International Electronic Journal of Algebra
Volume 2 (2007) 106-113

TOTALLY COFINITELY SUPPLEMENTED MODULES

Gökhan Bilhan

Received: 9 February 2007; Revised: 18 April 2007
Communicated by Patrick F. Smith

Abstract. It is proved that, over a Dedekind domain, a module is torsion if and only if every submodule is cofinitely supplemented.

Mathematics Subject Classification (2000): 16D99
Keywords: Cofinitely supplemented module

1. Introduction

Throughout R will denote an associative ring with identity and all modules are unital left R-modules. A module M is called *supplemented,* if every submodule of M has a supplement; that is, let N be a submodule of M, if M is supplemented, then there exists a submodule K of M such that $M = N + K$ and K is minimal in this sum; or equivalently, $N \cap K$ is a small submodule of K.

M is called *amply supplemented*, if whenever $M = N + K$, where N and K are submodules of M, then N has a supplement contained in K. Obviously, amply supplemented modules are supplemented but converse is not true.

A module N is called a *cofinite* submodule of M, if $\frac{M}{N}$ is finitely generated. If every cofinite submodule of M has a supplement in M, then M is called cofinitely supplemented, see [1]. Every finitely generated cofinitely supplemented module is supplemented.

In [7], totally supplemented modules are defined. A module M is called *totally supplemented*, if every submodule of M is supplemented. Let us say, a module is *totally cofinitely supplemented* if every submodule of M is cofinitely supplemented. By definition, every submodule of a totally (cofinitely) supplemented module is totally (cofinitely) supplemented.

This definition is not meaningless, that is not every submodule of a cofinitely supplemented module is cofinitely supplemented. Let us consider the $\mathbb{Z}$-modules $\mathbb{Z}$ and $\mathbb{Q}$, where $\mathbb{Z}$ is the set of integers and $\mathbb{Q}$ is rational integers. $\mathbb{Q}$ is cofinitely supplemented, since it's only cofinite submodule is itself but obviously $\mathbb{Z}$ is not cofinitely supplemented.

Theorem 2.13 of [1] can be rewritten after this new definition.

Theorem 1. *The following statements are equivalent:*

1- R is semiperfect.

2- Every left(right) R-module is cofinitely supplemented.

3- Every left(right) R-module is amply cofinitely supplemented.

4- Every left(right) R-module is totally cofinitely supplemented.

Corollary 2. *Let R be a discrete valuation ring, then the following statements are equivalent for an R-module M:*

1- M is cofinitely supplemented.

2- M is amply cofinitely supplemented.

3- M is totally cofinitely supplemented.

Proof. Since a DVR is local, then R is semiperfect by [8, 42.6]. So, by above Theorem result follows. □

The definition of totally cofinitely supplemented module could be weakened as it is seen in the next result:

Proposition 3. *Let M be a module such that every (cyclic)finitely generated submodule is cofinitely supplemented, then M is totally cofinitely supplemented.*

Proof. Let N be a submodule of M, then for any $n \in N$, Rn is cofinitely supplemented and by [1, Lemma 2.3], $N = \sum_{n \in N} Rn$ is cofinitely supplemented. □

Proposition 4. *Every totally cofinitely supplemented module is amply cofinitely supplemented and every amply cofinitely supplemented module is cofinitely supplemented.*

Proof. Let M be a totally cofinitely supplemented module, then every (cyclic) submodule of M is cofinitely supplemented. So, by [1, Corollary 2.11], M is amply cofinitely supplemented. Clearly amply cofinitely supplemented modules are cofinitely supplemented. □

Note that, converse statements of Proposition 4 are not true because of Example 1.7 of [7] (for finitely generated modules, cofinitely supplemented modules are supplemented) and Corollary 4.9 of [1].

Corollary 5. *Every factor module of a totally cofinitely supplemented module is totally cofinitely supplemented.*

Proof. Let M be a totally cofinitely supplemented module and K be a submodule of M. Let N be a submodule of M containing K, then by assumption N is cofinitely supplemented and by [1, Lemma 2.1], $\frac{N}{K}$ is cofinitely supplemented. □

The following result is the analog of [7, Theorem 2.8] and a generalization of Theorem 3.4 of the same paper.

