International Electronic Journal of Algebra
Volume 1 (2007) 55–97

Q-CHARACTERS AND MINIMAL AFFINIZATIONS

M. G. Moakes and A. N. Pressley

Received: 11 October 2006; Revised: 9 January 2007

Communicated by R.J. Marsh

Abstract. Every finite-dimensional irreducible representation of a (classical) affine Lie algebra has quantum analogues, but these are generally 'larger' than their classical counterparts. Among the quantum analogues of a particular classical representation, some (usually one) are 'minimal' in a certain precise sense. This paper studies the structure of these minimal representations when the underlying finite-dimensional Lie algebra is of rank 2. We also compute their q-characters in some cases.

Mathematics Subject Classification (2000): 17B37, 20G42, 81R50
Keywords: Quantum group, affine algebra, representation, affinization.

Introduction

Quantum affine algebras are the simplest infinite-dimensional examples of quantum groups. The classification of the finite-dimensional irreducible representations of quantum affine algebras is well known, but their structure is still not well understood. An important tool for studying the representations of quantum affine algebras is their q-character; in particular, this invariant is sufficiently fine to distinguish different representations up to isomorphism. The purpose of this paper is to describe a technique for computing q-characters.

The method is applied to the 'minimal' affinizations: every finite-dimensional irreducible representation of a (classical) affine Lie algebra has quantum analogues, but these are generally 'larger' than their classical counterparts. Among the quantum analogues of a particular classical representation, some (usually one) are 'minimal' in a certain precise sense. It is known that every finite-dimensional irreducible representation is a subquotient of a tensor product of minimal affinizations.

After some preliminary material, we treat the affine A_2 and C_2 cases in Sections 2 and 3, respectively. The methods used involve the construction of exact sequences of representations involving the representation to be studied together with certain so-called 'fundamental' representations. The structure and q-character of the fundamental representations is known, hence the exact sequences enable those of the representation under consideration to be obtained. The exact sequences we obtain enable us to compute the q-characters of minimal representations in the A_2 and C_2 cases by an inductive argument. In the A_2 case a closed formula for the q-character is obtained; in the C_2 case, an algorithm is given which enables the q-character to be computed for any given minimal representation.

For the affine G_2 case, treated in Section 4, the methods used in the preceding two chapters are more difficult to apply, essentially because of the complexity of the tensor product decompositions of the representations involved. Nevertheless, we have been able to determine the structure of the minimal affinizations in some cases that have not been analyzed before.

1. Preliminaries

1.1 Kac-Moody algebras

Let $n \geq 1$ be an integer, $I = \{1, 2, \ldots, n\}$. Let $\mathfrak{g}$ be the finite-dimensional complex simple Lie algebra of rank n with Cartan matrix $(a_{ij})_{i,j \in I}$. Thus, $\mathfrak{g}$ is the Lie algebra over the field $\mathbb{C}$ of complex numbers with generators $x_i^{\pm}$, h_i $(i \in I)$, and the following defining relations:

$$[h_i, h_j] = 0, \quad \left[h_i, x_j^{\pm}\right] = \pm a_{ij} x_j^{\pm}, \quad \left[x_i^{+}, x_j^{-}\right] = \delta_{ij} h_i,$$
$$(\operatorname{ad} x_i^{\pm})^{1-a_{ij}}(x_j^{\pm}) = 0, \quad \text{if } i \neq j.$$

Let $\mathfrak{h}$ be the linear span of the h_i $(i \in I)$. Denote by $(\ ,\ )$ a non-zero invariant bilinear form on $\mathfrak{g}$. Since the restriction of $(\ ,\ )$ to $\mathfrak{h}$ is non-degenerate, it induces a bilinear form on the dual space $\mathfrak{h}^*$, which we also denote by $(\ ,\ )$.

