International Electronic Journal of Algebra
Volume 5 (2009) 169-181

NONCOMMUTING GRAPH CHARACTERIZATION OF SOME SIMPLE GROUPS WITH CONNECTED PRIME GRAPHS

Liangcai Zhang and Wujie Shi

Received: 18 September 2008; Revised:10 December 2008
Communicated by Abdullah Harmancı

Abstract. Let G be a nonabelian group and associate a noncommuting graph $\nabla(G)$ with G as follows: The vertex set of $\nabla(G)$ is $G\backslash Z(G)$, where $Z(G)$ is the center of G, and two vertices are adjacent by an edge whenever they do not commute. In 2006, A. Abdollahi, S. Akbari and H. R. Maimani put forward a conjecture called AAM's Conjecture in [1] as follows: If M is a finite nonabelian simple group and G is a group such that $\nabla(G) \cong \nabla(M)$, then $G \cong M$. Even though this conjecture is known to hold for all simple groups with nonconnected prime graphs and the alternating group A_{10} (see [11]), it is still unknown for all simple groups with connected prime graphs except A_{10}. In the present paper, we prove that the conjecture is also true for $L_4(8)$, the projective special linear group of degree 4 over the finite field of order 8. The new method used in this paper also works well in the case $L_4(4)$, $L_4(7)$, $U_4(7)$, etc.

Mathematics Subject Classification (2000): 20D05, 20D06, 20D60
Keywords: finite group, noncommuting graph, projective special linear simple group

1. Introduction

It is well known that *the prime graph* $\Gamma(G)$ associated with a finite group G is defined as follows: The vertex set of $\Gamma(G)$ is $\pi(G)$, the set of prime divisors of the order of G. The primes p and q, considered as vertices of $\Gamma(G)$, are adjacent by an edge (we write $p \sim q$) if and only if G contains an element of order pq. Denote by $t(G)$ the number of connected components of $\Gamma(G)$ (see [12]).

Given a finite group G, we construct its *noncommuting graph* $\nabla(G)$ as follows: The vertex set of $\nabla(G)$ is $G\backslash Z(G)$, where $Z(G)$ is the center of G, and two vertices are adjacent by an edge whenever they do not commute(see [1,8]).

This work was supported by the NNSF of China (No.10871032), the SRFDP of China (No.20060285002) and a subproject of the NNSF of China (No.50674008)(Chongqing University, No.104207520080834; 104207520080968).

For a graph X, we denote the sets of vertices and edges of X by $V(X)$ and $E(X)$, respectively. Two graphs X and Y are said to be *isomorphic* if there exists a bijective map ϕ: $V(X) \to V(Y)$ such that x and y are adjacent in X if and only if $\phi(x)$ and $\phi(y)$ are adjacent in Y. If two graphs X and Y are isomorphic, we denote it by $X \cong Y$. It is easy to see that if $X \cong Y$, then $|V(X)| = |V(Y)|$ and $|E(X)| = |E(Y)|$.

In 2006, A. Abdollahi, S. Akbari and H. R. Maimani put forward a conjecture in [1] as follows.

***AAM*'s Conjecture**: If M is a finite nonabelian simple group and G is a group such that $\nabla(G) \cong \nabla(M)$, then $G \cong M$.

In [11], it has been proved that *AAM*'s Conjecture is true for all finite simple groups with nonconnected prime graphs and A_{10}, where A_{10} is the alternating group of degree 10. In the present paper, we will give another example to show that *AAM*'s Conjecture is also true for some simple groups with connected prime graphs. In fact, we prove that if G is a finite group such that $\nabla(G) \cong \nabla(L_4(8))$, then $G \cong L_4(8)$, where $L_4(8)$ is the projective special linear group of degree 4 over the finite field of order 8. The new method used in this paper also works well in the case $L_4(4)$, $L_4(7)$, $U_4(7)$, etc.

All further unexplained notations are standard and we refer the reader to [1,8].

2. Preliminaries and Lemmas

For any group G, we denote by $\pi_e(G)$ the set of orders of its elements. The set $\pi_e(G)$ is *closed* and *partially ordered* by the divisibility relation. Hence, it is uniquely determined by $\mu(G)$, the subset of its elements which are *maximal* under the divisibility relation.

Lemma 2.1. ([7], Lemma 1) *Let $L_4(q)$ be a projective special linear simple group, where $q = 2^m$ and m is a natural number. Then* $\mu(L_4(q)) = \{(q^2+1)(q+1), q^3-1, 2(q^2-1), 4(q-1)\}$. *In particular,* $\mu(L_4(8)) = \{3^2 \cdot 5 \cdot 13, 7 \cdot 73, 2 \cdot 3^2 \cdot 7, 2^2 \cdot 7\}$.

Let n be a natural number. We say that a finite group G is a K_n-group if $|\pi(G)| = n$. Now we quote some useful results on simple K_n-groups.

Lemma 2.2. ([3], Theorem 1) *Let G be a finite simple K_3-group. Then G is isomorphic to one of the following simple groups:* $A_5, A_6, L_2(7), L_2(8), L_2(17), L_3(3)$, $U_3(3)$ *or* $U_4(2)$.

Lemma 2.3. ([10], Theorem 2) *Let G be a finite simple K_4-group. Then G is isomorphic to one of the following simple groups:*

1. $A_7,\ A_8,\ A_9,\ A_{10},\ M_{11},\ M_{12},\ J_2,\ L_2(16),\ L_2(25),\ L_2(49),\ L_2(81),\ L_3(4),$ $L_3(5),\ L_3(7),\ L_3(8),\ L_3(17),\ L_4(3),\ S_4(4),\ S_4(5),\ S_4(7),\ S_4(9),\ S_6(2),\ O_8^+(2),$ $G_2(3),\ U_3(4),\ U_3(5),\ U_3(7),\ U_3(8),\ U_3(9),\ U_4(3),\ U_5(2),\ Sz(8),\ Sz(32),\ {}^3D_4(2),$ ${}^2F_4(2)'$;

2. $L_2(r)$, *where* r *is a prime satisfying the equation* $r^2-1=2^a\cdot 3^b\cdot u^c$, *where* $a\geq 1$, $b\geq 1$, $c\geq 1$, *and* $u>3$ *is a prime;*

3. $L_2(2^m)$, *where* $m\geq 1$ *satisfies the equations* $2^m-1=u$ *and* $2^m+1=3t^b$, *where* $t>3$, u *are primes and* $b\geq 1$*;*

4. $L_2(3^m)$, *where* $m\geq 1$ *satisfies the equations* $3^m-1=2u^b$ *and* $3^m+1=4t$, *or* $3^m-1=2u$ *and* $3^m+1=4t^c$, *where* u, t *are odd primes and* $b\geq 1$, $c\geq 1$.