Proposition 6. *Let K be a linearly compact submodule of M, then M is totally cofinitely supplemented if and only if $\frac{M}{K}$ is totally cofinitely supplemented.*

Proof. Necessity is by Corollary 5. For sufficiency: Let $\frac{M}{K}$ be a totally cofinitely supplemented module where K is a linearly compact submodule of M. Let N be a submodule of M. $N \cap K$ is linearly compact by [8, 29.8(2)]. Since $\frac{M}{K}$ is totally cofinitely supplemented and $\frac{N}{N \cap K} \cong \frac{N+K}{K}$, then $\frac{N}{N \cap K}$ is cofinitely supplemented. By [7, Theorem 3.4], N is cofinitely supplemented. □

We do not know if any direct sum of totally cofinitely supplemented modules is totally cofinitely supplemented over any ring, although as we will see soon, it is true over some commutative domains. But we can say the analogs of Theorem 2.9 and Corollary 2.10 of [7]. Since the proof of the first statement is virtually the same, we do not include it here.

Proposition 7. *Let $M = K \bigoplus L$ be a direct sum of submodules K and L such that K is totally cofinitely supplemented and L is semisimple. Then M is totally cofinitely supplemented if and only if K is totally cofinitely supplemented.*

Corollary 8. *Let $M = M_1 \bigoplus M_2 \bigoplus M_3$ be a direct sum of modules such that M_2 is linearly compact and M_3 is semisimple, then M is totally cofinitely supplemented if and only if M_1 is totally cofinitely supplemented.*

Proof. By Corollary 5, Propositions 6 and 7. □

An R-module M is called *locally artinian*, if every finitely generated submodule of M is artinian. Artinian modules are supplemented.

Proposition 9. *Any direct sum of locally artinian modules is totally cofinitely supplemented.*

Proof. Let $\{M_i\}_{i \in I}$ be a family of locally artinian modules for any index set I and $\bigoplus_{i \in I} M_i$ be direct sum of these modules. Let N be a submodule of M. Let us take any nonzero $n \in N$ and consider Rn. Clearly $Rn \subseteq Rm_1 + ... + Rm_k$ for some $m_1, ..., m_k$ from the family $\{M_i\}_{i\in I}$ and by assumption $Rm_1 + ... + Rm_k$ is artinian, then so is Rn and consequently Rn is cofinitely supplemented. Hence by Proposition 3, N is cofinitely supplemented. □

M is called $\bigoplus$-*cofinitely supplemented* if every cofinite submodule of M has a supplement that is a direct summand and M is cofinitely weak supplemented if every cofinite submodule of M has a weak supplement; that is, for any cofinite submodule A of M, there is a submodule B of M such that $M = A + B$ and $A \cap B$ is small in M. For these modules, see [3], [4] and [2]. A ring R is said to be a *left V-ring* if every simple left R-module is injective (see [8, page 192]).

Theorem 10. *Let R be a left V-ring and let M be an R-module, then the following statements are equivalent:*

$1-M$ *is cofinitely supplemented.*

$2-M$ *is amply cofinitely supplemented.*

$3-$ M *is totally cofinitely supplemented.*

$4-M$ *is* $\bigoplus -$*cofinitely supplemented.*

$5-$ *Every cofinite submodule of M is a direct summand.*

$6-M$ *is semisimple.*

$7-M$ *is cofinitely weak supplemented.*

Proof. (6) ⇒ (5) ⇒ (4) are straightforward implications by definitions.

(6) ⇒ (3) If M is semisimple then every submodule of M is semisimple and so cofinitely supplemented.

(2) ⇒ (1) and (3) ⇒ (2) are by Proposition 4.

(1) ⇒ (5) Let N be a cofinite submodule of M. then by assumption $M = N + K$ and $N \cap K << K$ for some submodule K of M. Then $N \cap K \subseteq RadK \subseteq RadM$. But by [8, 23.1], $RadM = 0$. Therefore $M = N \bigoplus K$.

(5) ⇒ (6) Let us suppose that M is not semisimple, then $M \neq SocM$. For any $m \in M - SocM$, we consider $\sigma(Rm)$ where $\sigma : M \to \dfrac{M}{SocM}$ is the canonical epimorphism. Since $\sigma(Rm) \cong \dfrac{Rm}{Rm \cap SocM}$, then it is finitely generated. Hence $\sigma(Rm)$ has a maximal submodule $\sigma(A)$, for some submodule A of $Rm + SocM$ containing $SocM$. By the isomorphism $\dfrac{\sigma(Rm)}{\sigma(A)} \cong \dfrac{Rm + SocM}{A}$ and assumption, $\dfrac{Rm + SocM}{A}$ is injective and then $\dfrac{M}{A} = \dfrac{Rm + SocM}{A} \bigoplus \dfrac{B}{A}$ for some submodule B of M containing A. Then B is a maximal submodule of M and consequently $\dfrac{B}{SocM}$ is a maximal submodule of $\dfrac{M}{SocM}$. A contradiction by Lemma 2.7 of [1], hence M is semisimple.

(4) $\Rightarrow$ (1) is by definition.

