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THE CARDINALITY OF AN ANNIHILATOR CLASS IN A VON NEUMANN REGULAR RING

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Abstract. One defines an equivalence relation on a commutative ring R by declaring elements $r_1, r_2 \in R$ to be equivalent if and only if $\mathrm{ann}_R(r_1) = \mathrm{ann}_R(r_2)$. If $[r]_R$ denotes the equivalence class of an element $r \in R$, then it is known that $|[r]_R| = |[r/1]_{T(R)}|$, where $T(R)$ denotes the total quotient ring of R. In this paper, we investigate the extent to which a similar equality will hold when $T(R)$ is replaced by $Q(R)$, the complete ring of quotients of R. The results are applied to compare the zero-divisor graph of a reduced commutative ring to that of its complete ring of quotients.

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1. Introduction

Let R be a commutative ring. One easily checks that an equivalence relation on R is given by declaring elements $r_1, r_2 \in R$ to be equivalent if and only if $\mathrm{ann}_R(r_1) = \mathrm{ann}_R(r_2)$. The cardinalities of such equivalence (annihilator) classes were considered in [13], where the authors were interested in ring-theoretic properties shared by von Neumann regular rings with identical zero-divisor structures. In [3], the authors show that every ring has the same zero-divisor structure as its total quotient ring. The proof of this result demonstrates that the cardinality of the annihilator class of an element does not change when the element is regarded as a member of its total quotient ring. We examine the degree to which this result can be generalized to a particular extension of a reduced total quotient ring.

Throughout, R will always be a commutative ring with $1 \neq 0$. Let $Z(R)$ denote the set of zero-divisors of R and $T(R) = R_{R \setminus Z(R)}$ its total quotient ring. A ring R will be called *reduced* if $\mathrm{nil}(R) = (0)$. A commutative ring R with $1 \neq 0$ is *von Neumann regular* if for each $x \in R$, there is a $y \in R$ such that $x = x^2y$ or, equivalently, R is reduced with Krull dimension zero [9, Theorem 3.1].

A subset $D \subseteq R$ is *dense in* R if $\operatorname{ann}_R(D) = (0)$. Let D_1 and D_2 be dense ideals of R and let $\varphi_i \in \operatorname{Hom}_R(D_i, R)$ $(i = 1, 2)$. Note that $\varphi_1 + \varphi_2$ is an R-module homomorphism on the dense ideal $D_1 \cap D_2$, and $\varphi_1 \circ \varphi_2$ is an R-module homomorphism on the dense ideal $\varphi_2^{-1}(D_1) = \{r \in R \mid \varphi_2(r) \in D_1\}$. Then $Q(R) = F/\sim$ is a commutative ring, where $F = \{\varphi \in \operatorname{Hom}_R(D, R) \mid D \subseteq R \text{ is a dense ideal}\}$ and $\sim$ is the equivalence relation defined by $\varphi_1 \sim \varphi_2$ if and only if there exists a dense ideal $D \subseteq R$ such that $\varphi_1(d) = \varphi_2(d)$ for all $d \in D$ [12, Proposition 2.3.1]. In [12], J. Lambek calls $Q(R)$ the *complete ring of quotients of* R.

Let $\overline{\varphi} \in Q(R)$ denote the equivalence class containing φ. For all $a/b \in T(R)$, the ideal bR of R is dense and $\varphi_{a/b} \in \operatorname{Hom}_R(bR, R)$, where $\varphi_{a/b}(br) = ar$. One checks that the mapping $a/b \mapsto \overline{\varphi_{a/b}}$ is a ring monomorphism, and that $\overline{\varphi_0}$ and $\overline{\varphi_1}$ are the additive and multiplicative identities of $Q(R)$, respectively. In particular, the mapping $R \to Q(R)$ defined by $r \mapsto \overline{\varphi_r}$ is an embedding. However, these mappings need not be onto (see [12]). If the mapping $R \to Q(R)$ *is* onto (i.e., $r \mapsto \overline{\varphi_r}$ is an isomorphism), then R is called *rationally complete*. Note that $Q(R)$ is von Neumann regular if and only if R is reduced [12, Proposition 2.4.1]. Thus every reduced rationally complete ring is von Neumann regular.

A ring extension $R \subseteq S$ is called a *ring of quotients of* R if $f^{-1}R = \{r \in R \mid fr \in R\}$ is dense in S for all $f \in S$. In particular, $T(R)$ is a ring of quotients of R. If S is a ring of quotients of R, then there exists an extension of the mapping $R \to Q(R)$ which embeds S into $Q(R)$ [12, Proposition 2.3.6]. Therefore, every ring of quotients of R can be regarded as a subring of $Q(R)$. It follows that a dense set in R is dense in every ring of quotients of R. Also, R has a *unique* maximal (with respect to inclusion) ring of quotients, which is isomorphic to $Q(R)$ [12, Proposition 2.3.6]. In recognition of this observation, we shall abuse notation and denote the maximal ring of quotients of R by $Q(R)$. It is not hard to check that $Q(R) = Q(T(R))$ for any ring R. In fact, if $R \subseteq S \subseteq Q$, then Q is a ring of quotients of R if and only if Q is a ring of quotients of S and S is a ring of quotients of R (e.g., see the comments prior to Lemma 1.5 in [8]).

Let $B(R) = \{e \in R \mid e^2 = e\}$, the set of idempotents of R. Then the relation "$\leq$" defined by $a \leq b$ if and only if $ab = a$ partially orders $B(R)$, and makes $B(R)$ a Boolean algebra with inf as multiplication in R, the largest element as 1, the smallest element as 0, and complementation defined by $a' = 1 - a$. One checks that $a \vee b = (a' \wedge b')' = a + b - ab$, where "+" is addition in R. A set $E \subseteq B(R)$ is called a *set of orthogonal idempotents* if $e_1e_2 = 0$ for all distinct $e_1, e_2 \in E$. For a reference on the Boolean algebra of idempotents, see [12].

A Boolean algebra B is *complete* if $\inf E$ exists for every subset $E \subseteq B$. If B is a complete Boolean algebra, then $\sup E = \inf\{b \mid b \in B \text{ and } b \geq e \text{ for all } e \in E\}$. It is well known that every Boolean algebra B is a subalgebra of a complete Boolean algebra $D(B)$, where the infimum of a set in B (when it exists) is the same as its infimum in $D(B)$. Here, $D(B)$ is the "so called" *Dedekind-MacNeille* completion of B [12, c.f. Section 2.4]. Note that $D(B(R)) = B(Q(R))$ for every von Neumann regular ring R [8, Theorem 11.9]. In particular, $B(Q(R))$ is complete. Moreover, $B(R) = B(Q(R))$ whenever $B(R)$ is complete.

In this paper, we continue the investigations of [3] and [11]. We will denote the annihilator class of an element r in R by $[r]_R$, i.e., $[r]_R = \{s \in R \mid \operatorname{ann}_R(s) = \operatorname{ann}_R(r)\}$. As in [4], we define the *zero-divisor graph of* R, $\Gamma(R)$, to be the (undirected) graph with vertices $V(\Gamma(R)) = Z(R) \setminus \{0\}$, such that distinct $v_1, v_2 \in V(\Gamma(R))$ are adjacent if and only if $v_1 v_2 = 0$. It is shown in [3, Theorem 2.2] that $\Gamma(R) \cong \Gamma(T(R))$ for any commutative ring R; the equality $|[r]_R| = |[r]_{T(R)}|$ for all $r \in R$ follows directly from the proof of this theorem (where we have identified R with its canonical image in $T(R)$). Both of these results fail when $T(R)$ is replaced by $Q(R)$ (e.g., Examples 2.10 and 2.11). In Section 2, we give necessary and sufficient conditions for the equality $|[r]_R| = |[r]_{Q(R)}|$ to hold, where R is a von Neumann regular ring such that $B(R)$ is complete and $2 \notin Z(R)$ (see Theorem 2.15). If either $B(R)$ is not complete or $2 \in Z(R)$, then the equality may or may not hold (see Examples 2.11, 2.17, and Corollary 2.16). This result is applied in Section 3 to give sufficient conditions for $\Gamma(R) \cong \Gamma(Q(R))$ to hold when R is a reduced ring. In particular, we provide a characterization of zero-divisor graphs which satisfy $\Gamma(R) \cong \Gamma(Q(R))$, where R is a reduced ring such that $|Z(R)| < \aleph_\omega$ and $2 \notin Z(R)$ (see Theorem 3.3).