In the sequel we denote by $\pi(n)$ the set of prime divisors of a natural number n.

Lemma 2.4. ([5], Theorem A) *Let* q *be a prime power. Then each finite simple* K_5*-group is isomorphic to one of the following simple groups:*

1. $L_2(q)$, *where* q *satisfies* $|\pi(q^2-1)|=4$*;*

2. $L_3(q)$, *where* q *satisfies* $|\pi((q^2-1)(q^3-1))|=4$*;*

3. $U_3(q)$, *where* q *satisfies* $|\pi((q^2-1)(q^3+1))|=4$*;*

4. $O_5(q)$, *where* q *satisfies* $|\pi(q^4-1)|=4$*;*

5. $Sz(2^{2m+1})$, *where* $|\pi((2^{2m+1}-1)(2^{4m+2}+1))|=4$*;*

6. $R(q)$, *where* $q=3^{2m+1}$ *satisfies* $|\pi(q^2-1)|=3$ *and* $|\pi(q^2-q+1)|=1$*;*

7. $A_{11}, A_{12}, M_{22}, J_3, HS, He, M^cL, L_4(4), L_4(5), L_4(7), L_5(2), L_5(3), L_6(2), O_7(3),$ $O_9(2), S_6(3), S_8(2), U_4(4), U_4(5), U_4(7), U_4(9), U_5(3), U_6(2), O_8^+(3), O_8^-(2), {}^3D_4(3),$ $G_2(4), G_2(5), G_2(7), G_2(8)$.

Lemma 2.5. ([5], Theorem B) *Let* q *be a prime power. Then each finite simple* K_6*-group is isomorphic to one of the following simple groups:*

1. $L_2(q)$, *where* q *satisfies* $|\pi(q^2-1)|=5$*;*

2. $L_3(q)$, *where* q *satisfies* $|\pi((q^2-1)(q^3-1))|=5$*;*

3. $L_4(q)$, *where* q *satisfies* $|\pi((q^2-1)(q^3-1)(q^4-1))|=5$*;*

4. $U_3(q)$, *where* q *satisfies* $|\pi((q^2-1)(q^3+1))|=5$*;*

5. $U_4(q)$, *where* q *satisfies* $|\pi((q^2-1)(q^3+1)(q^4-1))|=5$*;*

6. $O_5(q)$, *where* q *satisfies* $|\pi(q^4-1)|=5$*;*

7. $G_2(q)$, *where* q *satisfies* $|\pi(q^6-1)|=5$*;*

8. $Sz(2^{2m+1})$, *where* $|\pi((2^{2m+1}-1)(2^{4m+2}+1))|=5$*;*

9. $R(3^{2m+1})$, *where* $|\pi((3^{2m+1}-1)(3^{6m+3}+1))|=5$*;*

10. $A_{13}, A_{14}, A_{15}, A_{16}, M_{23}, M_{24}, J_1, Suz, Ru, Co_2, Co_3, Fi_{22}, HN, L_5(7), L_6(3),$ $L_7(2), O_7(4), O_7(5), O_7(7), O_9(3), S_6(4), S_6(5), S_6(7), S_8(3), U_5(4), U_5(5), U_5(9), U_6(3),$ $U_7(2), F_4(2), O_8^+(4), O_8^+(5), O_8^+(7), O_{10}^+(2), O_8^-(3), O_{10}^-(2), {}^3D_4(4), {}^3D_4(5)$.

Now we quote some preliminary results from elementary number theory which will play an important role in proving Lemma 2.9.

Lemma 2.6. ([4], Theorems 2.3.3 and 4.6.3) *Let a, x, m and n be natural numbers. Then the following assertions hold.*

1. If p is a prime number, then $a^p \equiv a(mod p)$.

2. If g.c.d.$\{m, n\} = d$, then g.c.d.$\{x^m - 1, x^n - 1\} = x^d - 1$. In particular, $m \big| n$ if and only if $x^m - 1 \big| x^n - 1$.

Lemma 2.7. ([13]) *If p is a prime and $n \geq 2$ is a natural number, then there exists a prime z such that $z \big| p^n - 1$ and $z \nmid p^m - 1$ for $1 \leq m < n$ unless either*

1. $n = 6$ and $p = 2$, or

2. $n = 2$ and $p = 2^q - 1$ is a Mersenne prime, where q is natural number.

Lemma 2.8. ([6]) *Let $n \geq 2$ and f be two natural numbers and $q = p^f$, where p is a prime number. Then*

1. $Out(L_n(q)) \cong Z_{(n,q-1)} : Z_f : Z_2$ if $n \geq 3$;

2. $Out(L_2(q)) \cong Z_{(2,q-1)} \times Z_f$.

Lemma 2.9. *Let G be a finite nonabelian simple group such that $\pi(G) \subseteq \{2, 3, 5, 7, 13, 73\}$. Then G is isomorphic to one of the groups listed in Table 1. In particular,*

1. $3 \in \pi(G)$ if and only if $G \neq Sz(8)$;

2. $\pi(Out(G)) \subseteq \{2, 3\}$ if $G \neq S_6(2)$.