Let α_i $(i \in I)$ be the simple roots of $\mathfrak{g}$, let Δ^+ be the set of positive roots, $\Delta^- = -\Delta^+$, and let

$$Q = \bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i, \qquad Q^+ = \sum_{i=1}^{n} \mathbb{N}\alpha_i.$$

A weight is a linear functional $\lambda \in \mathfrak{h}^*$. Weights can be identified with maps $I \to \mathbb{C}$ in the obvious way: to $\tilde{\lambda} : I \to \mathbb{C}$ corresponds the unique weight $\lambda \in \mathfrak{h}^*$ such that

$$\lambda(h_i) = \tilde{\lambda}(i).$$

We shall often confuse these two notions of weights in the following.

Denote by

$$P = \{\lambda : I \to \mathbb{C} \,|\, \lambda(i) \in \mathbb{Z} \ \ \forall\, i \in I\}$$

the set of integral weights, and by

$$P^+ = \{\lambda : I \to \mathbb{C} \,|\, \lambda(i) \in \mathbb{N} \ \ \forall\, i \in I\}$$

the set of dominant integral weights. The fundamental weights $\lambda_i \in P^+$ $(i \in I)$ are the dominant integral weights such that

$$\lambda_i(h_j) = \delta_{ij} \ \ \forall i, j \in I.$$

Denote by $\leq$ the partial ordering on P given by $\lambda, \mu \in P$,

$$\lambda \geq \mu \ \text{ if and only if } \ \ \lambda - \mu \in Q^+.$$

With respect to this ordering, there is a unique maximal root $\theta \in \Delta^+$.

1.2 Affine algebras

Define

$$L(\mathfrak{g}) = \mathfrak{g} \otimes_{\mathbb{C}} L,$$

where $L = \mathbb{C}\left[t, t^{-1}\right]$ is the algebra of Laurent polynomials in an indeterminate t. For $x \in \mathfrak{g}$, $n \in \mathbb{Z}$, write xt^n for $x \otimes t^n \in L(\mathfrak{g})$.

The following bracket defines a Lie algebra structure on $L(\mathfrak{g})$:

$$[xt^m, yt^n]_0 = [x, y]\, t^{m+n}, \ \ \forall\, m, n \in \mathbb{Z},\, x, y \in \mathfrak{g}.$$

The bilinear form $(\ ,\ )$ on $\mathfrak{g}$ is extended to $L(\mathfrak{g})$ by

$$(xt^m, yt^n) = \delta_{m+n,0}(x, y).$$

Let d be the derivation of $L(\mathfrak{g})$ defined by

$$d(xt^m) = mxt^m \qquad \forall\, m \in \mathbb{Z},\, x \in \mathfrak{g}.$$

Define a bilinear map $\psi : L(\mathfrak{g}) \times L(\mathfrak{g}) \to \mathbb{C}$ by

$$\psi(a, b) = (d(a), b), \qquad \forall\, a, b \in L(\mathfrak{g}).$$

It can be shown that ψ is a 2-cocycle on the Lie algebra $L(\mathfrak{g})$, so we can define a central extension

$$\overline{L(\mathfrak{g})} = L(\mathfrak{g}) \oplus \mathbb{C}c$$

of $L(\mathfrak{g})$ with bracket

$$[a+\lambda c, b+\mu c] = [a,b]_0 + \psi(a,b)c, \qquad \forall\ a,b \in L(\mathfrak{g}), \lambda, \mu \in \mathbb{C}.$$

The centre of $\overline{L(\mathfrak{g})}$ is the 1-dimensional subspace spanned by the element c.

It is known that the affine Lie algebra $\overline{L(\mathfrak{g})}$ is isomorphic to the Kac-Moody algebra $\hat{\mathfrak{g}}$ with generalized Cartan matrix $(a_{ij})_{i,j\in\hat{I}}$, where $\hat{I} = \{0,1,\dots,n\}$, $a_{00} = 2$ and

$$a_{0i} = -\frac{2(\theta,\alpha_i)}{(\alpha_i,\alpha_i)}, \qquad a_{i0} = -\frac{2(\theta,\alpha_i)}{(\theta,\theta)}, \quad \forall i \in I.$$

Thus, $\hat{\mathfrak{g}}$ is given by generators $x_i^{\pm}$, h_i $(i \in \hat{I})$ and the same defining relations as $\mathfrak{g}$, but with I replaced by $\hat{I}$.