2. The Cardinality of $[e]_{Q(R)}$

The investigation in this section involves a set-theoretic treatment of elements in a ring. The main theorems are numbered 2.4, 2.8, 2.15, and 2.16. The results numbered 2.1 through 2.8 develop useful relations within $Q(R)$, and ultimately provide an interpretation of elements in $Q(R)$ as subsets of a set. The results numbered 2.9 through 2.17 provide answers regarding the cardinalities of $[e]_R$ and $[e]_{Q(R)}$.

Throughout this section, R will always be a von Neumann regular ring unless stated otherwise. If $r \in R$, say $r = r^2 s$, then $e_r = rs$ is the *unique* idempotent that satisfies $[r]_R = [e_r]_R$ (c.f. the discussion prior to Theorem 4.1 in [3], or Remark 2.4 of [11]). Moreover, $r = ue_r$ for some unit u of R [9, Corollary 3.3].

The following proposition shows that a nonzero element of a ring of quotients of R will map some idempotent of R into R nontrivially. Recall that $f^{-1}R$ is dense in S whenever f is a nonzero element of a ring of quotients S of R. In particular, there is an $r \in R$ such that $fr \in R \setminus \{0\}$.

Proposition 2.1. *Let R be a von Neumann regular ring. If $R \subseteq S$ is a ring of quotients of R, then for all $0 \neq f \in S$ there exists an $e \in B(R)$ such that $e \leq e_f$ and $0 \neq fe \in R$.*

Proof. Let $0 \neq f \in S$. Choose $r \in R$ such that $0 \neq fr \in R$. There is a unit u of R such that $r = ue_r$, and hence $fe_r = u^{-1}fr \in R \setminus \{0\}$. Let $e = e_f e_r$ (note that it makes sense to talk about e_f since $S \subseteq Q(R)$ and $Q(R)$ is von Neumann regular). Let $s \in Q(R)$ and $t \in R$ be elements such that $f = f^2 s$ and $r = r^2 t$. Then

$$e = e_f e_r = (fs)(rt) = (fr)(st) = e_{fr} \in R.$$

Moreover, $e \leq e_f$ and $fe = fe_r \in R \setminus \{0\}$. □

For any set $A \subseteq R$, let $E_A = \{e_r \in B(R) \mid r \in A\}$. If $e \in B(R)$, then consider the set $\mathcal{R}_e(R) = \{\emptyset \neq A \subseteq R \mid e_{r_1} e_{r_2} = 0$ for all distinct $r_1, r_2 \in A$, and $\sup E_A = e\}$. Note that $\mathcal{R}_e(R) \neq \emptyset$ since $\{e\} \in \mathcal{R}_e(R)$. Also, if $\sup E_A = e$ and $0 \neq e' \in B(R)$ with $e' \leq e$, then there exists an $e'' \in E_A$ such that $e'e'' \neq 0$. Otherwise, $e'' \leq 1 - e'$ for all $e'' \in E_A$, and thus $e = \sup E_A \leq 1 - e'$. But this implies that $e'e = 0$, a contradiction. This fact is generalized in (1) of the following proposition.

Proposition 2.2. *Suppose that $E \subseteq B(R)$ is a set of orthogonal idempotents in a von Neumann regular ring R.*

(1) *Let $e' \in B(R)$. Then $e' \sup E = 0$ if and only if $E \cup \{e'\}$ is a set of orthogonal idempotents. In particular, $r \sup E = 0$ if and only if $re' = 0$ for all $e' \in E$ ($r \in R$).*
(2) *Suppose that E is finite; say $E = \{e_1, ..., e_n\}$. Then $\sup E = \sum_{j=1}^{n} e_j$.*
(3) *Let $e' \in B(R)$. If $f \in Q(R)$ such that $e' \leq e_f$, then $fe' \in [e']_{Q(R)}$.*
(4) *Let $e', e \in B(R)$ such that $e' \leq e$ and $2e' \in [e']_R$. Then $e' + e \in [e]_R$.*

Proof. Note that $\sup E \in B(Q(R))$.

(1) If $e'e'' \neq 0$ for some $e'' \in E$, then $e'e'' \leq e'' \leq \sup E$ implies that $e'e'' \sup E = e'e'' \neq 0$; in particular, $e' \sup E \neq 0$. Conversely, suppose that $e' \sup E \neq 0$. Since $e' \sup E \leq \sup E$, the above comments show there exists an $e'' \in E$ such that

$(e' \sup E)e'' \neq 0$; in particular, $e'e'' \neq 0$. Thus $E \cup \{e'\}$ is *not* a set of orthogonal idempotents.

The "in particular" statement holds since $[r]_R = [e_r]_R$ for all $r \in R$.

(2) It is easy to check that $e = \sum_{j=1}^{n} e_j \in B(R)$. Also, $e_j e = e_j$ for all $j \in \{1, ..., n\}$. Hence $\sup E \leq e$. But $e_j \leq \sup E$ for all $j \in \{1, ..., n\}$, and thus $e \sup E = e$; that is, $e \leq \sup E$. Therefore, $e = \sup E$.

(3) Clearly $\mathrm{ann}_{Q(R)}(e') \subseteq \mathrm{ann}_{Q(R)}(fe')$. Let $a \in \mathrm{ann}_{Q(R)}(fe')$. Then $ae' \in \mathrm{ann}_{Q(R)}(f) = \mathrm{ann}_{Q(R)}(e_f)$. Thus $0 = ae'e_f = ae'$; that is, $a \in \mathrm{ann}_{Q(R)}(e')$. Hence $\mathrm{ann}_{Q(R)}(e') = \mathrm{ann}_{Q(R)}(fe')$, i.e., $fe' \in [e']_{Q(R)}$.

(4) If $r \in \mathrm{ann}_R(e)$, then $re = 0$ and $re' = ree' = 0$. Hence $r \in \mathrm{ann}_R(e' + e)$, and therefore $\mathrm{ann}_R(e) \subseteq \mathrm{ann}_R(e'+e)$. To show the reverse inclusion, let $r \in \mathrm{ann}_R(e'+e)$. Note that $0 = re'(e' + e) = r(2e')$. Then $2e' \in [e']_R$ implies that $re' = 0$, and therefore $re = re' + re = r(e' + e) = 0$. Hence $\mathrm{ann}_R(e' + e) \subseteq \mathrm{ann}_R(e)$. $\square$

In order to investigate cardinality, we shall translate the elements of an equivalence class $[e]_{Q(R)}$ into sets of elements of $\mathcal{R}_e(R)$. Such a correspondence is given in Theorem 2.4, and is motivated by the following example.

Example 2.3. *Let F be an infinite field and J an infinite indexing set. Let $F_j = F$ for all $j \in J$. Define $R = \{(r_j) \in \prod_{j \in J} F_j \mid \{r_j\}_{j \in J} \subseteq \{s_1, ..., s_n\}$ for some $\{s_1, ..., s_n\} \subseteq F$, for some $n \in \mathbb{N}\}$ (c.f.* [11, Example 3.5]*). Note that R is von Neumann regular. Let D be the dense ideal of R generated by the minimal nonzero idempotents of R (that is, the elements with a 1 in precisely one coordinate and 0 elsewhere). Then D is contained in $f^{-1}R$ for all $f \in \prod_{j \in J} F_j$. Thus $\prod_{j \in J} F_j$ is a ring of quotients of R. Moreover, $\prod_{j \in J} F_j$ is rationally complete* [12, Proposition 2.3.8]*. Therefore, $Q(R) = \prod_{j \in J} F_j$.*

Consider R from Example 2.3. Suppose that $F = \mathbb{Q}$, $J = \mathbb{N}$, and let e be the multiplicative identity of R (the largest element of $B(R)$). Note that there is a correspondence between $\mathcal{R}_e(R)$ and $[e]_{Q(R)}$, which is defined by taking the "sum" of the elements of a set in $\mathcal{R}_e(R)$. For example, the set

$$\{(1, 0, 0, ...), (0, 2, 0, ...), (0, 0, 3, ...), ...\} \in \mathcal{R}_e(R)$$

corresponds to the element $(1, 2, 3, ...) \in Q(R)$. This correspondence is generalized in the following theorem.