G	$\lvert G \rvert$	$Out(G)$	G	$\lvert G \rvert$	$Out(G)$
A_5	$2^2 \cdot 3 \cdot 5$	2	$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2
A_6	$2^3 \cdot 3^2 \cdot 5$	2^2	$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3	$L_3(3)$	$2^4 \cdot 3^3 \cdot 13$	2
$U_4(2)$	$2^6 \cdot 3^4 \cdot 5$	2	A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	D_{12}
A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	2	$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	2^2
A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	2	$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	S_3
$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	D_8	J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2
$O_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	S_3	$S_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1
$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2	$L_2(25)$	$2^3 \cdot 3 \cdot 5^2 \cdot 13$	2^2
$L_2(13)$	$2^2 \cdot 3 \cdot 5 \cdot 13$	2	$U_3(4)$	$2^6 \cdot 3 \cdot 5^2 \cdot 13$	4
$L_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 13$	2^2	$S_4(5)$	$2^6 \cdot 3^2 \cdot 5^4 \cdot 13$	2
${}^2F_4(2)'$	$2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$	2	$L_2(27)$	$2^2 \cdot 3^3 \cdot 7 \cdot 13$	6
$G_2(3)$	$2^6 \cdot 3^6 \cdot 7 \cdot 13$	2	$L_2(64)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	6
$L_3(9)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13$	2^2	$Sz(8)$	$2^6 \cdot 5 \cdot 7 \cdot 13$	3
$G_2(4)$	$2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$	2	$S_4(8)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$	6
$S_6(3)$	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	2	$O_7(3)$	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	2
${}^3D_4(2)$	$2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$	3	$U_4(5)$	$2^7 \cdot 3^4 \cdot 5^6 \cdot 7 \cdot 13$	2^2
$O_8^+(3)$	$2^{12} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13$	S_4	$L_3(8)$	$2^9 \cdot 3^2 \cdot 7^2 \cdot 73$	6
$U_3(9)$	$2^5 \cdot 3^6 \cdot 5^2 \cdot 73$	4	$L_2(3^6)$	$2^3 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13 \cdot 73$	2×6
${}^3D_4(3)$	$2^6 \cdot 3^{12} \cdot 7^2 \cdot 13^2 \cdot 73$	3	$G_2(9)$	$2^8 \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13 \cdot 73$	2
$S_4(27)$	$2^6 \cdot 3^{12} \cdot 5 \cdot 7^2 \cdot 13^2 \cdot 73$	6	$L_4(8)$	$2^{18} \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 13 \cdot 73$	6

Table 1 Finite Nonabelian Simple Groups with $\pi(G) \subseteq \{2, 3, 5, 7, 13, 73\}$

Proof. If G is a simple K_3-group, then $G \cong A_5, L_2(7), L_2(8), A_6, L_3(3), U_3(3)$ or $U_4(2)$ by Lemma 2.2.

If G is a simple K_4-group, then $G \cong L_2(13), A_7, L_2(25), L_2(27), A_8, A_9, Sz(8), L_2(49), U_3(4), U_3(5), U_3(9), L_3(4), J_2, S_6(2), A_{10}, U_4(3), G_2(3), S_4(5), L_4(3), {}^2F_4(2)', L_3(8), S_4(7), O_8^+(2), {}^3D_4(2)$ by Lemma 2.3.

If G is a simple K_5-group or a simple K_6-group, then $\pi(G) \subseteq \{2, 3, 5, 7, 13, 73\}$. By Lemmas 2.4 and 2.5, G is isomorphic to one of the groups listed in Lemmas 2.4 and 2.5. Now we assert all candidates for G must be $L_2(64), L_2(3^6), L_3(9), G_2(4), G_2(9), S_4(8), S_4(27), S_6(3), O_7(3), U_4(5), O_8^+(3), {}^3D_4(3)$ and $L_4(8)$.

Suppose G is isomorphic to $L_2(q)$, where q is a prime power satisfying $|\pi(q^2 - 1)| = 4$ or 5. Let $q = p^s$, where p is a prime number and s is a natural number. Then $|G| = |L_2(q)| = \frac{q(q^2-1)}{(2,q-1)} = \frac{p^s(p^{2s}-1)}{(2,p^s-1)}$ (see [2]). Therefore $p\big||G|$. Since $\pi(G) \subseteq \{2, 3, 5, 7, 13, 73\}$, it follows that p must be one of the numbers $2, 3, 5, 7, 13$ or 73.

Suppose $p = 2$. Since $\{3, 5, 7, 13, 73\} \subseteq \pi(\prod_{i=1}^{12}(2^i - 1))$, we assert that $2s \leq 12$. In fact, if $2s > 12$, then there exists a prime z such that $z\big|2^{2s} - 1$ and $z \nmid 2^i - 1$ for $1 \leq i < 2s$ by Lemma 2.7. On one side, we have that $z \in \pi(G) \subseteq \{2, 3, 5, 7, 13, 73\}$ since $z\big|2^{2s} - 1$. On the other side, we have that $z \notin \{2, 3, 5, 7, 13, 73\} \supseteq \pi(G)$ since $z \nmid 2^i - 1$ for $1 \leq i < 2s$. This is again a contradiction. Therefore $1 \leq s \leq 6$.

If $1 \leq s \leq 5$, then G is a simple K_n-group according to its order, where $n = 3$ or 4, an obvious contradiction. If $s = 6$, then $|G| = \frac{2^6(2^{12}-1)}{(2,2^6-1)} = 2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$, hence a candidate $L_2(64)$ arises in this subcase.

We omit the remaining proof, which is made by a case by case analysis by a similar argument as done above in term of the changes of the values of p and the types of the involved groups. If necessary, one may apply Lemmas 2.4−2.8. Finally, we find that our assertion is true as desired. □

In the sequel a completely reducible group will be called a CR-group. The center of a CR-group is the direct product of the abelian factors in the decomposition. Hence a CR-group is centerless, that is has trivial center, if and only if it is a direct product of nonabelian simple groups. The following lemma determines the structure of the automorphism group of a centerless CR-group.

