$. Then $q(r_i) = m_i\prod_{j\neq i}(r_i - r_j)$. If D has characteristic 0 or if D has characteristic $p > 0$ and m_i is not a multiple of p, then $m_i \neq 0$ in D. Thus, $q(r_i) \neq 0$, which proves that r_i is a root of $p'(x)$ of multiplicity $m_i - 1$. Otherwise, $m_i = 0$ in D. Thus, $q(r_i) = 0$, which proves that the multiplicity of the root r_i of $p'(x)$ is m_i or greater. □

We conclude Section 4 by counting equivalence classes of D-nice polynomials $p(x)$ as described in Theorem 4.13.

Corollary 4.16. *Let D be an integral domain of characteristic* 3*, and suppose $m_0 = m_3 = 2$ mod 3 and $m_1 = m_3 = 1$ mod 3. Let $d = m_0 + m_1 + m_2 + m_3$. Then the number of equivalence classes of D-nice polynomials $p(x)$ of the type (m_0, m_1, m_2, m_3) where $p'(x)$ has degree $d-2$ is $\frac{1}{8}(3^n - 3)(3^n - 5)$ if D has 3^n $(n \geq 2)$ elements and is infinite if D is infinite.*

Proof. We may assume $m_0 = m_3 = 2$ and $m_1 = m_2 = 1$. Let $S^* = \{p_{a,b}(x) = x^2(x-a)(x-b)(x-1)^2 : a, b \neq 0, 1; a \neq b, p_{a,b}(x)$ is QF(D)-nice$\}$. Every such D-nice polynomial is equivalent to a polynomial in S^*. Under a horizontal translation by 1 and a reflection over the y-axis, $p_{a,b}(x)$ is equivalent to $p_{1-a,1-b}(x)$ and to no other polynomials in S^*. These two polynomials are never equal if $p_{a,b}(x)$ has four distinct roots. To see this, note that these polynomials are equal iff $1 - b = a$ and $1 - a = b$ iff $a + b + 2 = 0$. However, $a + b + 2 \neq 0$. Thus, every QF(D)-nice polynomial $p_{a,b}(x) \in S^*$ is equivalent to exactly one other polynomial in S^*, so half of these form a set S of representatives of the equivalence classes of D-nice polynomials of these types. It is easy to see that the number of equivalence classes is infinite if D is.

By Proposition 4.14, for every $c_2 \neq 0, \pm 1$, there are exactly five values of r_2 so that the polynomial constructed by formulas (4.9)-(4.10) fails to have four distinct roots. Thus, if D has 3^n $(n \geq 2)$ elements, there are $(3^n - 3)(3^n - 5)$ choices of ordered pairs (c_2, r_2) so that the polynomials constructed by the formulas have four distinct roots. If we construct one polynomial by choosing $c_2 = a$ and $r_2 = b$ then construct another by choosing $c_2 = b$ and $r_2 = a$, then the two polynomials we constructed, if they have four distinct roots, are different. If they were equal, then this polynomial would have a single root a and a critical point a. But, by Proposition 4.15, this is impossible, so the number of distinct polynomials constructed by formulas (4.9)-(4.10) is the number of ordered pairs (c_2, r_2) if distinct choices of ordered pairs give distinct polynomials. But this is not true since the multiplicities of r_1 and r_2 are 1, and the multiplicities of c_1 and c_2 are also 1 (switching the labels of r_1 and r_2 gives the same polynomial; switching the labels of c_1 and c_2 also gives the same polynomial). It is easy to see that c_1 and c_2 are never equal since the only solutions to the equation $c_1 = c_2$ using (4.9) for the value of c_1 are $c_2 = 0$ and $c_2 = 1$, which are values we exclude. And, by our choices of c_2, $r_1 \neq r_2$. Then it follows that for any QF(D)-nice polynomial $p_{a,b}(x)$ there exist exactly four ordered pairs (c_2, r_2) so that $p_{a,b}(x)$ is constructed by the formulas. Hence, the number of distinct polynomials is $\frac{1}{4}(3^n - 3)(3^n - 5)$ if D has 3^n elements. Then we divide this number by 2 since every polynomial is equivalent to exactly one other, as we showed above. This completes the proof. $\square$

5. Open Problems

The following four problems about constructing, describing, and classifying D-nice polynomials with four distinct roots remain unsolved.

Problem 5.1. Find a formula for constructing all D-nice polynomials with four distinct roots over integral domains of characteristic 0 by solving relations (3.1)-(3.3) in D. Use this formula to count equivalence classes of such D-nice polynomials. In particular, prove or disprove Conjecture 3.4.

Problem 5.2. Let D be an integral domain of characteristic $p > 0$. Find a formula for constructing all D-nice polynomials of the type (m_0, m_1, m_2, m_3) where none of m_0, m_1, m_2, m_3, and $d = m_0 + m_1 + m_2 + m_3$ are multiples of p by solving relations (3.1)-(3.3) in D. Use this formula to count equivalence classes of such D-nice polynomials. In particular, prove or disprove Conjecture 3.4.

Problem 5.3. Determine the number of equivalence classes of D-nice polynomials $p(x)$ over finite fields of characteristic 2 where the four multiplicities of the roots of $p(x)$ are odd.

Problem 5.4. Let D be an integral domain of characteristic $p > 3$. Suppose m_0, m_1, m_2, and m_3 are not multiples of p but that $d = m_0 + m_1 + m_2 + m_3$ is. Find a formula for constructing all such D-nice polynomials by solving relations (4.1)-(4.2) in D, and use this formula to count equivalence classes of such D-nice polynomials.

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