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ON VNL-RINGS AND n-VNL-RINGS

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Abstract. A ring R is called a *VNL-ring* if a or $1-a$ is regular for every $a \in R$. We call a ring R a *right n-VNL-ring* if whenever $a_1R + a_2R + \cdots + a_nR = R$ for some elements $a_1, a_2, \cdots, a_n$ of R, then there exists at least one element a_i (von Neumann) regular. It is proven that there exists a right 2-VNL-ring but not right 3-VNL, which gives a negative answer to a question raised by Chen and Tong in 2006. We prove that R is regular iff the $n \times n$ matrix ring over R is a VNL-ring for some $n \geq 2$. It is also proven that a ring R is a division ring iff R is semilocal and the 2×2 upper triangular matrix ring over R is a VNL-ring.

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1. Introduction

Throughout this paper, all rings are associative with identity. For a ring R, let $M_n(R)$ and $T_n(R)$ be the rings of all $n \times n$ matrices and all $n \times n$ upper triangular matrices over R respectively. The ring of integers modulo n is denoted by $\mathbb{Z}_n$.

An element a of a ring R is said to be *(von Neumann) regular* if $a = aba$ for some $b \in R$. Moreover, if $ab = ba$ then a is *strongly regular*. A ring is *regular* (resp., *strongly regular*) if all of its elements are regular (resp., strongly regular). Contessa called a ring R a *VNL-ring* (von Neumann regular local ring) if a or $1-a$ is regular for every $a \in R$ [2]. Such rings are of interest since they constitute a subclass of the PM-rings (that is, every prime ideal is contained in a unique maximal ideal). A ring is *abelian* if all idempotents are contained in the center. Some properties of commutative VNL-rings and abelian VNL-rings are studied in [3], [4] and [1].

Osba et al. [3] called a commutative ring R an *SVNL-ring* if whenever $(S) = R$ for some nonempty subset S of R, at least one of the elements in S is regular, where

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(S) is an ideal generated by S. In the last paragraph of their paper, they asked whether there is a VNL-ring that is not an SVNL-ring. Chen and Tong introduced noncommutative SVNL-rings. According to [1], a ring R is a *right SVNL-ring* if whenever $(S)_r = R$ for some nonempty subset S of R, at least one element in S is regular, where $(S)_r$ is a right ideal generated by S. Left SVNL-rings are defined analogously, and a ring R is SVNL if it is a left and right SVNL-ring. Chen and Tong proved that every abelian VNL-ring is an SVNL-ring [1, Theorem 2.8], and raised a question whether every VNL-ring is an SVNL-ring [1, Question 3.17]. We prove that there exists a VNL-ring which is neither left nor right 3-VNL-ring. It answers the question of Chen and Tong in the negative. VNL-rings have not been related to other more familiar classes of rings before. In this paper, it is proven that R is regular iff $M_n(R)$ is a VNL-ring for some $n \geq 2$ iff $M_n(R)$ is a VNL-ring for every $n \geq 2$. A ring R is a division ring iff R is semilocal and $T_2(R)$ is a VNL-ring. It is also proven that the trivial extension of R is an abelian VNL-ring iff R is local.

2. VNL-rings and SVNL-rings

In this section, the properties and several extensions of VNL-rings and SVNL-rings are investigated. VNL-rings are also related to more familiar classes of rings.

Proposition 2.1. *If R is a VNL-ring, then eRe is also a VNL-ring for every $e = e^2 \in R$.*

Proof. For $a \in eRe$, a or $1 - a$ is regular in R. If a is regular, then there exists $b \in R$ such that $a = aba$. So $a = (ae)b(ea) = a(ebe)a$. Thus, a is regular in eRe. If $1 - a$ is regular, then there exists $c \in R$ such that $1 - a = (1 - a)c(1 - a)$. So $e - a = e(1 - a)e = e(1 - a)c(1 - a)e = (e - a)ece(e - a)$. Thus, $e - a$ is regular in eRe. Therefore, eRe is a VNL-ring. □

According to Proposition 2.1, one may ask whether the property "VNL" is Morita invariant. To answer it, we need the following useful fact, the proof of which is trivial.