(1) $\Rightarrow$ (7) is by [2, Theorem 2.19]. $\square$

2. Totally Cofinitely Supplemented Modules over Commutative Rings

Throughout this section R will denote a commutative ring.

Lemma 11. *Let R be a noetherian ring and $M = M_1 \bigoplus M_2 \bigoplus ... \bigoplus M_n$ be a direct sum of totally cofinitely supplemented submodules $M_i (1 \leq i \leq n)$ for some $n \geq 2$. Let $R = ann(M_i) + ann(M_j)$ for all $1 \leq i < j \leq n$, then M is totally cofinitely supplemented.*

Proof. Let U and V be two submodules of M such that V is cofinite in U. By [6, Lemma 4.1], $U = (U \cap M_1) \bigoplus ... \bigoplus (U \cap M_n)$ and $V = (V \cap M_1) \bigoplus ... \bigoplus (V \cap M_n)$. For every $1 \leq i \leq n$, $V \cap M_i$ is cofinite submodule of $U \cap M_i$. Because, $\frac{U}{V} \cong \bigoplus \frac{U \cap M_i}{V \cap M_i}$. By assumption, $U \cap M_i$ is cofinitely supplemented, then there exists a supplement K_i of $V \cap M_i$ in $U \cap M_i$. Let $K = K_1 \bigoplus K_2 \bigoplus ... \bigoplus K_n$. Then $U = (V \cap M_1 + K_1) \bigoplus ... \bigoplus (V \cap M_n + K_n) = (V \cap M_1 + ... + V \cap M_n) + (K_1 \bigoplus ... \bigoplus K_n) = V + K$. Also, $V \cap K = V \cap K_1 \bigoplus ..... \bigoplus V \cap K_n$. Hence by [8, 19.3], K is supplement of V in U. Therefore U is cofinitely supplemented. $\square$

Theorem 12. *Let R be a noetherian ring and $\{M_i\}_{i \in I}$ be a family of totally cofinitely supplemented R-modules, then $\bigoplus_{i \in I} M_i$ is totally cofinitely supplemented in case, $R = ann(M_i) + ann(M_j)$ for any $i \neq j \in I$.*

Proof. Let $\bigoplus_{i \in I} M_i = M$ and N be any submodule of M. By taking a nonzero $n \in N$ and considering Rn, we can say $Rn \subseteq Rm_1 + ... + Rm_k$ for some $m_i's$ from $M_i's$ $(1 \leq i \leq k)$. Since each M_i is totally cofinitely supplemented, then every Rm_i $(1 \leq i \leq k)$ is totally cofinitely supplemented. Hence by Lemma11, $Rm_1 \bigoplus ... \bigoplus Rm_k$ is totally cofinitely supplemented. Therefore Rn and consequently $N = \sum_{n \in N} Rn$ is cofinitely supplemented. $\square$

The following two results are easy adaptations of [7, Lemma 4.3. and Lemma 4.4]. Therefore the proofs are not given in detail.

Let M be a local R-module. Then M is cyclic and hence $M \cong \frac{R}{ann(M)}$. There exists a unique maximal ideal P of R such that $ann(M) \subseteq P$. Then we call the module M, *p-local.*

Lemma 13. *Let P be a maximal ideal of R and let an R-module $M = M_1 \bigoplus ... \bigoplus M_n$ be a finite direct sum of p-local submodules $M_i (1 \leq i \leq n)$ for some positive integer n. Then M is totally cofinitely supplemented.*

Proof. By the proof of Lemma 4.3 of [7], R is semiperfect and by Theorem 1, M is totally cofinitely supplemented. □

Corollary 14. *Let $M = M_1 \bigoplus M_2 \bigoplus ... \bigoplus M_n$ be a finite direct sum of local submodules $M_i (1 \leq i \leq n)$ for some positive integer n. Then M is totally cofinitely supplemented.*

Proof. Clear by Lemma 13. □

A module M is called *coatomic*, if every proper submodule of M is contained in a maximal submodule or equivalently, for a submodule N of M, whenever $Rad(\frac{M}{N}) = \frac{M}{N}$, then $M = N$. Finitely generated and semisimple modules are coatomic. For properties of coatomic modules, the papers [5] and [9] can be seen.

$LocM$ will represent the sum of all local submodules of M and $TorM$ is the torsion submodule of M.

Proposition 15. *Let R be a commutative ring and M be a coatomic R-module, then M is cofinitely supplemented if and only if M is amply cofinitely supplemented.*

Proof. Let M be cofinitely supplemented and coatomic, then by [1, Theorem 2.8], $\frac{M}{LocM}$ does not contain a maximal submodule where $LocM$ is the sum of all local submodules of M. Since M is coatomic, then $M = LocM$, so by [7, Theorem 4.5] and by [1, Theorem 2.10] result follows. □

Theorem 16. *Let R be a commutative noetherian ring and M be a coatomic R-module. The following statements are equivalent:*

$1-$ *M is cofinitely supplemented.*

$2-$ *M is amply cofinitely supplemented.*

$3-$ *M is totally cofinitely supplemented.*

Proof. (1) ⇒ (2) is by Proposition 15.