Lemma 2.10. ([9], Theorem 3.3.20) *Let R be a finite centerless CR-group and write $R = R_1 \times R_2 \times \cdots \times R_k$ where R_i is a direct product of n_i isomorphic copies of a simple group H_i, and H_i and H_j are not isomorphic if $i \neq j$. Then $Aut(R) \cong Aut(R_1) \times Aut(R_2) \times \cdots \times Aut(R_k)$ and $Aut(R_i) \cong Aut(H_i) \wr S_{n_i}$ where in this wreath product $Aut(H_i)$ appears in its right regula representation and the* *symmetric group S_{n_i} in its natural permutation representation. Moreover these isomorphisms induce isomorphisms $Out(R) \cong Out(R_1) \times Out(R_2) \times \cdots \times Out(R_k)$ and $Out(R_i) \cong Out(H_i) \wr S_{n_i}$.*

At the end of this section, we quote two lemmas on noncommuting graph. Let g be an element of a finite group G. We denote by g^G the conjugacy class of G containing g. Also, we denote by $|g^G|$ the size of the conjugacy class g^G.

Lemma 2.11. ([8], Lemma 2) *Let G be a finite group such that $Z(G) = 1$. If H is a group such that $\nabla(H) \cong \nabla(G)$, then $|Z(H)|\big|(|C_G(g_i)| - 1)$ and $|Z(H)|\big|(|g_i^G| - 1)$ for every $g_i \in G^*$, where $G^* = G \setminus \{1\}$ and $1 \le i \le |G^*|$. In particular, if one of the following two conditions holds:*

1. *$g.c.d.\{|C_G(g_1)| - 1, |C_G(g_2)| - 1, \ldots, |C_G(g_{|G^*|})| - 1\} = 1$, or*
2. *$g.c.d.\{|g_1^G| - 1, |g_2^G| - 1, \ldots, |g_{|G^*|}^G| - 1\} = 1$,*

then $|H| = |G|$.

Lemma 2.12. ([11], Lemma 2.4) *Let G and H be finite groups. If $\nabla(H) \cong \nabla(G)$, then*

$$|C_H(x) \setminus Z(H)| = |C_G(\phi(x)) \setminus Z(G)|$$

for all $x \in H \setminus Z(H)$, where ϕ is a graph isomorphism from $\nabla(H)$ to $\nabla(G)$.

3. A new characterization of $L_4(8)$ by its noncommuting graph

Theorem : *Let $L_4(8)$ be the projective special linear group of degree 4 over the finite field of order 8. If G is a finite group with $\nabla(G) \cong \nabla(L_4(8))$, then $G \cong L_4(8)$.*

Proof. First we suppose that $M := L_4(8)$. Now we want to prove that $G \cong M$. We divide the proof into the following seven lemmas.

Lemma 3.1. *If $\nabla(G) \cong \nabla(M)$, then $|G| = |M|$. In particular, $Z(G) = 1$.*

Proof. By Lemma 2.11, it is sufficient to find a pair of elements in M, say u and v, such that $g.c.d.\{|C_M(u)| - 1, |C_M(v)| - 1\} = 1$ or $g.c.d.\{|u^M| - 1, |v^M| - 1\} = 1$. By Lemma 2.1, it follows that $\mu(M) = \{3^2 \cdot 5 \cdot 13, 7 \cdot 73, 2 \cdot 3^2 \cdot 7, 2^2 \cdot 7\}$. Therefore there exist two elements in M, say x and y, such that $o(x) = 3^2 \cdot 5 \cdot 13$ and $o(y) = 7 \cdot 73$ respectively. Since $|M| = 2^{18} \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 13 \cdot 73$ and $\{2 \cdot 5, 2 \cdot 13, 2 \cdot 73, 3 \cdot 73, 5 \cdot 7, 5 \cdot 73, 7 \cdot 13, 13 \cdot 73\} \cap \pi_e(M) = \varnothing$ by Lemma 2.1, it follows that $|C_M(x)| = 3^i \cdot 5 \cdot 13$ and $|C_M(y)| = 7^j \cdot 73$, where $2 \le i \le 4$, $1 \le j \le 3$.

In the sequel, we will use the following symbols, where x and y are those elements mentioned above.

(1) See Table 2: $Z_x := |C_M(x)|$; $N_x := \frac{|M|}{Z_x} = |x^M|$; $N'_x := N_x - 1$.
(2) See Table 3: $Z_y := |C_M(y)|$; $N_y := \frac{|M|}{Z_y} = |y^M|$; $N'_y := N_y - 1$.

Z_x	N'_x
$3^2 \cdot 5 \cdot 13$	$229 \cdot 257966867$
$3^3 \cdot 5 \cdot 13$	$31 \cdot 635208737$
$3^4 \cdot 5 \cdot 13$	$3^2 \cdot 5 \cdot 145862747$

Table 2 Possible Values of Z_x and N'_x

From Table 2, we have $\pi(N'_x) \subseteq \{3, 5, 31, 229, 257966867, 635208737, 145862747\}$.

Z_y	N'_y
$7 \cdot 73$	67629219839
$7^2 \cdot 73$	$71 \cdot 239 \cdot 431 \cdot 1321$
$7^3 \cdot 73$	$7 \cdot 83 \cdot 269 \cdot 8831$

Table 3 Possible Values of Z_y and N'_y

From Table 3, we have $\pi(N'_y) \subseteq \{7, 71, 83, 239, 269, 431, 1321, 8831, 67629219839\}$.

If Z_x is equal to one of the values listed in Table 2, then N'_x satisfies the condition $\pi(N'_x) \cap \pi(N'_y) = \varnothing$ for any y satisfying $o(y) = 7 \cdot 73$ by Tables 2 and 3. Let $u = x$ and $v = y$, where N'_x and N'_y satisfy the condition mentioned above. Then $g.c.d.\{|u^M| - 1, |v^M| - 1\} = g.c.d.\{N'_x, N'_y\} = 1$, which implies that $|G| = |M|$ by Lemma 2.11. Since $|G \setminus Z(G)| = |M \setminus Z(M)|$ by the definition of the noncommuting graph, it follows immediately that $Z(G) = Z(M) = 1$. □

Lemma 3.2. *If $\nabla(G) \cong \nabla(M)$, then $5 \cdot 73 \notin \pi_e(G)$ and $13 \cdot 73 \notin \pi_e(G)$.*