Theorem 2.4. *Let R be a von Neumann regular ring and suppose that $e \in B(R)$. The mapping $\sigma_e : \mathcal{R}_e(R) \to [e]_{Q(R)}$ defined by*

$$\sigma_e(A) = f \text{ if and only if } f \in [e]_{Q(R)} \text{ with } fe_r = r \text{ for all } r \in A$$

is a well-defined function. Moreover, $\sigma_e(A) \in R$ *if and only if* $\sigma_e(A) = \sigma_e(A')$ *for some* $A' \in \mathcal{R}_e(R)$ *with* $|A'| < \infty$*.*

Proof. Fix $e \in B(R)$. To show that σ_e is well-defined, we first show that every element of $\mathcal{R}_e(R)$ corresponds to some element in $[e]_{Q(R)}$. Let $A \in \mathcal{R}_e(R)$. Note that $D = (1-e, E_A)$ is a dense ideal of R: Any element $r \in R\setminus\{0\}$ that annihilates $1-e$ satisfies $re = r \neq 0$, and therefore does not annihilate all of E_A by Proposition 2.2 (1). Define $\varphi \in \mathrm{Hom}_R(D, R)$ by

$$\varphi\big(t(1-e) + \sum_{e_r \in E_A} t_r e_r\big) = \sum_{e_r \in E_A} t_r r.$$

(Indeed, φ is well-defined since multiplication by the appropriate idempotent will show that equal elements of D have equal "like terms," and clearly $t_r e_r = t'_r e_r$ implies that $t_r r = t'_r r$.) Then $\varphi(1-e) = 0$ and $\varphi(e_r) = r$ for all $r \in A$. Therefore, there exists an element $f \in Q(R)$ such that $f(1-e) = 0$ and $fe_r = r$ for all $r \in A$. It follows that $e_f \leq e$ (in $B(Q(R))$). To prove the reverse inequality, let $r \in A$. Then

$$r = fe_r = e_f f e_r = e_f r.$$

Thus $r(1-e_f) = 0$, which implies that $e_r \leq e_f$. Hence $e = \sup E_A \leq e_f$, and therefore $e = e_f$. This shows that $f \in [e]_{Q(R)}$, and therefore $\sigma_e(A) = f$. It remains to show that σ_e is single-valued. Suppose that A maps to both f and g. Then $(f-g)$ annihilates D. But D is dense in $Q(R)$, and thus $f - g = 0$, i.e., $f = g$. Therefore, σ_e is well-defined.

To see that the "moreover" statement is true, suppose that $\sigma_e(A) \in R$. Then $\sigma_e(A) = \sigma_e(A')$, where $A' = \{\sigma_e(A)\}$. Conversely, suppose that $A' \in \mathcal{R}_e(R)$ with $|A'| < \infty$; say $A' = \{r_1, ..., r_n\}$. Then

$$\sigma_e(A') = \sigma_e(A')e = \sigma_e(A')(\sum_{j=1}^{n} e_{r_j}) = \sum_{j=1}^{n}(\sigma_e(A')e_{r_j}) = \sum_{j=1}^{n} r_j \in R,$$

where the second equality follows from Proposition 2.2 (2) (c.f. the last paragraph prior to the statement of this theorem). □

By the last part of the previous proof, we have

Corollary 2.5. *Let* R *be a von Neumann regular ring. If* $A \in \mathcal{R}_e(R)$ *is a finite set, then* $\sigma_e(A) = \sum_{r\in A} r \in [e]_R$*.*

Let $e \in B(R)$ and define the set

$$\mathcal{E}_e(R) = \{E_A \mid A \in \mathcal{R}_e(R)\}.$$

We shall write $f \prec E$ whenever $E \in \mathcal{E}_e(R)$ and $\sigma_e(A) = f$ for some $A \in \mathcal{R}_e(R)$ with $E_A = E$. By Proposition 2.2 (3), this is equivalent to declaring $f \prec E$ if and only if E is a set of orthogonal idempotents such that $\sup E = e_f = e$ and $fe' \in R$ for all $e' \in E$ (indeed, let $A = \{fe'\}_{e' \in E}$, c.f. the second paragraph in the proof of Theorem 2.8). In particular, if $r \in [e]_R$, then $r \prec E$ for all $E \in \mathcal{E}_e(R)$, i.e., $\{r \in [e]_R \mid r \prec E\} = [e]_R$ for all $E \in \mathcal{E}_e(R)$.

Corollary 2.6. *Let R be a von Neumann regular ring and suppose that $e \in B(R)$. If $E \in \mathcal{E}_e(R)$, then*

$$|\{A \in \mathcal{R}_e(R) \mid E_A = E\}| = |\{f \in [e]_{Q(R)} \mid f \prec E\}|.$$

Proof. The mapping $\{A \in \mathcal{R}_e(R) \mid E_A = E\} \to \{f \in [e]_{Q(R)} \mid f \prec E\}$ defined by $A \mapsto \sigma_e(A)$ is a well-defined surjection by Theorem 2.4 and the definition of $\prec$. It is injective since if $A_1, A_2 \in \{A \in \mathcal{R}_e(R) \mid E_A = E\}$ with $\sigma_e(A_1) = \sigma_e(A_2)$, then

$$A_1 = \{\sigma_e(A_1)e'\}_{e' \in E} = \{\sigma_e(A_2)e'\}_{e' \in E} = A_2.$$

Therefore,

$$|\{A \in \mathcal{R}_e(R) \mid E_A = E\}| = |\{f \in [e]_{Q(R)} \mid f \prec E\}|.$$

□

Suppose that R is a reduced ring. Then the mapping $\operatorname{ann}_{Q(R)}(J) \mapsto \operatorname{ann}_R(J \cap R)$ ($J \subseteq Q(R)$) is a well-defined bijection of $\operatorname{Ann}(Q(R))$ onto $\operatorname{Ann}(R)$, where $\operatorname{Ann}(R) = \{\operatorname{ann}_R(J) \mid J \subseteq R\}$ [12, Proposition 2.4.3]; in particular, $[r]_R \subseteq [r]_{Q(R)}$ for all $r \in R$. Alternatively, suppose that R is a von Neumann regular ring. Then $[e]_R = \{r \in [e]_R \mid r \prec E\}$ for all $E \in \mathcal{E}_e(R)$. Since $\sigma_e(\mathcal{R}_e(R)) \subseteq [e]_{Q(R)}$, we have

Proposition 2.7. *Let R be a von Neumann regular ring and suppose that $e \in B(R)$. Then $[e]_R \subseteq \{f \in [e]_{Q(R)} \mid f \prec E\}$ for all $E \in \mathcal{E}_e(R)$. In particular, $[e]_R \subseteq [e]_{Q(R)}$.*

Of course, the "in particular" statement of the above proposition can be justified by the simpler argument that $r = ue$ for some unit u of R (and hence of $Q(R)$), for all $r \in [e]_R$. However, we will apply the first part of the proposition in the proof of Lemma 2.12.

Note that Theorem 2.4 implies that *some* of the elements of $[e]_{Q(R)}$ correspond to elements of $\mathcal{R}_e(R)$. The next theorem shows that *every* element in $[e]_{Q(R)}$ is of this type.

Theorem 2.8. *Let R be a von Neumann regular ring. Suppose that $e \in B(R)$. Then σ_e is surjective. In particular, $|[e]_{Q(R)}| \leq |\mathcal{R}_e(R)|$.*

Proof. Fix $e \in B(R)$. The result is trivial for the case $e = 0$. Suppose that $e \neq 0$. To show that σ_e is onto, choose any $f \in [e]_{Q(R)}$. Let $\mathcal{C} = \{\emptyset \neq E \subseteq B(R) \mid e'e'' = 0$ for all distinct $e', e'' \in E$, $e' \leq e$ for all $e' \in E$, and $fe' \in R$ for all $e' \in E\}$. Note that $\mathcal{C} \neq \emptyset$ since $\{0\} \in \mathcal{C}$. Let $\mathcal{C}$ be partially ordered by inclusion; then an application of Zorn's lemma shows that $\mathcal{C}$ has a maximal element, call it E. We will show that $\sup E = e$. If not, then consider $0 \neq e' = e - \sup E \in B(Q(R))$. Note that $fe' \in [e']_{Q(R)}$ by Proposition 2.2 (3). Hence Proposition 2.1 implies that there exists an $e'' \in B(R)$ such that $e'' \leq e'$ and $fe'' = fe'e'' \in R \setminus \{0\}$. Also, $e' \leq e$ implies $e'' \leq e$, and thus

$$e'' \sup E = e''(e - e') = e'' - e'' = 0.$$

But then $E \cup \{e''\} \in \mathcal{C}$ by Proposition 2.2 (1), contradicting the maximality of E. Therefore, $\sup E = e$.