Lemma 2.2. *Let $diag(a_1, a_2, \cdots, a_n)$ be the $n \times n$ diagonal matrix with a_i in each entry on the main diagonal. $diag(a_1, a_2, \cdots, a_n)$ is regular in $M_n(R)$ (resp., $T_n(R)$) iff $a_1, a_2, \cdots, a_n$ are all regular in R.*

Theorem 2.3. *For a ring R, the following are equivalent:*

(1) R is a regular ring.

(2) $M_n(R)$ is a right SVNL-ring for every $n \geq 2$.

(3) $M_n(R)$ is a VNL-ring for every $n \geq 2$.

(4) $M_n(R)$ is a right SVNL-ring for some $n \geq 2$.

(5) $M_n(R)$ is a VNL-ring for some $n \geq 2$.

Proof. It is a well-known fact that R is regular iff $M_n(R)$ is regular. Hence "(1)⇒ (2)⇒ (4)⇒ (5)" and "(1)⇒ (3)⇒ (5)" are clear.

"(5)⇒ (1)" Let $A =\mathrm{diag}(a, 1-a, 1, \cdots, 1) \in M_n(R)$ and I_n be the identity matrix. Then $I_n - A = \mathrm{diag}(1-a, a, 0, \cdots, 0)$. Because $M_n(R)$ is a VNL-ring, either A or $I_n - A$ is regular. For any case, a is regular by Lemma 2.2. Thus R is regular. □

By Theorem 2.3, if R is a VNL-ring which is not regular, then $M_n(R)$ is not a VNL-ring for every $n \geq 2$. Therefore, the property "VNL" is not Morita invariant. By a similar way to prove "(5)⇒ (1)" in Theorem 2.3, it also follows that if $T_n(R)$ is a VNL-ring for some $n \geq 2$, then R is regular. But the converse is not true. For example, $T_2(\mathbb{Z}_6)$ is not a VNL-ring since neither $\left(\begin{smallmatrix} 3 & 1 \\ 0 & 3 \end{smallmatrix}\right)$ nor $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 3 & 1 \\ 0 & 3 \end{smallmatrix}\right)$ is regular, though $\mathbb{Z}_6$ is regular.

Proposition 2.4. *For any ring R and $n \geq 4$, $T_n(R)$ is not a VNL-ring.*

Proof. Applying Proposition 2.1, we may assume that $n = 4$. Let $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Then $\mathrm{diag}(A, I_2 - A)$ and $\mathrm{diag}(I_2 - A, A)$ are not regular. Hence $T_n(R)$ is not a VNL-ring for any $n \geq 4$. □

Proposition 2.5. *For a division ring D, $T_2(D)$ is an SVNL-ring.*

Proof. Let $R = T_2(D)$. If $R = (S)_r$ for some nonempty subset S of R, then there exist $A_1, A_2, \cdots, A_m \in S$ such that $A_1R + A_2R + \cdots + A_mR = R$, where $A_i = \left(\begin{smallmatrix} a_i & c_i \\ 0 & b_i \end{smallmatrix}\right)$. Thus there exists $b_k \neq 0$ for some k, whence A_k is regular. Similarly, R is a left SVNL-ring and so R is an SVNL-ring. □

Recall that a ring R is said to be *semilocal* if $R/J(R)$ is semisimple, where $J(R)$ is the Jacobson radical of R. The following fact was proven in [3, Theorem 3.1] and [1, Lemma 2.6].

Lemma 2.6. *Let $R = \prod_{\alpha\in\Lambda} R_\alpha$ be a ring. Then R is a VNL-ring if and only if there exists $\alpha_0 \in \Lambda$ such that R_{α_0} is a VNL-ring and for each $\alpha \in \Lambda - \{\alpha_0\}$, R_α is a regular ring.*

Proposition 2.7. *R is a division ring if and only if R is semilocal and $T_2(R)$ is a VNL-ring.*

Proof. The fact that if $T_2(R)$ is a VNL-ring then R is regular was pointed out in the paragraph after Theorem 2.3. Since $J(R)=0$, R is semisimple. Hence, $R \cong \prod_{i=1}^m R_i$, where $R_i \cong M_{n_i}(D_i)$ and D_i is a division ring. Thus $T_2(R) \cong \prod_{i=1}^m T_2(R_i)$. Suppose $m \geq 2$. Because $T_2(R)$ is VNL, there exists at least one $T_2(R_i)$ regular by Lemma 2.6. But $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is not regular, a contradiction. Therefore, we may assume $R = M_n(D)$ for some division ring D. Suppose $n \geq 2$. Then $M_n(T_2(D)) \cong T_2(M_n(D)) = T_2(R)$ is a VNL-ring. It implies that $T_2(D)$ is regular by Theorem 2.3, but $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is not regular, a contradiction. So $n=1$. Therefore, $R=D$ is a division ring. The converse is clear by Proposition 2.5. □