Proof. Let ϕ be a graph isomorphism from $\nabla(M)$ to $\nabla(G)$. If $5 \cdot 73 \in \pi_e(G)$, then there exists an element, say x, such that $o(x) = 5$. Therefore, $5 \cdot 73 \big| |C_G(x)|$. By Lemmas 2.12 and 3.1, $5 \cdot 73 \big| |C_M(\phi^{-1}(x))|$. If $2 \in \pi(o(\phi^{-1}(x)))$, then there exists a natural number i such that $x_1 := (\phi^{-1}(x))^i \in M$ is of order 2. Then $5 \cdot 73 \big| |C_M(x_1)|$, too. Let $y_1 \in C_M(x_1)$ such that $o(y_1) = 5$. Hence $o(x_1 y_1) = 2 \cdot 5$. Thus $2 \cdot 5 \in \pi_e(M)$, which is a contradiction since $2 \nsim 5$ in $\Gamma(M)$. Therefore $2 \notin \pi(o(\phi^{-1}(x)))$. By a similar argument, we have that $\{3, 5, 7, 13, 73\} \cap \pi(o(\phi^{-1}(x))) = \varnothing$, too. Thus $\pi(M) \cap \pi(o(\phi^{-1}(x))) = \varnothing$. It follows that $\phi^{-1}(x) = 1$, a contradiction.

By a similar argument as done above, we get that $13 \cdot 73 \notin \pi_e(G)$. □

Lemma 3.3. *If $\nabla(G) \cong \nabla(M)$, then G is nonsolvable.*

Proof. Suppose that G is solvable. Since $|G| = |M|$ by Lemma 3.1, it follows that G has a Hall $\{5, 73\}$-subgroup H of order $5 \cdot 73$. Therefore H is a cyclic subgroup, which implies that $5 \cdot 73 \in \pi_e(G)$. This is a contradiction by Lemma 3.2. □

Lemma 3.4. *Let K be the maximal normal solvable subgroup of G. Then K is a $\{2,3,7\}$-subgroup.*

Proof. First we assume that $\{5,13,73\} \subseteq \pi(K)$. Let T be a Hall $\{5,73\}$-subgroup of K. It is easy to see that T is a cyclic subgroup of order $5 \cdot 73$ since $|G| = 2^{18} \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 13 \cdot 73$ by Lemma 3.1. Thus $5 \cdot 73 \in \pi_e(K) \subseteq \pi_e(G)$, a contradiction.

Now we suppose that $\{p,q,r\} = \{5,13,73\}$.

Next we assume that $\{p,q\} \subseteq \pi(K)$ and $r \notin \pi(K)$. Let T be a Hall $\{p,q\}$-subgroup of K. It is easy to see that T is a cyclic subgroup of order $p \cdot q$. If $\{p,q\} \neq \{5,13\}$, then $p \cdot q \in \pi_e(K) \subseteq \pi_e(G)$, a contradiction. If $\{p,q\} = \{5,13\}$, then K is a $\{2,3,5,7,13\}$-subgroup. Let $R_p \in Syl_p(K)$. By Frattini argument $G = KN_G(R_p)$. Therefore, the normalizer $N_G(R_p)$ contains an element of order 73, say x. Obviously, $< x > R_p$ is a subgroup of G of order $73 \cdot p$, which is abelian. Hence $73 \cdot p \in \pi_e(G)$, a contradiction.

Finally, if $r \in \pi(K)$ and $\{p,q\} \cap \pi(K) = \varnothing$, then K is a $\{2,3,7,r\}$-subgroup. Let $R_r \in Syl_r(K)$. By Frattini argument $G = KN_G(R_r)$. Therefore, the normalizer $N_G(R_r)$ contains two elements of orders p and q, say x and y, respectively. Obviously, $< x > R_r$ and $< y > R_r$ are subgroups of G of orders $p \cdot r$ and $q \cdot r$, respectively. It is clear that both of them are abelian. Hence $\{p \cdot r, q \cdot r\} \subseteq \pi_e(G)$, a contradiction. □

In the sequel, we will use the following notation. Given a finite group G, denote by $Soc(G)$ the socle of G, which is the subgroup generated by the set of all minimal normal subgroups of G.

Lemma 3.5. *Let $\overline{G} := G/K$ and $S := Soc(\overline{G})$. Then the following assertions are true:*

(1) If $r \in \{5,13,73\}$ and $r \notin \pi(S)$, then $r \notin \pi(Aut(S))$.

(2) If $S \neq M$, then $\pi(S) = \{2,3,7\}$.

(3) S is isomorphic to one of the following groups: $L_2(7), L_2(8), U_3(3), L_2(7) \times L_2(7), L_2(7) \times L_2(8), L_2(7) \times U_3(3), L_2(8) \times L_2(8), L_2(7) \times L_2(7) \times L_2(7), L_2(7) \times L_2(7) \times L_2(8)$ or M.

Proof. Since G is nonsolvable by Lemma 3.3, we have that S is a centerless CR-group. Put $S = P_1 \times P_2 \times \cdots \times P_m$, where P_i's are nonabelian finite simple groups. Since $|G| = |M| = 2^{18} \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 13 \cdot 73$ by Lemma 3.1, it is obvious that P_i is isomorphic to one of the groups listed in Table 1. Moreover, $\{5,13,73\} \subseteq \pi(\overline{G})$ by Lemma 3.4.

In the sequel, we assume that $\{p,q,r\} = \{5,13,73\}$.