Let $A = \{fe' \mid e' \in E\}$. Then Proposition 2.2 (3) implies $E_A = E$, and thus $A \in \mathcal{R}_e(R)$. Also, $e_{fe'} = e'$ implies that $fe_{fe'} = fe'$ for all $fe' \in A$. Hence $\sigma_e(A) = f$.

The "in particular" statement is clear. □

We now turn our attention to the cardinality of $[e]_R$. The previous theorem allows one to derive information about the cardinality of $[e]_{Q(R)}$ from the set $\mathcal{R}_e(R)$. We will be able to relate the cardinalities of $[e]_{Q(R)}$ and $[e]_R$ if we can find a way to use the set $\mathcal{R}_e(R)$ to reveal information about $|[e]_R|$. The next three lemmas accomplish this by considering elements of the subset $\mathcal{E}_e(R)$ of $\mathcal{R}_e(R)$.

Lemma 2.9. *Let R be a von Neumann regular ring. Suppose that $E \subseteq B(R) \setminus \{0\}$ is a set of orthogonal idempotents with $\sup E = e$. Moreover, assume that $B(R)$ is complete and $2e' \in [e']_R$ for all $e' \in E$. Then $|[e]_R| \geq 2^{|E|}$.*

Proof. Define the mapping $\rho : \mathcal{P}(E) \to [e]_R$ by

$$\rho(E') = \sup E' + e,$$

where $\mathcal{P}(E)$ is the "power set" of E. Let $E' \subseteq E$. It is clear that $\operatorname{ann}_R(\sup E') \subseteq \operatorname{ann}_R(2 \sup E')$. Conversely, let $r \in \operatorname{ann}_R(2 \sup E')$. Then $2r \in \operatorname{ann}_R(\sup E')$, and hence $2re' = 0$ for all $e' \in E'$ by Proposition 2.2 (1). Thus $r \in \operatorname{ann}_R(2e') = \operatorname{ann}_R(e')$ for all $e' \in E'$, and therefore Proposition 2.2 (1) implies that $r \in \operatorname{ann}_R(\sup E')$. This shows that $\operatorname{ann}_R(\sup E') = \operatorname{ann}_R(2 \sup E')$, i.e., $2 \sup E' \in [\sup E']_R$. Hence ρ is well-defined by Proposition 2.2 (4). To show that ρ is injective, suppose that $E_1, E_2 \subseteq E$ with $E_1 \neq E_2$; say $0 \neq e' \in E_1 \setminus E_2$. Then

$$e' \sup E_1 = e' \neq 0 = e' \sup E_2,$$

where the last equality holds by Proposition 2.2 (1). It follows that $\sup E_1 \neq \sup E_2$. Thus $E_1 \neq E_2$ implies that $\rho(E_1) \neq \rho(E_2)$. Therefore, ρ is injective, and hence

$$|[e]_R| \geq |\mathcal{P}(E)| = 2^{|E|}.$$

□

For the remainder of this section, it will be necessary to recall some facts from set theory. In what follows, we will assume the generalized continuum hypothesis. Given any cardinal $\mathfrak{m}$, let $\mathrm{cf}(\mathfrak{m})$ denote the cofinality of $\mathfrak{m}$. Note that $\mathrm{cf}(\mathfrak{m}) \leq \mathfrak{m}$, and $\mathrm{cf}(\mathfrak{m})$ is infinite whenever $\mathfrak{m}$ is infinite (e.g., see [15, Theorem 21.10]). An infinite cardinal $\mathfrak{m}$ is called *regular* if $\mathfrak{m} = \mathrm{cf}(\mathfrak{m})$. If $\mathfrak{m}$ is not regular, then it is called *singular*. Note that every successor cardinal is regular. Recall that $\mathfrak{m}^{\mathfrak{m}'}$ is defined to be the cardinal number $|A^B|$, where A and B are sets of cardinality $\mathfrak{m}$ and $\mathfrak{m}'$, respectively, and A^B is the set of all functions from B into A. If $\aleph_\alpha$ and $\aleph_\beta$ are infinite cardinals, then

$$\aleph_\alpha^{\aleph_\beta} = \begin{cases} \aleph_\alpha, & \aleph_\beta < \mathrm{cf}(\aleph_\alpha) \\ \aleph_{\alpha+1}, & \mathrm{cf}(\aleph_\alpha) \leq \aleph_\beta \leq \aleph_\alpha \\ \aleph_{\beta+1}, & \aleph_\alpha < \aleph_\beta \end{cases}$$

[15, Theorem 23.9]. Also, $m^{\aleph_\beta} = \aleph_{\beta+1}$ for every $2 \leq m < \infty$ [15, Theorem 22.13]. The notation $\sum_{i\in I} \mathfrak{m}_i$ is used to express the cardinality of the disjoint union $\coprod_{i\in I} A_i$, where $|A_i| = \mathfrak{m}_i$ for each $i \in I$. If I is an infinite indexing set with $\mathfrak{m}_i$ infinite for some $i \in I$, then $\sum_{i\in I} \mathfrak{m}_i = |I| \sup_{i\in I} \mathfrak{m}_i$. A detailed exposition of cardinal numbers can be found in chapter four of [15].

It is our goal to find conditions that ensure the equality $|[e]_{Q(R)}| = |[e]_R|$. We will see that it suffices to impose restrictions on the elements of the set $\mathcal{E}_e(R)$. The next two examples motivate such restrictions.

Example 2.10. *Let F be a field such that $|F| = \aleph_\omega$ and set $J = \mathbb{N}$. Suppose that R is the ring in Example 2.3. Choose an infinite subset I of $\mathbb{N}$, and let e be the idempotent with 1 in all coordinates $i \in I$ and 0 elsewhere. Then*

$$|[e]_R| = \aleph_\omega < \aleph_{\omega+1} = \aleph_\omega^{\aleph_0} = |[e]_{Q(R)}|,$$

where the second equality holds since cf($\aleph_\omega$) $= \aleph_0$ [15, Theorem 22.11].

Example 2.11. *Let $K = \mathbb{Z}_2(X)$, and define the ring $R = \prod_{\mathbb{N}} \mathbb{Z}_2 + \bigoplus_{\mathbb{N}} K$. As in the Example 2.3, we have $Q(R) = \prod_{\mathbb{N}} K$. Choose an infinite subset I of $\mathbb{N}$, and let e be the idempotent with 1 in all coordinates $i \in I$ and 0 elsewhere. Then $|[e]_R| = \aleph_0 < \aleph_1 = |[e]_{Q(R)}|$.*

In Example 2.10, we found an element $e \in B(R)$ with an infinite set $E \in \mathcal{E}_e(R)$ such that $\mathrm{cf}(|[e']_R|) \leq |E| < |[e']_R|$ for some $e' \in E$ (namely, E was the set of minimal nonzero idempotents less than e, and e' could have been any element of E). In Example 2.11, we found an element $e \in B(R)$ with a set $E \in \mathcal{E}_e(R)$ such that $2e' \notin [e']_R$ for some $e' \in E$ (as before, E was the set of minimal nonzero idempotents less than e, and e' could have been any element of E). As a result, Lemma 2.9 fails for the element e. When R is a von Neumann regular ring such that $B(R)$ is complete, the desired equality will necessarily be obtained in the absence of such scenarios.

We shall say that an element $E \in \mathcal{E}_e(R)$ is *regular* if the relation $|E| < \sup\{|[e']_R| \mid e' \in E\}$ implies that either $\sup\{|[e']_R| \mid e' \in E\}$ is finite or $|E| < \mathrm{cf}(\sup\{|[e']_R| \mid e' \in E\})$. As a special case, $E \in \mathcal{E}_e(R)$ is regular if $|E| < \sup\{|[e']_R| \mid e' \in E\}$ implies that $\sup\{|[e']_R| \mid e' \in E\}$ is either finite or a regular cardinal. Clearly E is regular if it is finite.