Given a ring R, the trivial extension of R is the ring $T(R,R)=\{(a,b) : a,b \in R\}$ with the usual addition and the multiplication $(a_1,b_1)(a_2,b_2)=(a_1a_2, a_1b_2+b_1a_2)$. In fact, $T(R,R)$ is isomorphic to the subring $\{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) : a,b \in R\}$ of $T_2(R)$.

Proposition 2.8. *(1) Let $R = T_n(S)$ for a ring S and $n \geq 2$. Then $T(R,R)$ is not a VNL-ring.*

(2) Let R_1, R_2 be rings and $R = R_1 \times R_2$. Then $T(R,R)$ is not a VNL-ring.

Proof. (1) Suppose $n=2$. Let $T=(A,I_2) \in T(R,R)$, where $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. If T is regular, then there exists $(X,Y) \in T(R,R)$ such that $(A,I_2)=(A,I_2)(X,Y)(A,I_2)$. So $AX+AYA+XA=I_2$. Write $X = \left(\begin{smallmatrix} x_1 & x_2 \\ 0 & x_3 \end{smallmatrix}\right)$ and $Y = \left(\begin{smallmatrix} y_1 & y_2 \\ 0 & y_3 \end{smallmatrix}\right)$. Thus

$$\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} x_1 & x_2 \\ 0 & x_3 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} y_1 & y_2 \\ 0 & y_3 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} x_1 & x_2 \\ 0 & x_3 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right),$$

a contradiction by comparing the (2,2) entry of matrices in two sides. It can also be proven that $(I_2,0)-T$ is not regular as above. Hence $T(T_2(S),T_2(S))$ is not VNL.

Suppose $n \geq 3$. Let $A = \left(\begin{smallmatrix} A_1 & \alpha_1 \\ 0 & A_2 \end{smallmatrix}\right), B = \left(\begin{smallmatrix} B_1 & \alpha_2 \\ 0 & B_2 \end{smallmatrix}\right) \in R$, where $A_1, B_1 \in T_2(S)$. If (A,B) is regular in $T(R,R)$, then (A_1,B_1) is regular in $T(T_2(S),T_2(S))$. Since $T(T_2(S),T_2(S))$ is not VNL, neither is $T(R,R)$.

Therefore, $T(R,R)$ is not a VNL-ring.

(2) We denote $[r_1,r_2]$ for the element of R to erase the ambiguity. Let $a = ([1,0],[1,1]) \in T(R,R)$. It is easy to verify that a and $(1,0)-a$ are not regular in $T(R,R)$. Hence $T(R,R)$ is not a VNL-ring. □

Lemma 2.9. *Let R be a ring. If $(a,1)$ is strongly regular in $T(R,R)$, then a is invertible in R.*

Proof. Since $(a,1)$ is strongly regular in $T(R,R)$, there exists $(x,y) \in T(R,R)$ such that $(a,1)=(a,1)(x,y)(a,1)$ and $(a,1)(x,y)=(x,y)(a,1)$. Thus $axa=a$, $ax=xa$ and $ax+aya+xa=1$. By the first two equations, $a=a^2x=xa^2$. Thus $aya=$

$ay(a^2x) = (aya)(ax) = (1 - ax - xa)ax = (1 - ax)ax - xa^2x = -x(a^2x) = -xa$, implying $ax = 1 = xa$. Therefore, a is invertible. □

Proposition 2.10. *For a ring R, $T(R,R)$ is an abelian VNL-ring if and only if R is a local ring.*

Proof. Let $A = (a, 1) \in T(R,R)$. If A is regular, then A is strongly regular since $T(R,R)$ is an abelian ring, whence a is invertible by Lemma 2.9. If $(1,0) - A$ is regular, then so is $A - (1,0)$. Hence $A - (1,0)$ is strongly regular. So $1 - a$ is invertible by Lemma 2.9. Therefore, R is a local ring. The converse is clear. □

Remark 2.11. *It is clear that if $T(R,R)$ is a VNL-ring (resp., an SVNL-ring), then R is a VNL-ring (resp., an SVNL-ring). But the converse is not true. By Proposition 2.5 and Proposition 2.8, although $R = T_2(\mathbb{Z}_2)$ is an SVNL-ring, $T(R,R)$ is not a VNL-ring.*

3. n-VNL-rings

In this section, the properties and some examples of n-VNL-rings are considered. The Example 3.4 below answers [1, Question 3.17].