1.3 Quantum groups

Let q be a transcendental complex number and, for non-zero positive integers m, n, $m \le n$, define

$$q_i = q^{d_i},\ [n]_q = \tfrac{q^n - q^{-n}}{q-q^{-1}},$$
$$[n]_q! = [n]_q[n-1]_q \dots [1]_q,\ \left[{m \atop n}\right]_q = \tfrac{[m]_q!}{[n]_q![m-n]_q!}.$$

Here, d_i $(i \in I)$ are the coprime positive integers such that the matrix $(d_i a_{ij})_{i,j\in I}$ is symmetric.

The quantum group $U_q(\mathfrak{g})$ is the algebra over $\mathbb{C}$ which is generated by elements $x_i^{\pm}, k_i^{\pm 1}$ $(i \in I)$, with the following defining relations:

$$k_i k_i^{-1} = k_i^{-1} k_i = 1,$$
$$k_i k_j = k_j k_i,$$
$$k_i x_j^{\pm} k_i^{-1} = q_i^{\pm a_{ij}} x_j^{\pm},$$
$$\left[x_i^+, x_j^-\right] = \delta_{ij} \tfrac{k_i - k_i^{-1}}{q_i - q_i^{-1}},$$
$$\textstyle\sum_{r=0}^{1-a_{ij}} (-1)^r \left[{1-a_{ij} \atop r}\right]_{q_i} (x_i^{\pm})^r x_j^{\pm} (x_i^{\pm})^{1-a_{ij}-r} = 0 \quad \text{if } i \neq j.$$

Then $U_q(\mathfrak{g})$ is a Hopf algebra with comultiplication Δ, counit ϵ and antipode S given by:

$$\Delta(x_i^+) = x_i^+ \otimes k_i + 1 \otimes x_i^+, \quad \Delta(x_i^-) = x_i^- \otimes 1 + k_i^{-1} \otimes x_i^-, \quad \Delta(k_i^{\pm 1}) = k_i^{\pm 1} \otimes k_i^{\pm 1},$$
$$\epsilon(x_i^{\pm}) = 0, \qquad \epsilon(k_i^{\pm 1}) = 1,$$
$$S(x_i^+) = -x_i^+ k_i^{-1}, \quad S(x_i^-) = -k_i x_i^+, \quad S(k_i^{\pm 1}) = k_i^{\mp 1}.$$

Let $\hat{\mathfrak{g}}$ be the affine Lie algebra associated to $\mathfrak{g}$. Then, $U_q(\hat{\mathfrak{g}})$ is defined to be the algebra over $\mathbb{C}$ which is generated by elements $x_i^{\pm}, k_i^{\pm 1}$ $(i \in \hat{I})$, with defining relations and Hopf algebra structure given by the same formulas as in the $U_q(\mathfrak{g})$ case but with I replaced by $\hat{I}$. Denote by $U_q(+)$ (resp $U_q(-)$) the subalgebra of $U_q(\hat{\mathfrak{g}})$ generated by $k_i^{\pm 1}, x_i^+$ (resp. $k_i^{\pm 1}, x_i^-$) $(i \in \hat{I})$.

In [1] and [9] there is a realization of $U_q(\hat{\mathfrak{g}})$ analogous to the realizations of $\hat{\mathfrak{g}}$ as a central extension of $L(\mathfrak{g})$.