Lemma 2.12. *Let R be a von Neumann regular ring, $e \in B(R)$, and $E \in \mathcal{E}_e(R)$. Assume that $B(R)$ is complete and $2e' \in [e']_R$ for all $e' \in E$. If $E \in \mathcal{E}_e(R)$ is regular, then $|[e]_R| = |\{f \in [e]_{Q(R)} \mid f \prec E\}|$.*

Proof. If E is finite, then $\{f \in [e]_{Q(R)} \mid f \prec E\} \subseteq [e]_R$ by Theorem 2.4. The reverse inclusion holds by Proposition 2.7, and hence the result follows. Suppose that E is infinite; say $|E| = \aleph_\alpha$ for some ordinal α. Let $\sup\{|[e']_R| \mid e' \in E\}| = \mathfrak{m}$. Define

$$F : \{A \in \mathcal{R}_e(R) \mid E_A = E\} \to \big(\cup\{[e']_R \mid e' \in E\}\big)^E$$

by the rule

$$F(A)(e) = r \text{ if and only if } e \in E \text{ with } e = e_r \text{ for some } r \in A.$$

Note that if $r_1, r_2 \in A$ with $r_1 \neq r_2$, then $e_{r_1} e_{r_2} = 0$. In particular, $r_1 \neq r_2$ implies that $e_{r_1} \neq e_{r_2}$, and therefore F is well-defined by definition. Also, F is injective since if $F(A_1) = F(A_2)$, then $r = F(A_1)(e_r) \in A_1$ for all $r \in A_2$, and similarly we have $A_1 \subseteq A_2$ so that $A_1 = A_2$. Hence

$$|\{A \in \mathcal{R}_e(R) \mid E_A = E\}| \leq |\big(\cup\{[e']_R \mid e' \in E\}\big)^E| = (\aleph_\alpha \mathfrak{m})^{\aleph_\alpha},$$

where the equality holds since the union is disjoint. Therefore,

$$|\{f \in [e]_{Q(R)} \mid f \prec E\}| \leq (\aleph_\alpha \mathfrak{m})^{\aleph_\alpha} = \left\{ \begin{array}{cc} \mathfrak{m}, & \mathfrak{m} > \aleph_\alpha \\ \aleph_{\alpha+1}, & \mathfrak{m} \leq \aleph_\alpha \end{array} \right.,$$

where the inequality follows by Corollary 2.6, and the equality follows since E is regular.

Let $e' \in E$ and $r \in [e']_R$. Then $e_r = e'$ and $e - e' \in B(R)$. Using Proposition 2.2 (2), it is easy to check that $\{r, e - e'\} \in \mathcal{R}_e(R)$, and thus Corollary 2.5 implies that $r + (e - e') \in [e]_R$. This shows that the mapping $[e']_R \to [e]_R$ given by $r \mapsto r + (e - e')$ is well-defined. Clearly it is also injective. Hence $|[e']_R| \leq |[e]_R|$ for all $e' \in E$, and therefore $\mathfrak{m} \leq |[e]_R|$. Also, $|E \setminus \{0\}| = \aleph_\alpha$, and thus

$$|[e]_R| \geq 2^{\aleph_\alpha} = \aleph_{\alpha+1}$$

by Lemma 2.9. Therefore, we have $|[e]_R| = |\{f \in [e]_{Q(R)} \mid f \prec E\}|$ since Proposition 2.7 implies that the reverse inequality always holds. $\square$

Remark 2.13. *Although the following arguments generalize to arbitrary Boolean algebras, we shall assume that B is the Boolean algebra of idempotents of a commutative ring. Suppose that B is complete, and let $b \in B$. Then $B|_b = \{e \in B \mid e \leq b\}$ is a complete Boolean algebra, where the partial order on $B|_b$ is inherited from B. Let $s(b)$ denote the least cardinal such that there is no set $E \subseteq B|_b$ of orthogonal idempotents with $|E| = s(b)$. Suppose that B is infinite. In* [1, Corollary 2.7]*, it is shown that there exists a finite set of orthogonal idempotents $\{b_1, ..., b_n\} \subseteq B$ with $\sup\{b_1, ..., b_n\} = 1$, such that $|B|_{b_i}| = \sum_{\mathfrak{m} < s(b_i)} |B|_{b_i}|^{\mathfrak{m}}$ for each $i = 1, ..., n$. (In* [1]*, this result is stated in the context of compact extremely disconnected topological spaces.) We will show that this implies $|\mathcal{E}_e(R)| \leq |B(R)|_e|$ whenever e is an element of a complete Boolean algebra $B(R)$ such that $|B(R)|_e|$ is infinite.*

Suppose that B is complete and infinite. Let $\mathcal{E} = \{E \subseteq B \mid e_1 e_2 = 0$ for all distinct $e_1, e_2 \in E$ and $\sup E = 1\}$. It suffices to show that $|\mathcal{E}| \leq |B|$. Note that the number of subsets of cardinality less than $\mathfrak{n}$ of a set J is at most $\sum_{\mathfrak{m} < \mathfrak{n}} |J|^{\mathfrak{m}}$. Using [1, Corollary 2.7]*, choose a set of orthogonal elements $\{b_1, ..., b_n\} \subseteq B$ such that $\sup\{b_1, ..., b_n\} = 1$, and*

$$|B|_{b_i}| = \sum_{\mathfrak{m} < \mathfrak{m}_i} |B|_{b_i}|^{\mathfrak{m}}$$

for each $i \in \{1, ..., n\}$, where $\mathfrak{m}_i$ is the least cardinal such that there is no set $E \subseteq B|_{b_i}$ of orthogonal elements with $|E| = \mathfrak{m}_i$. By the choice of $\mathfrak{m}_i$ together with the fourth sentence of this paragraph, we have $|\mathcal{E}_{b_i}| \leq \sum_{\mathfrak{m} < \mathfrak{m}_i} |B|_{b_i}|^{\mathfrak{m}} = |B|_{b_i}|$, where $\mathcal{E}_{b_i} = \{E \subseteq B|_{b_i} \mid e_1 e_2 = 0$ for all distinct $e_1, e_2 \in E$, and $\sup E = b_i\}$. Let $\mathfrak{m}_j = \max_{1 \leq i \leq n}\{\mathfrak{m}_i\}$. Note that $\mathfrak{m}_j$ is infinite (and hence so is $B|_{b_j}$) since B is infinite (this is an application of König's Lemma, e.g., see [10, Exercise 25.12]*). Let $\mathcal{E}^* = \{E \in \mathcal{E} \mid e \leq b_i$ for some $i \in \{1, ..., n\}$ for all $e \in E\}$. Noting that $\{b_1, ..., b_n\} \subseteq B$ is a set of orthogonal idempotents such that $\sup\{b_1, ..., b_n\} = 1$, we see that $E \in \mathcal{E}^*$ if and only if $E = \cup_{i=1}^n E_i$ for some $E_i \in \mathcal{E}_{b_i}$ (namely, $E_i =$*

$B|_{b_i} \cap E$). Then $|\mathcal{E}^*| \leq \Pi_{i=1}^n |\mathcal{E}_{b_i}| \leq \max_{1\leq i\leq n}(|B|_{b_i}|)$, where the second inequality follows since $|\mathcal{E}_{b_i}| \leq |B|_{b_i}|$ for each $i \in \{1, ..., n\}$.