Definition 3.1. A ring R is called a *right n-VNL-ring* if whenever $a_1R + a_2R + \cdots + a_nR = R$ for some elements $a_1, a_2, \cdots, a_n$ of R, then there exists at least one element a_i regular.

Left n-VNL-rings are defined analogously, and a ring R is an *n-VNL-ring* if it is a left and right n-VNL-ring. It is clear that regular rings are n-VNL-rings, and R is an SVNL-ring iff R is an n-VNL ring for every $n \geq 1$. Examples below will show that n-VNL-rings constitute a subclass of regular rings and SVNL-rings. According to the definition, every ring is a 1-VNL-ring. Thus it will be convenient and cause no misunderstanding if we say that a ring is n-VNL only when $n \geq 2$.

From [1, Theorem 2.8], we can infer that for an abelian ring R, R is left n-VNL iff R is right n-VNL. But $T_2(D)$, which is not abelian, is also n-VNL by Proposition 2.5. We do not know whether every left n-VNL ring is right n-VNL.

Proposition 3.2. *The following statements hold:*

(1) Every right 2-VNL-ring R is a VNL-ring.

(2) For $n \geq m$, every right n-VNL-ring R is a right m-VNL-ring.

Proof. (1) Let $a \in R$. Since $aR + (1-a)R = R$ and R is a right 2-VNL-ring, a or $1 - a$ is regular. Therefore R is a VNL-ring.

(2) It is obvious. □

Corollary 3.3. *For a ring R, the following are equivalent:*

(1) R is a regular ring.

(2) $M_n(R)$ is a right m-VNL-ring for every $n \geq 2$.

(3) $M_n(R)$ is a right m-VNL-ring for some $n \geq 2$.

Proof. "(1)⇒ (2)⇒ (3)" is trivial, and "(3) ⇒ (1)" is clear by Proposition 3.2 and Theorem 2.3. □

Example 3.4. *(1) For every division ring D, $T_3(D)$ is a 2-VNL-ring.*

(2) For any ring R and $m, n \geq 3$, $T_m(R)$ is neither left nor right n-VNL.

Proof. (1) Denote $A = \begin{pmatrix} A_1 & \alpha \\ 0 & a \end{pmatrix} \in T_3(D)$, where $A_1 \in T_2(D)$ and α is a 2×1 matrix. It is obvious that A_1 is not regular in $T_2(D)$ iff $A_1 \in \{\begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix} : d \in D\}$. If $a \neq 0$ and A_1 is regular, then there exists $B_1 \in T_2(D)$ such that $A_1 B_1 A_1 = A_1$. Thus, A is regular since

$$\begin{pmatrix} A_1 & \alpha \\ 0 & a \end{pmatrix} = \begin{pmatrix} A_1 & \alpha \\ 0 & a \end{pmatrix} \begin{pmatrix} B_1 & -B_1 \alpha a^{-1} \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} A_1 & \alpha \\ 0 & a \end{pmatrix}.$$

If $a = 0$ and A_1 is invertible, then A is regular since $\begin{pmatrix} A_1 & \alpha \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A_1 & \alpha \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_1 & \alpha \\ 0 & 0 \end{pmatrix}$. Therefore, if A is not regular, then

(i) $A = \begin{pmatrix} A_1 & \alpha \\ 0 & a \end{pmatrix}$, where A_1 is not regular and $a \neq 0$; or

(ii) $A = \begin{pmatrix} A_1 & \alpha \\ 0 & 0 \end{pmatrix}$, where A_1 is not invertible.

Sequentially, assume that there exist non-regular matrices $B = (b_{ij}), C = (c_{kl}) \in T_3(D)$ such that $BT_3(D) + CT_3(D) = T_3(D)$. Then there exist $X = (x_{ij}), Y = (y_{kl}) \in T_3(D)$ such that $BX + CY = I_3$. We only need to consider three cases.