(1) Assume the first assertion to be false, then $r\big||Aut(S)|$. Since $Inn(S) \cong S$, we have that $r \nmid |Inn(S)|$. Because $Aut(S)/Inn(S) = Out(S)$, it follows that $r\big||Out(S)|$. But $Out(S) = Out(S_1)\times Out(S_2)\times\cdots\times Out(S_k)$, where the groups S_j's are direct products of some isomorphic copies of the simple groups belonging to the set $\{P_1, P_2, \ldots, P_m\}$ such that $S = P_1\times P_2\times\cdots\times P_m = S_1\times S_2\times\cdots\times S_k$. Therefore $r\big||Out(S_j)|$ for some j, where $1 \leq j \leq k$. Suppose that S_j is a direct product of t isomorphic copies of a simple group P_i, where $P_i \in \{P_1, P_2, \ldots, P_m\}$. By Lemma 2.10, we obtain that $|Out(S_j)| = |Out(P_i)|^t \cdot t!$. Since $\pi(P_i) \subseteq \pi(S) \subseteq \{2, 3, 7, p, q\}$, we have that $\pi(Out(P_i)) \subseteq \{2, 3\}$ or $|Out(P_i)| = 1$ by Lemma 2.9, which implies that $r \nmid |Out(P_i)|$. Therefore $r\big|t!$, which implies that $t \geq r \geq 5$. If $P_i \ncong Sz(8)$, then $3\big||P_i|$ by Lemma 2.9, which implies that $3^5\big||S_j|\big||S|\big||G|$, a contradiction. If $P_i \cong Sz(8)$, then $5\big||P_i|$ by Lemma 2.9, which implies that $5^5\big||S_j|\big||S|\big||G|$, a contradiction. Hence $r \notin \pi(Aut(S))$, as desired.

(2) First, we assume that $\{p, q\} \subseteq \pi(S)$ and $r \notin \pi(S)$. It is easy to see that $r \notin \pi(Aut(S))$ by the first assertion. Thus $r \in \pi(C_{\overline{G}}(S))$ since $\overline{G}/C_{\overline{G}}(S) \lesssim Aut(S)$. This implies that $\{p \cdot r, q \cdot r\} \subseteq \pi_e(G)$, a contradiction.

Next, we assume that $r \in \pi(S)$ and $\{p, q\} \cap \pi(S) = \varnothing$. It is easy to see that $\{p, q\} \cap \pi(Aut(S)) = \varnothing$ by the first assertion. Thus $\{p, q\} \subseteq \pi(C_{\overline{G}}(S))$ since $\overline{G}/C_{\overline{G}}(S) \lesssim Aut(S)$. This implies that $\{p \cdot r, q \cdot r\} \subseteq \pi_e(G)$, a contradiction.

Finally, we assume that $\{p, q, r\} \subseteq \pi(S)$. Since $|G| = |M| = 2^{18}\cdot 3^4\cdot 5\cdot 7^3\cdot 13\cdot 73$, it follows that $S \cong M$ by Lemma 2.9. However, this contradicts our hypothesis.

(3) Recall that $\pi(S) = \{2, 3, 7\}$ if $S \neq M$ by the second assertion. Therefore, using Table 1, we have that S is isomorphic to one of the following groups: $L_2(7), L_2(8), U_3(3), L_2(7)\times L_2(7), L_2(7)\times L_2(8), L_2(7)\times U_3(3), L_2(8)\times L_2(8), L_2(7)\times L_2(7)\times L_2(7), L_2(7)\times L_2(7)\times L_2(8)$ or M. □

Lemma 3.6. *Let S be a group mentioned in Lemma 3.5(3). Then the following assertions are true.*

(1) $|S|\big|\frac{|\overline{G}|}{|C_{\overline{G}}(S)|}\big||Aut(S)|$.

(2) If $S \neq M$, then $\{5, 13, 73\} \subseteq \pi(C_{\overline{G}}(S))$ and $C_{\overline{G}}(S)$ is nonsolvable.

Proof. (1) Since $S \cap C_{\overline{G}}(S) = Z(S) = 1$, we have that $S \cong S/S \cap C_{\overline{G}}(S) \cong C_{\overline{G}}(S)S/C_{\overline{G}}(S) \leq \overline{G}/C_{\overline{G}}(S) \lesssim Aut(S)$. Thus $|S|\big|\frac{|\overline{G}|}{|C_{\overline{G}}(S)|}\big||Aut(S)|$.

(2) If $S \cong L_2(7)\times L_2(7), L_2(7)\times L_2(8), L_2(7)\times U_3(3), L_2(8)\times L_2(8), L_2(7)\times L_2(7)\times L_2(7)$ or $L_2(7)\times L_2(7)\times L_2(8)$, then $|Aut(S)| = |Aut(L_2(7))|^2\cdot|S_2| = 2^9\cdot 3^2\cdot 7^2, |Aut(L_2(7))|\cdot|Aut(L_2(8))| = 2^7\cdot 3^4\cdot 7^2, |Aut(L_2(7))|\cdot|Aut(U_3(3))| = 2^{10}\cdot 3^4\cdot 7^2, |Aut(L_2(8))|^2\cdot|S_2| = 2^7\cdot 3^6\cdot 7^2, |Aut(L_2(7))|^3\cdot|S_3| = 2^{13}\cdot 3^4\cdot 7^3$ or $|Aut(L_2(7))|^2\cdot|S_2|\cdot|Aut(L_2(8))| = 2^{12}\cdot 3^5\cdot 7^3$ by Lemma 2.10, where S_r is a

symmetry group on $\{1,2,\ldots,r\}$. Thus $\pi(Aut(S)) = \{2,3,7\}$ if $S \neq M$. It follows that $\{5,13,73\} \subseteq \pi(C_{\overline{G}}(S))$ since $\overline{G}/C_{\overline{G}}(S) \lesssim Aut(S)$. Suppose $C_{\overline{G}}(S)$ is solvable. Then $C_{\overline{G}}(S)$ has a Hall $\{5,73\}$-subgroup T_1. It is easy to see that T_1 is an abelian subgroup of order $5\cdot 73$. Thus $5\cdot 73 \in \pi_e(C_{\overline{G}}(S))$. Hence $5\cdot 73 \in \pi_e(G)$ too, a contradiction. □

Let n be a natural number and p a prime. In the sequel, $e(n_p)$ denotes a non-negative integer such that $p^{e(n_p)}\big| n$ but $p^{e(n_p)+1} \nmid n$.

Lemma 3.7. *S is not isomorphic to any of the following groups: $L_2(7), L_2(8), U_3(3)$, $L_2(7)\times L_2(7), L_2(7)\times L_2(8), L_2(7)\times U_3(3), L_2(8)\times L_2(8), L_2(7)\times L_2(7)\times L_2(7)$ or $L_2(7)\times L_2(7)\times L_2(8)$. Therefore, $S \cong M = L_4(8)$.*

Proof. Suppose $S \cong L_2(7), L_2(8), U_3(3), L_2(7)\times L_2(7), L_2(7)\times L_2(8), L_2(7)\times U_3(3), L_2(8)\times L_2(8), L_2(7)\times L_2(7)\times L_2(7)$ or $L_2(7)\times L_2(7)\times L_2(8)$. In the sequel, we also assume that $\{p,q,r\} = \{5,13,73\}$ if p,q or r appears.