The mapping $\psi : \mathcal{E} \to \mathcal{E}^*$ given by $E \mapsto \{bb_i \mid b \in E$ and $i \in \{1, ..., n\}\}$ is well-defined. Let $E^* \in \mathcal{E}^*$; say $E^* = \cup_{i=1}^n E_i$, ($E_i \in \mathcal{E}_{b_i}$). Note that every element belonging to a member of $\psi^{-1}(\{E^*\})$ is a sum *(that is, supremum), $b = bb_1 + \cdots + bb_n$, with $bb_i \in E_i \cup \{0\}$ for each $i \in 1, ..., n$. Then an element $b \in B$ belongs to a member of $\psi^{-1}(\{E^*\})$ if and only if $b = e_1 + \cdots + e_n$ for some $e_i \in E_i \cup \{0\}$. This shows that the mapping*

$$(E_1 \cup \{0\}) \times \cdots \times (E_n \cup \{0\}) \to \cup\{E \mid E \in \psi^{-1}(\{E^*\})\}$$

given by the rule $(e_1, ..., e_n) \mapsto \sum_{i=1}^n e_i$ is a well-defined surjection. Since the elements of E^* are orthogonal, this mapping is also injective. Hence,

$$|\cup\{E \mid E \in \psi^{-1}(\{E^*\})\}| = \Pi_{i=1}^n |E_i \cup \{0\}| < \mathfrak{m}_j \leq |B|_{b_j}|,$$

where the last inequality holds since a complete Boolean algebra is always strictly larger than any of its sets of orthogonal elements. (Indeed, if $E \subseteq B$ is a set of nonzero orthogonal elements, then the mapping $E' \mapsto \sup E'$ defines an injection from the power set of E into B. In particular, $|E| < |B|$ for any set $E \subseteq B$ of orthogonal elements.) Also, if $E' \in \psi^{-1}(\{E^*\})$, then

$$|E'| \leq |\cup\{E \mid E \in \psi^{-1}(\{E^*\})\}| < \mathfrak{m}_j.$$

Since $|\cup\{E \mid E \in \psi^{-1}(\{E^*\})\}| < |B|_{b_j}|$, it follows that the number of subsets of cardinality less than $\mathfrak{m}_j$ of $\cup\{E \mid E \in \psi^{-1}(\{E^*\})\}$ is at most $\sum_{\mathfrak{m}<\mathfrak{m}_j} |B|_{b_j}|^{\mathfrak{m}}$. But it has been shown that every member of $\psi^{-1}(\{E^*\})$ has cardinality strictly less than $\mathfrak{m}_j$, and thus

$$|\psi^{-1}(\{E^*\})| \leq \sum_{\mathfrak{m}<\mathfrak{m}_j} |B|_{b_j}|^{\mathfrak{m}} = |B|_{b_j}|.$$

Therefore,

$$\begin{aligned}
|\mathcal{E}| &= |\cup_{E^*\in\mathcal{E}^*} \psi^{-1}(\{E^*\})| \\
&= \sum_{E^*\in\mathcal{E}^*} |\psi^{-1}(\{E^*\})| \\
&= |\mathcal{E}^*| \sup\{|\psi^{-1}(\{E^*\})| \mid E^* \in \mathcal{E}^*\} \\
&\leq \big(\max_{1\leq i\leq n}(|B|_{b_i}|)\big)|B|_{b_j}| \\
&= \max_{1\leq i\leq n}(|B|_{b_i}|) \\
&\leq |B|.
\end{aligned}$$

If $2e' \in [e']_R$ for all $e' \leq e$, then Proposition 2.2 (4) implies that the mapping $B(R)|_e \to [e]_R$ given by $e' \mapsto e' + e$ is well-defined. It is clearly injective, and thus the following lemma holds by the above remark.

Lemma 2.14. *Let R be a von Neumann regular ring. Suppose that $B(R)$ is complete and choose $e \in B(R)$. Assume that $2e' \in [e']_R$ for all $e' \leq e$. If $|B(R)|_e|$ is infinite, then $|\mathcal{E}_e(R)| \leq |[e]_R|$.*

Note that Lemma 2.14 can fail if $B(R)|_e$ is finite. For example, let $R = \prod_{i=1}^{5} \mathbb{Z}_3$ and let $e = (1, 1, 1, 1, 1)$. Then $\mathcal{E}_e(R) = 52$ (the fifth *Bell number*), but $|[e]_R| = 32$.

Given any element e of the complete Boolean algebra $B(R)$, we will say that a cardinal $\mathfrak{m}$ is *achieved by regular elements of* $\mathcal{E}_e(R)$ if there exists a set of regular elements $\{E_i\}_{i\in I} \subseteq \mathcal{E}_e(R)$ with $| \cup_{i\in I} \{f \in [e]_{Q(R)} \mid f \prec E_i\}| = \mathfrak{m}$. Let R be the ring in Example 2.10. Note that the regular elements of $\mathcal{E}_e(R)$ are precisely the finite elements. Letting $\{E_i\}_{i\in I}$ denote the family of all regular elements of $\mathcal{E}_e(R)$, we have $\cup_{i\in I}\{f \in [e]_{Q(R)} \mid f \prec E_i\} = [e]_R$, and hence $|[e]_{Q(R)}|$ is *not* achieved by regular elements.

We now state and prove the main theorem of this section.

Theorem 2.15. *Suppose that R is a von Neumann regular ring such that $B(R)$ is complete. Let $e \in B(R)$ be an element such that $2e' \in [e']_R$ for all $e' \leq e$. Then $|[e]_{Q(R)}| = |[e]_R|$ if and only if $|[e]_{Q(R)}|$ is achieved by regular elements of $\mathcal{E}_e(R)$.*

Proof. The necessity is clear since $|[e]_{Q(R)}| = |[e]_R|$ implies that $|[e]_{Q(R)}|$ is achieved by the regular element $E = \{e\}$ (indeed, $[e]_R \subseteq [e]_{Q(R)}$ and $r \prec \{e\}$ for all $r \in [e]_R$).

To prove the converse, note that if $|E| < \infty$ for all $E \in \mathcal{E}_e(R)$, then $[e]_{Q(R)} = [e]_R$ by Theorems 2.4 and 2.8. In particular, $|[e]_{Q(R)}| = |[e]_R|$.

Suppose that $\mathcal{E}_e(R)$ contains an infinite element. Then $|[e]_R|$ is infinite by Lemma 2.9. Suppose that I is an indexing set such that $\{E_i\}_{i\in I} \subseteq \mathcal{E}_e(R)$ is a family of regular elements with $| \cup_{i\in I} \{f \in [e]_{Q(R)} \mid f \prec E_i\}| = |[e]_{Q(R)}|$. Then

$$\begin{aligned}
|[e]_{Q(R)}| &= | \cup_{i\in I} \{f \in [e]_{Q(R)} \mid f \prec E_i\}| \\
&\leq |I| \sup_{i\in I} |\{f \in [e]_{Q(R)} \mid f \prec E_i\}| \\
&= |I||[e]_R| \\
&\leq |\mathcal{E}_e(R)||[e]_R| \\
&= |[e]_R|,
\end{aligned}$$

where the second equality follows by Lemma 2.12 and the last equality follows by Lemma 2.14. Thus $|[e]_{Q(R)}| = |[e]_R|$ since Proposition 2.7 implies that the reverse inequality is always true. □

It is known that $Q(\prod_{i\in I} R_i) = \prod_{i\in I} Q(R_i)$ for any family of rings $\{R_i\}_{i\in I}$ [12, Proposition 2.3.8]. It is easy to see that $|[(e_i)]_{\prod_{i\in I} R_i}| = \prod_{i\in I} |[e_i]_{R_i}|$ for any such product. Therefore, if $|[e_i]_{R_i}| = |[e_i]_{Q(R_i)}|$ for all $i \in I$, then $|[(e_i)]_{\prod_{i\in I} R_i}| = |[(e_i)]_{Q(\prod_{i\in I} R_i)}|$.

Note that a ring may have $|[e]_R| = |[e]_{Q(R)}|$ without satisfying the condition "$2e' \in [e]_R$ for all $e' \leq e$." For example, the equality is automatic whenever $R = Q(R)$. The following application of the previous theorem shows that a ring $R \neq Q(R)$ can have an idempotent e such that $2e \notin [e]_R$, and yet $|[e]_R| = |[e]_{Q(R)}|$. Moreover, this equality can hold even if $B(R)$ is not complete.

Recall that a ring R is *Boolean* if $x = x^2$ for all $x \in R$, i.e., $R = B(R)$ (as sets). In particular, a Boolean ring is von Neumann regular, and has characteristic 2. Moreover, a ring R is Boolean if and only if $Q(R)$ is Boolean [12, Lemma 2.4.4].

Corollary 2.16. *Suppose that I is an indexing set, $\{F_i\}_{i\in I}$ is a family of rationally complete rings, A is a von Neumann regular ring with $B(A)$ complete, $|A| < \aleph_\omega$, $2 \notin Z(A)$, and B is a Boolean ring. Let $\mathcal{S}$ be a nonempty subset of $\{A, B, \prod_{i\in I} F_i\}$. If $R \cong \prod_{S\in\mathcal{S}} S$, then $|[e]_R| = |[e]_{Q(R)}|$ for all $e \in B(R)$.*

Proof. We might as well assume that $R = \prod_{S\in\mathcal{S}} S$. By the above comments, it suffices to show that the result is true when $\mathcal{S}$ is a singleton set. Clearly it is true when $\mathcal{S} = \{\prod_{i\in I} F_i\}$ since $\prod_{i\in I} F_i = \prod_{i\in I} Q(F_i) = Q\big(\prod_{i\in I} F_i\big)$. To see that it holds for $\mathcal{S} = \{B\}$, recall that each equivalence class of a von Neumann regular ring is represented by a *unique* idempotent. Thus, since $Q(B)$ is Boolean, $|[e]_B| = 1 = |[e]_{Q(B)}|$ for all $e \in B$.