Case 1. $B = \begin{pmatrix} 0 & b_{12} & b_{13} \\ 0 & 0 & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix}$ and $C = \begin{pmatrix} 0 & c_{12} & c_{13} \\ 0 & 0 & c_{23} \\ 0 & 0 & c_{33} \end{pmatrix}$.

Since $BX + CY = I_3$, $\begin{pmatrix} 0 & b_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{pmatrix} + \begin{pmatrix} 0 & c_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} \\ 0 & y_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. By comparing the second row of matrices in two sides, we have $(0,\ 0) = (0,\ 1)$, a contradiction.

Case 2. $B = \begin{pmatrix} 0 & b_{12} & b_{13} \\ 0 & 0 & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix}$ and $C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & 0 & 0 \end{pmatrix}$, where $\begin{pmatrix} c_{11} & c_{12} \\ 0 & c_{22} \end{pmatrix}$ is not invertible.

Since $BX + CY = I_3$, $\begin{pmatrix} 0 & b_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{pmatrix} + \begin{pmatrix} c_{11} & c_{12} \\ 0 & c_{22} \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} \\ 0 & y_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence $c_{22} y_{22} = 1$. Because $\begin{pmatrix} c_{11} & c_{12} \\ 0 & c_{22} \end{pmatrix}$ is not invertible, $c_{11} = 0$. It is a contradiction by comparing the first column of matrices in two sides.

Case 3. $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & 0 & 0 \end{pmatrix}$, where $\left(\begin{smallmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} c_{11} & c_{12} \\ 0 & c_{22} \end{smallmatrix}\right)$ are not invertible.

Since $BX + CY = I_3$, it is a contradiction by comparing the (3,3) entry of matrices in two sides.

Therefore, $T_3(D)$ is a right 2-VNL-ring. Using the same way, we can prove that $T_3(D)$ is a left 2-VNL-ring.

(2) If $m \geq 4$, then $T_m(R)$ is not VNL by Proposition 2.4. Thus, $T_m(R)$ is neither left nor right n-VNL by Proposition 3.2. Suppose $m = 3$. Let

$$B_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then B_1, B_2, B_3 are non-regular matrices, but $B_1 + B_2 + B_3$ is invertible in $T_3(R)$. Hence $T_3(R)$ is neither left nor right 3-VNL. Therefore, $T_m(R)$ is neither left nor right n-VNL. □

Remark 3.5. *Let D be a division ring. According to Example 3.4, $T_3(D)$ is a VNL-ring by Proposition 3.2 (1), but $T_3(D)$ is not an SVNL-ring. It gives a negative answer to Question 3.17 in* [1].

If $m \geq 2$ and $m = \prod_{i=1}^{t} p_i^{k_i}$ is a prime power decomposition of m, then $\mathbb{Z}_m$ is an SVNL-ring iff $k_i > 1$ for at most one value of i [3, Proposition 2.9]. This fact also determines when $\mathbb{Z}_m$ is a VNL-ring by [1, Theorem 2.8]. It is completely characterized here when $T_n(\mathbb{Z}_m)$ is a VNL-ring.

Example 3.6. *Let $n \geq 2$ and $m \geq 2$. $T_n(\mathbb{Z}_m)$ is a VNL-ring iff $n = 2$ or 3 and m is a prime number.*

Proof. The "if" part follows by Proposition 2.5 and Example 3.4. For the "only if" part, $n = 2$ or 3 by Proposition 2.4. Because $\mathbb{Z}_m$ is a semilocal ring and $T_2(\mathbb{Z}_m)$ is a VNL-ring by Proposition 2.1, $\mathbb{Z}_m$ is a division ring by Proposition 2.7. Therefore, m is a prime number. □

References

[1] W. X. Chen and W. T. Tong, *On noncommutative VNL-rings and GVNL-rings*, Glasgow Math. J., 48 (2006), 11-17.

[2] M. Contessa, *On certain classes of PM-rings*, Comm. Algebra, 12 (1984), 1447-1469.

[3] E. A. Osba, M. Henriksen and O. Alkam, *Combining local and von Neumann regular rings*, Comm. Algebra, 32(7) (2004), 2639-2653.

[4] E. A. Osba, M. Henriksen, O. Alkam and F. A. Smith, *The maximal regular ideal of some commutative rings*, Comment. Math. Univ. Carolinae, 47(1) (2006), 1-10.

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