(1) ⇒ (3) By [1, Theorem 2.8] and assumption, $M = LocM$. So, $M = \bigoplus_{i \in I} L_i$ where L_i's are local submodules of M. Let N be any submodule of M, then for any nonzero $n \in N$, Rn will be a submodule of $\bigoplus_{i \in F} L_i$ where F is a finite subset of I. By Theorem 12, $\bigoplus_{i \in F} L_i$ is totally cofinitely supplemented and so Rn is cofinitely supplemented. Consequently N is cofinitely supplemented by Proposition 3.

(3) ⇒ (2) ⇒ (1) is by Corollary 2. □

Proposition 17. *If R is a commutative domain, then every coatomic cofinitely supplemented R-module is torsion.*

Proof. Let M be a coatomic cofinitely supplemented R-module, then $\frac{M}{LocM}$ does not contain a maximal submodule by [1, Theorem 2.8]. But then $M = LocM$ because M is coatomic. By [1, Lemma 4.1], $M \leq TorM$, that is $M = TorM$. Hence M is torsion. □

Corollary 18. *If R is a commutative domain, then every totally cofinitely supplemented R-module is torsion.*

Proof. Let M be totally cofinitely supplemented, then Rm is cofinitely supplemented and coatomic, for any $m \in M$. By Proposition 17, Rm is torsion. Therefore $M = \sum_{m \in M} Rm$ is torsion. □

A commutative domain R is called *h-local*, if every non-zero non-unit of R belongs to only finitely many maximal ideals and $\frac{R}{P}$ is a local ring for every prime ideal P of R. By [6, p.148], Dedekind domains are h-local.

Corollary 19. *Let R be an h-local domain, then an R-module M is totally cofinitely supplemented if and only if M is torsion.*

Proof. (⇒) By Corollary 18.

(⇐) Every submodule of M is torsion. So, cofinitely supplemented by Theorem 4.6 of [1], hence M is totally cofinitely supplemented. □

Corollary 20. *Let R be an h-local domain, then any sum of totally cofinitely supplemented modules is totally cofinitely supplemented.*

Proof. Any sum of torsion modules is torsion, so result follows by Corollary 19. □

Theorem 21. *Let R be a domain. The following statements are equivalent:*

$1-$ *R is h-local.*

$2-$ *Every torsion R-module is totally cofinitely supplemented.*

$3-$ *Every cyclic torsion R-module is totally cofinitely supplemented.*

$4-$ *An R-module M is totally cofinitely supplemented if and only if M is torsion.*

Proof. (1) ⇒ (2) By Corollary 19.

(2) ⇒ (3) Clear.

(3) ⇒ (1) Every cyclic torsion R-module is totally cofinitely supplemented, then cofinitely supplemented. By [1, Theorem 4.6].

(1) ⇒ (4) By Corollary 19.

(4) ⇒ (2) Clear. □

References

[1] R. Alizade, G. Bilhan and P.F. Smith, Modules Whose Maximal Submodules have Supplements, Comm. Algebra, 29(6), (2001), 2389-2405.

[2] R. Alizade and E. Büyükaşık, Cofinitely Weak Supplemented Modules, Comm. Algebra, 31(11), (2003), 5377-5390.

[3] H. Çalışıcı and A. Pancar, $\bigoplus$-cofinitely supplemented modules, Czech. Mathematical Journal, 54(129) (2004), 1083-1088

[4] H. Çalışıcı and A. Pancar, Cofinitely Semiperfect Modules, Siberian Math. Journal, 46(2) (2005), 359-363.

[5] G. Güngöroğlu, Coatomic Modules, Far East J.Math. Sci. Special Volume, Part II, (1998), 153-162.

[6] E. Matlis, Cotorsion Modules, Mem. Amer. Math. Soc., 49 (1964), pp. 66.

[7] P.F. Smith, Finitely Generated modules are Amply Supplemented, The Arabian Journal for Science and Engineering, 25(2C) (2000), 69-79.

[8] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Philedelphia, 1991.

[9] H. Zöschinger, Koatomare Moduln, Math. Z. 170(3) (1980), 221-232.

Gökhan Bilhan
Dokuz Eylül University, Fen Edebiyat Fakültesi,
Matematik B., Tınaztepe Y. 35160,
Buca, İzmir, Türkiye
E-mail: gokhan.bilhan@deu.edu.tr