It remains to show that the result holds for $\mathcal{S} = \{A\}$. Since $\aleph_\omega$ is the smallest singular cardinal, every element of $\mathcal{E}_e(A)$ is regular. In particular, Theorem 2.8 implies that $|[e]_{Q(A)}|$ is achieved by regular elements of $\mathcal{E}_e(A)$ for all $e \in A$. Finally, 2 is a unit of A since it is not a zero-divisor, and thus $2e \in [e]_A$ for all $e \in B(A)$. Therefore, $|[e]_A| = |[e]_{Q(A)}|$ for all $e \in B(A)$ by Theorem 2.15. □

It is easy to illustrate the convenience of the previous corollary. For example, let $F = \mathbb{Q}$ and $J = \mathbb{N}$ in Example 2.3. Then R is a von Neumann regular ring, $B(R)$ is complete, $|R| = \aleph_1 < \aleph_\omega$, and $2 \notin Z(R)$ (here, 2 is the element of R with the *integer* 2 in all coordinates). Therefore, $|[e]_R| = |[e]_{Q(R)}|$ for all $e \in B(R)$ by

Corollary 2.16. Note that we were able to draw this conclusion without knowing anything about $Q(R)$.

Several of the previous results were proved under the assumption that "$B(R)$ is complete." We conclude this section with an example showing that this "completeness" statement must be included in all of those results. However, we only emphasize the necessity for Theorem 2.15.

Example 2.17. *Let $F_n = \mathbb{Q}$ for all $n \in \mathbb{N}$. Define $R \subseteq \prod_{n\in\mathbb{N}} F_n$ to be the ring such that $(r_n) \in R$ if and only if there exists $N \in \mathbb{N}$ such that $r_n = r_N$ for all $n \geq N$. As in Example 2.3, one shows that $Q(R) = \prod_{n\in\mathbb{N}} F_n$. For any $n \in \mathbb{N}$, let $e(n)$ be the element of $B(R)$ with 1 in the coordinate n and 0 elsewhere. Let $N \in \mathbb{N}$ and define $E = \{e(n) \in B(R) \mid n \geq N\}$. Then the idempotent $e = \sup E$ is clearly an element of $B(R)$ (e is the element with 1 in all coordinates $n \geq N$ and 0 elsewhere). Note that $B(R)$ is not complete since the set $\{e(2n+1)\}_{n=N}^{\infty} \subseteq B(R)$ has no supremum in $B(R)$. It is easy to see that $E \in \mathcal{E}_e(R)$ is regular and $|[e]_{Q(R)}|$ is achieved by E. However,*

$$|[e]_R| = \aleph_0 < \aleph_1 = |[e]_{Q(R)}|.$$

3. Zero-Divisor Graphs

The idea of a zero-divisor graph was introduced by I. Beck in [6]. While he was mainly interested in colorings, we shall investigate the interplay between ring-theoretic properties and graph-theoretic properties. This approach begun in a paper by D.F. Anderson and P.S. Livingston [4], and has since continued to evolve (e.g., [2], [3], [5], [7], [11], [13], [14], and [16]).

Let Γ be a graph and let $v \in V(\Gamma)$. As in [3], a vertex $w \in V(\Gamma)$ is called a *complement* of v if v is adjacent to w, and the edge $v - w$ is not an edge of any triangle in Γ. In ring-theoretic terms, this is the same as saying that w is a complement of v in $\Gamma(R)$ if and only if $0 \neq v, w \in R$ are distinct, $vw = 0$, and $\mathrm{ann}(v) \cap \mathrm{ann}(w) \subseteq \{0, v, w\}$. As in [3], we will say that Γ is *complemented* if every vertex has a complement, and is *uniquely complemented* if it is complemented and any two complements of a vertex are adjacent to the same vertices. Note that $\Gamma(R)$ is uniquely complemented if and only if either R is nonreduced and $\Gamma(R)$ is a *star graph* (i.e., a graph with at least two vertices such that there exists a vertex which is adjacent to every other vertex, and these are the only adjacency relations), or R is reduced and $T(R)$ is von Neumann regular [3, Theorems 3.5 and 3.9]. Moreover, [3, Theorem 3.5] shows that a reduced ring is uniquely complemented if and only if it is complemented.

Suppose that R is a von Neumann regular ring. Let $x \in R$. Then there is a unit $u \in R$ such that $xu = e_x$, the unique idempotent satisfying $[x]_R = [e_x]_R$. Hence $1 - e_x$ is a complement of x since $(1 - e_x)x = 0$, and $tx = 0 = t(1 - e_x)$ implies $t = te_x = t(xu) = (tx)u = 0$. By [3, Theorem 3.5], $\Gamma(R)$ is uniquely complemented. Thus $\operatorname{ann}(x') = \operatorname{ann}(1 - e_x)$ for every complement x' of x.

In this section, we explore some consequences of the results given in Section 2. Theorem 3.2 gives sufficient conditions to conclude that a reduced ring R satisfies $\Gamma(R) \cong \Gamma(Q(R))$. In Theorem 3.3, we explain precisely when $\Gamma(R) \cong \Gamma(Q(R))$ for "small" reduced rings with $2 \notin Z(R)$. Finally, Theorem 3.4 shows that a Boolean ring R satisfies $\Gamma(R) \cong \Gamma(Q(R))$ if and only if $R \cong Q(R)$. Moreover, the zero-divisor graphs of such Boolean rings are completely characterized.

Let $S \subseteq V(\Gamma(R))$ be a family of vertices. As in [11], we shall call v a *central vertex of* S if v is adjacent to s for all $s \in S$. The following lemma is implicit in the proofs of Lemma 3.3 and Theorem 3.4 of [11].

Lemma 3.1. *Let R be a von Neumann regular ring. Then $B(R)$ is complete if and only if whenever $\emptyset \neq S \subseteq V(\Gamma(R))$ is a family of vertices that has a central vertex, there exists a central vertex v of S possessing a complement that is adjacent to all of the central vertices of S (and hence, since $\Gamma(R)$ is uniquely complemented, every complement of v is adjacent to every central vertex of S).*

Proof. To prove the necessity of the stated conditions, suppose that there is a $\emptyset \neq S \subseteq V(\Gamma(R))$ with central vertices such that, if v is any central vertex of S, then there exists a central vertex w of S with $(1 - e_v)w \neq 0$. Let $S' = \{1 - e_s \in B(R) \mid s \in S\}$, and let $C = \{b \in B(R) \setminus \{0\} \mid be_s = 0 \text{ for all } s \in S\}$. Note that $C \neq \emptyset$ since $e_v \in C$ whenever v is a central vertex of S. Moreover, every element of C is a central vertex of S. Therefore, to every $b \in C$ there corresponds a central vertex w of S such that $(1 - b)w \neq 0$. In particular, $(1 - b)e_w \neq 0$. Let $f = \inf S'$ (in $D(B(R))$). Note that $f \neq 0$ since $b \leq f$ whenever $b \in C$. Thus, if $f \in B(R)$, then $f \in C$ and hence there is a central vertex w of S such that $fe_w \neq e_w$. But $e_w \in C$, and hence $e_w \leq f$. This is a contradiction, and therefore $f \notin B(R)$. Since the infimum of a set taken in $B(R)$ agrees with the infimum taken in $D(B(R))$, we have that $B(R)$ is not complete.

Conversely, suppose that the stated conditions on $V(\Gamma(R))$ are satisfied. Let $\emptyset \neq S \subseteq B(R)$ be any family of elements. It is clear that $\inf S = 0$ if $0 \in S$. Suppose that $0 \notin S$. If $S = \{1\}$, then $\inf S = 1$. If $S \neq \{1\}$ and contains 1, then we may remove 1 from S without changing $\inf S$. Thus we may assume $0, 1 \notin S$.