Step 1. Suppose $G_1 := C_{\overline{G}}(S)$, $\overline{G_1} := G_1/K_1$ and $S_1 := Soc(\overline{G_1})$, where K_1 is the maximal normal solvable subgroup of G_1. Then the following assertions are true.

(a) By Lemma 3.2, $\{5\cdot 73, 13\cdot 73\}\cap \pi_e(G_1) = \varnothing$ and $\{5\cdot 73, 13\cdot 73\}\cap\pi_e(\overline{G_1}) = \varnothing$.

(b) By Lemma 3.6(2), $\{5,13,73\} \subseteq \pi(G_1)$ and G_1 is nonsolvable. By a similar argument in Lemma 3.4, we obtain that K_1 is a $\{2,3,7\}$-subgroup, too.

(c) Moreover, $\{5,13,73\} \subseteq \pi(\overline{G_1})$ by (b) and so $\overline{G_1}$ is nonsolvable.

(d) By Lemma 3.6(1), $e(|G_1|_2) \le 15$, $e(|G_1|_3) \le 3$, $e(|G_1|_7) \le 2$, $e(|G_1|_5) = e(|G_1|_{13}) = e(|G_1|_{73}) = 1$.

(e) By the choice of K_1 and (c), S_1 is a direct product of some finite nonabelian simple groups listed in Table 1.

By Table 1 and (d), we have that $\{5,13,73\}\cap\pi(S_1) \neq \{5,13,73\}$ and so $S_1 \neq M$. Now we assert that $\pi(S_1) = \{2,3,7\}$.

First we assume that $\{p,q\} \subseteq \pi(S_1)$ and $r \notin \pi(S_1)$. It is easy to see $r \notin \pi(Aut(S_1))$ by a similar argument in Lemma 3.5(1). Thus $r \in \pi(C_{\overline{G_1}}(S_1))$ since $\overline{G_1}/C_{\overline{G_1}}(S_1) \lesssim Aut(S_1)$. It implies that $\{p\cdot r, q\cdot r\} \subseteq \pi_e(\overline{G_1})$, which contradicts (a).

Next we assume that $r \in \pi(S_1)$ and $\{p,q\}\cap\pi(S_1) = \varnothing$. It is easy to see $\{p,q\}\cap\pi(Aut(S_1)) = \varnothing$ by a similar argument in Lemma 3.5(1). Thus $\{p,q\} \subseteq \pi(C_{\overline{G_1}}(S_1))$ since $\overline{G_1}/C_{\overline{G_1}}(S_1) \lesssim Aut(S_1)$. It implies that $\{p\cdot r, q\cdot r\} \subseteq \pi_e(\overline{G_1})$, which contradicts (a).

Hence we obtain that $\pi(S_1) = \{2,3,7\}$. It follows that $S_1 \cong L_2(7), L_2(8), U_3(3)$, $L_2(7)\times L_2(7)$ or $L_2(7)\times L_2(8)$ by (d) and (e).

Step 2. Suppose $G_2 := C_{\overline{G_1}}(S_1)$, $\overline{G_2} := G_2/K_2$ and $S_2 := Soc(\overline{G_2})$, where K_2 is the maximal normal solvable subgroup of G_2. Then the following assertions are true.

(f) By Lemma 3.2, $\{5\cdot 73, 13\cdot 73\}\cap\pi_e(G_2) = \varnothing$ and $\{5\cdot 73, 13\cdot 73\}\cap\pi_e(\overline{G_2}) = \varnothing$.

(g) By a similar argument in Lemma 3.6, $\{5, 13, 73\} \subseteq \pi(G_2)$ and G_2 is nonsolvable. By a similar argument in Lemma 3.4, we obtain that K_2 is a $\{2, 3, 7\}$-subgroup, too.

(h) Moreover, $\{5, 13, 73\} \subseteq \pi(\overline{G_2})$ by (g) and so $\overline{G_2}$ is nonsolvable.

(i) $e(|G_2|_2) \leq 12$, $e(|G_2|_3) \leq 2$, $e(|G_2|_7) \leq 1$, $e(|G_2|_5) = e(|G_2|_{13}) = e(|G_2|_{73}) = 1$ since $|S_1|\Big|\frac{|\overline{G_1}|}{|C_{\overline{G_1}}(S_1)|}\Big||Aut(S_1)|$.

(j) By the choice of K_2 and (i), S_2 is a direct product of some finite nonabelian simple groups listed in Table 1.

By Table 1 and (i), we have that $\{5, 13, 73\}\cap\pi(S_2) \neq \{5, 13, 73\}$ and so $S_2 \neq M$. Now we assert that $\pi(S_2) = \{2, 3, 7\}$.

First we assume that $\{p, q\} \subseteq \pi(S_2)$ and $r \notin \pi(S_2)$. It is easy to see $r \notin \pi(Aut(S_2))$ by a similar argument in Lemma 3.5(1). Thus $r \in \pi(C_{\overline{G_2}}(S_2))$ since $\overline{G_2}/C_{\overline{G_2}}(S_2) \lesssim Aut(S_2)$. It implies that $\{p\cdot r, q\cdot r\} \subseteq \pi_e(\overline{G_2})$, which contradicts (f).

Next we assume that $r \in \pi(S_2)$ and $\{p, q\} \cap \pi(S_2) = \varnothing$. It is easy to see $\{p, q\} \cap \pi(Aut(S_2)) = \varnothing$ by a similar argument in Lemma 3.5(1). Thus $\{p, q\} \subseteq \pi(C_{\overline{G_2}}(S_2))$ since $\overline{G_2}/C_{\overline{G_2}}(S_2) \lesssim Aut(S_2)$. It implies that $\{p\cdot r, q\cdot r\} \subseteq \pi_e(\overline{G_2})$, which contradicts (f).