Since R is reduced, $D = \mathrm{ann}_R(\{1-s\}_{s\in S})+(\{1-s\}_{s\in S})$ is dense in R, and hence in $Q(R)$. Let $\inf S = f \in B(Q(R))$. Then $f(\{1-s\}_{s\in S}) = (0)$. Suppose that S has no infimum in $B(R)$. Then $f \neq 0$ since $f \notin B(R)$. Evidently, $\mathrm{ann}_R(\{1-s\}_{s\in S}) \neq (0)$ since otherwise $fD = (0)$. That is, $C = \{v \in V(\Gamma(R)) \mid v \text{ is adjacent to } 1 - s \text{ for all } s \in S\} \neq \emptyset$. By hypothesis, there is a $v^* \in C$ whose complements are adjacent to every element of C. In particular, $v(1 - e_{v^*}) = 0$ for all $v \in C$. Since $v^* \in C$, it follows that $e_{v^*}(1 - s) = 0$ for all $s \in S$; that is, $e_{v^*} \leq s$ for all $s \in S$. Moreover, if $0 \neq v \in B(R)$ with $v \leq s$ for all $s \in S$, then $v \in C$ so that $v(1 - e_{v^*}) = 0$; that is, $v \leq e_{v^*}$. But this shows that $f = \inf S = e_{v^*} \in B(R)$, a contradiction. Thus every $\emptyset \neq S \subseteq B(R)$ has an infimum, and hence $B(R)$ is a complete Boolean algebra. $\square$

Let R be any ring. We shall say that $\Gamma(R)$ is *central vertex complete*, or *c.v.-complete*, if $\Gamma(R)$ satisfies the condition of Lemma 3.1. Thus Lemma 3.1 can be restated as follows:

Let R be a von Neumann regular ring. Then $B(R)$ is complete if and only if $\Gamma(R)$ is c.v.-complete.

As already noted, every ring R satisfies $\Gamma(R) \cong \Gamma(T(R))$ by [3, Theorem 2.2]. In [3, Theorem 4.1], it is shown that the zero-divisor graphs of two von Neumann regular rings R and S are isomorphic if and only if there is a Boolean algebra isomorphism $\varphi : B(R) \to B(S)$ such that $|[e]_R| = |[\varphi(e)]_S|$ for all $1 \neq e \in B(R)$. Therefore, Examples 2.10 and 2.11 illustrate that a von Neumann regular ring R may fail to satisfy the condition $\Gamma(R) \cong \Gamma(Q(R))$. (Indeed, if R is the ring in Example 2.10, then $|[e]_R| < \aleph_{\omega+1}$ for all $e \in B(R)$. If R is the ring in Example 2.11, then $|[e]_R| < \aleph_1$ for all $e \in B(R)$.)

Recall that a von Neumann regular ring R satisfies $B(R) = B(Q(R))$ if and only if $B(R)$ is complete [8, Theorem 11.9].

Theorem 3.2. *Let R be a reduced ring. Suppose that $\Gamma(R)$ is a complemented c.v.-complete graph. If $2e \in [e]_{T(R)}$ and $|[e]_{Q(T(R))}|$ is achieved by regular elements of $\mathcal{E}_e(T(R))$ for all $e \in B(T(R)) \setminus \{1\}$, then $\Gamma(R) \cong \Gamma(Q(R))$.*

Proof. Suppose that $\Gamma(R)$ is a complemented c.v.-complete graph. Note that it makes sense to speak of $\mathcal{E}_e(T(R))$ since $T(R)$ is von Neumann regular by [3, Theorem 3.5]. Also, $\Gamma(R) \cong \Gamma(T(R))$ implies that $B(T(R))$ is complete by Lemma 3.1. Thus $B(T(R)) = B(Q(T(R)))$ by [8, Theorem 11.9]. Suppose that $2e \in$

$[e]_{T(R)}$ and $|[e]_{Q(T(R))}|$ is achieved by regular elements of $\mathcal{E}_e(T(R))$ for all $1 \neq e \in B(T(R))$. Then Theorem 2.15 implies that $|[e]_{T(R)}| = |[e]_{Q(T(R))}|$ for all $1 \neq e \in B(T(R))$. Thus $\Gamma(T(R)) \cong \Gamma(Q(T(R))) = \Gamma(Q(R))$, where the isomorphism follows by [3, Theorem 4.1] and the equality follows since $Q(T(R)) = Q(R)$; hence $\Gamma(R) \cong \Gamma(Q(R))$ by [3, Theorem 2.2]. $\square$

To apply the previous result, one must have information regarding the zero-divisor graph of R, as well as information about the total quotient ring of R. However, information regarding $T(R)$ is unnecessary when R is "small."

Theorem 3.3. *Let R be a reduced ring. Suppose that $|V(\Gamma(R))| < \aleph_\omega$. If $2 \notin Z(R)$, then $\Gamma(R) \cong \Gamma(Q(R))$ if and only if $\Gamma(R)$ is a complemented c.v.-complete graph.*

Proof. Note that $|V(\Gamma(T(R)))| < \aleph_\omega$ since $\Gamma(R) \cong \Gamma(T(R))$. Also, 2 is a unit in $T(R)$ since $2 \notin Z(R)$ implies that $2 \notin Z(T(R))$. Finally,

$$|T(R)| \leq |Z(T(R))|^2 = (|V(\Gamma(T(R)))| + 1)^2 < \aleph_\omega$$

(the first inequality is an application of the first isomorphism theorem on the $T(R)$-module homomorphism $T(R) \to T(R)r$ defined by $s \mapsto sr$, where $0 \neq r \in Z(T(R))$).

If $\Gamma(R) \cong \Gamma(Q(R))$, then $\Gamma(R)$ is complemented since $Q(R)$ is von Neumann regular, and is c.v.-complete since $B(Q(R))$ is complete. Conversely, suppose that $\Gamma(R)$ is a complemented c.v.-complete graph. Then $\Gamma(R) \cong \Gamma(T(R))$ implies that $T(R)$ is von Neumann regular and $B(T(R))$ is complete. Therefore, $B(T(R)) = B(Q(T(R)))$. Moreover, $|[e]_{T(R)}| = |[e]_{Q(T(R))}|$ for all $e \in B(T(R))$ by Corollary 2.16. Thus

$$\Gamma(R) \cong \Gamma(T(R)) \cong \Gamma(Q(T(R))) = \Gamma(Q(R)),$$

where the second isomorphism follows from [3, Theorem 4.1]. $\square$

Note that a Boolean ring R is rationally complete if and only if $B(R)$ is a complete Boolean algebra [17, Theorem 12.3.4]. The following theorem was proved in [11, Theorem 3.4 and Theorem 4.1]. However, a simpler proof is available with the aid of Lemma 3.1.

Theorem 3.4. *Let R be a Boolean ring. Then the following conditions are equivalent:*

(1) *R is rationally complete.*
(2) *$\Gamma(R)$ is c.v.-complete.*
(3) *$\Gamma(R) \cong \Gamma(Q(R))$.*

Proof. (1)$\Leftrightarrow$(2) Lemma 3.1 implies that $B(R)$ is complete if and only if $\Gamma(R)$ is c.v.-complete; that is, R is rationally complete if and only if $\Gamma(R)$ is c.v.-complete.

(3)$\Rightarrow$(2) Since $B(Q(R))$ is complete, (3) implies that $\Gamma(R)$ is c.v.-complete by Lemma 3.1.

(1)$\Rightarrow$(3) This is obvious. $\square$

We end this section by observing that the zero-divisor graphs of rationally complete Boolean rings are completely characterized: It is known that a ring R is Boolean if and only if either $R \cong \mathbb{Z}_2$ or $R \notin \{\mathbb{Z}_9, \mathbb{Z}_3[X]/(X^2)\}$ and $\Gamma(R) \neq \emptyset$ has the property that every vertex has a unique complement [11, Theorem 2.5]. Taking this together with the previous theorem, we have the following corollary.

Corollary 3.5. *Suppose that R is a ring which is not isomorphic to either of the rings in the set $\{\mathbb{Z}_9, \mathbb{Z}_3[X]/(X^2)\}$. Then R is a rationally complete Boolean ring if and only if either $R \cong \mathbb{Z}_2$ or $\Gamma(R) \neq \emptyset$ is a c.v.-complete graph such that every vertex has a unique complement.*

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