Hence we get that $\pi(S_2) = \{2, 3, 7\}$ by (i). It follows that $S_1 \cong L_2(7)$ or $L_2(8)$ by (i) and (j).

Step 3. Suppose $G_3 := C_{\overline{G_2}}(S_2)$, $\overline{G_3} := G_3/K_3$ and $S_3 := Soc(\overline{G_3})$, where K_3 is the maximal normal solvable subgroup of G_3. Then the following assertions are true.

(k) By Lemma 3.2, $\{5\cdot 73, 13\cdot 73\}\cap\pi_e(G_3) = \varnothing$ and $\{5\cdot 73, 13\cdot 73\}\cap\pi_e(\overline{G_3}) = \varnothing$.

(l) By a similar argument in Lemma 3.6, $\{5, 13, 73\} \subseteq \pi(G_3)$ and G_3 is nonsolvable. By a similar argument in Lemma 3.4, we obtain that K_3 is a $\{2, 3, 7\}$-subgroup, too.

(m) Moreover, $\{5, 13, 73\} \subseteq \pi(\overline{G_3})$ by (l) and so $\overline{G_3}$ is nonsolvable.

(n) $e(|G_3|_2) \leq 9$, $e(|G_3|_3) \leq 1$, $e(|G_3|_7) = 0$, $e(|G_3|_5) = e(|G_3|_{13}) = e(|G_3|_{73}) = 1$ since $|S_2|\Big|\frac{|\overline{G_2}|}{|C_{\overline{G_2}}(S_2)|}\Big||Aut(S_2)|$.

(o) By the choice of K_3 and (n), S_3 is a direct product of some finite nonabelian simple groups listed in Table 1. In particular, S_3 is nonsolvable.

By Table 1 and (n), we have that $\{5, 13, 73\}\cap\pi(S_3) \neq \{5, 13, 73\}$ and so $S_3 \neq M$.

First we assume that $\{p,q\} \subseteq \pi(S_3)$ and $r \notin \pi(S_3)$. It is easy to see $r \notin \pi(Aut(S_3))$ by a similar argument in Lemma 3.5(1). Thus $r \in \pi(C_{\overline{G_3}}(S_3))$ since $\overline{G_3}/C_{\overline{G_3}}(S_3) \lesssim Aut(S_3)$. It implies that $\{p \cdot r, q \cdot r\} \subseteq \pi_e(\overline{G_3})$, which contradicts (k).

Next we assume that $r \in \pi(S_3)$ and $\{p,q\} \cap \pi(S_3) = \varnothing$. It is easy to see $\{p,q\} \cap \pi(Aut(S_3)) = \varnothing$ by a similar argument in Lemma 3.5(1). Thus $\{p,q\} \subseteq \pi(C_{\overline{G_3}}(S_3))$ since $\overline{G_3}/C_{\overline{G_3}}(S_3) \lesssim Aut(S_3)$. It implies that $\{p \cdot r, q \cdot r\} \subseteq \pi_e(\overline{G_3})$, which contradicts (k).

Hence we get that $\pi(S_3) = \{2,3\}$ by (n) and so S_3 is solvable, which contradicts (o).

By Steps 1-3, we have obtained that $S \cong M$. By Lemma 3.6(1), it follows that $|\overline{G}| = |G| = |M|$. Hence $K = 1$ and $G \cong M = L_4(8)$. □

By Lemmas 3.1-3.7, we complete the proof of Theorem. □

Remark 3.8. *Although we have not found a general method to deal with all simple groups on AAM's conjecture, it is evident that the method used in the present paper also works well in the cases* $L_4(4)$, $L_4(7)$, $U_4(7)$, *etc.*

Acknowledgment. The authors would like to thank the referee for the valuable suggestions and comments.

References

[1] A. Abdollahi, S. Akbari and H. R. Maimani, *Noncommuting graph of a group*, J. Algebra, 298(2) (2006), 468-492.

[2] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, Atlas of Finite Groups, Clarendon Press(Oxford), London / New York, (1985).

[3] M. Herzog, *On finite simple groups of order divisible by three primes only*, J. Algebra, 10 (1968), 383-388.

[4] L. G. Hua, A Guide to the Number Theory (in Chinese), Science Press, Beijing, China, (1979).

[5] A. Jafarzadeh and A. Iranmanesh, *On simple K_n-groups for $n = 5, 6$*, London Mathematical Society Lecture Note Series 340 (Groups St Andrews 2005 Volume 2), Cambridge University Press, (2007), 668-680.

[6] P. Kleidman and M. Liebeck, The subgroup structure of finite classical groups, Cambridge University Press, (1990).

[7] V. D. Mazurov and G. Y. Chen, *Recognizability of finite simple groups $L_4(2^m)$ and $U_4(2^m)$ by spectrum*, Algebra Logic, 47(1) (2008), 49-55.

[8] A. R. Moghaddamfar, W. J. Shi, W. Zhou and A. R. Zokayi, *On the noncommuting graph associated with a finite group*, Sib. Math. J., 46(2) (2005), 325-332.

[9] Derek J. S. Robinson, A Course in the Theory of Groups (2nd ed), Springer-Verlag, New York/Heidelberg/Berlin, 2003.

[10] W. J. Shi, *On simple K_4-groups* (in Chinese), Chinese Sci. Bull., 36(7) (1991), 1281-1283.

[11] L. L. Wang and W. J. Shi, *A new characterization of A_{10} by its noncommuting graph*, Comm. Algebra, 36(2) (2008), 523-528.

[12] J. S. Williams, *Prime graph components of finite groups*, J. Algebra, 69(2) (1981), 487-513.

[13] K. Zsigmondy, *Zur theorie der potenzreste*, Monatsh. Math. Phys., 3 (1892), 265-284.

Liangcai Zhang
College of Mathematics and Physics
Chongqing University
Shapingba, Chongqing 400044
People's Republic of China
e-mail: zlc213@163.com

Wujie Shi
School of Mathematical Sciences
Suzhou University
Suzhou, Jiangsu 215006
People's Republic of China
e-mail: wjshi@suda.edu.cn