Theorem 1.1. *Let $\mathcal{A}_q(\mathfrak{g})$ be the algebra over $\mathbb{C}$ with generators $x_{i,r}^{\pm}$ $(i \in I, r \in \mathbb{Z})$, $k_i^{\pm 1}$ $(i \in I)$, $h_{i,r}$ $(i \in I, r \in \mathbb{Z}/\{0\})$ and $c^{\pm 1/2}$, and the following defining relations:*

$$c^{\pm 1/2} \text{ are central,}$$
$$c^{1/2}c^{-1/2} = c^{-1/2}c^{1/2} = 1, \quad k_i k_i^{-1} = k_i^{-1} k_i = 1,$$
$$k_i k_j = k_j k_i, \; k_i h_{j,r} = h_{j,r} k_i,$$
$$k_i x_{j,r}^{\pm} k_i^{-1} = q_i^{\pm a_{ij}} x_{j,r}^{\pm},$$
$$[h_{i,r}, h_{j,s}] = \delta_{r,-s} \tfrac{1}{r}[r a_{ij}]_{q_i} \tfrac{c^r - c^{-r}}{q_j - q_j^{-1}},$$
$$\left[h_{i,r}, x_{j,s}^{\pm}\right] = \pm \tfrac{1}{r}[r a_{ij}]_{q_i} c^{\mp |r|/2} x_{i,r+s}^{\pm},$$
$$x_{i,r+1}^{\pm} x_{j,s}^{\pm} - q^{\pm a_{ij}} x_{j,s}^{\pm} x_{i,r+1}^{\pm} = q^{\pm a_{ij}} x_{i,r}^{\pm} x_{j,s+1}^{\pm} - x_{j,s+1}^{\pm} x_{i,r}^{\pm},$$
$$\left[x_{i,r}^{+}, x_{j,s}^{-}\right] = \delta_{i,j} \frac{c^{(r-s)/2}\Phi_{i,r+s}^{+} - c^{-(r-s)/2}\Phi_{i,r+s}^{-}}{q_i - q_i^{-1}},$$
$$\sum_{\pi \in \Sigma_m} \sum_{k=0}^{m} (-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_{q_i} x_{i,r_{\pi(1)}}^{\pm} \ldots x_{i,r_{\pi(k)}}^{\pm} x_{j,s}^{\pm} x_{i,r_{\pi(k+1)}}^{\pm} \ldots x_{i,r_{\pi(m)}}^{\pm} = 0 \quad \text{if } i \neq j.$$

In the last relation, $m = 1 - a_{ij}$, Σ_m is the symmetric group on m letters and the $\Phi_{i,r}^{\pm}$ are determined by equating powers of u in the formal power series

$$\Phi_i^{\pm}(u) = \sum_{r=0}^{\infty} \Phi_{i,\pm r}^{\pm} u^{\pm r} = k_i^{\pm 1} exp\left(\pm(q_i - q_i^{-1}) \sum_{s=1}^{\infty} h_{i,\pm s} u^{\pm s}\right);$$

the relation is to hold for all sequences of integers $r_1, \ldots, r_m$.

Then there is an isomorphism of algebras $f : U_q(\hat{\mathfrak{g}}) \to \mathcal{A}_q(\mathfrak{g})$ such that

$$f(k_0) = k_\theta^{-1}, f(k_i) = k_i, \; f(x_i^{\pm}) = x_{i,0}^{\pm}.$$

Here, we have $k_\theta = \prod_i k_i^{r_i}$ if $\theta = \sum_i r_i \alpha_i$.

We remark that explicit formulas are known [1] for the images under f of $x_0^{\pm}$, but we shall not need them in this work. On the other hand, explicit expressions are not known for the comultiplication, counit and antipode on the generators $h_{i,s}$ $(i \in I, s \in \mathbb{Z}/\{0\})$. The following result will be sufficient for our purposes.

Lemma 1.2 [11, Lemma 1] *On representations of* $U_q(\hat{\mathfrak{g}})$ *on which* $c^{1/2}$ *acts as the identity we have, for all* $i \in I, r > 0$,

$$\Delta(h_{i,r}) = h_{i,r} \otimes 1 + 1 \otimes h_{i,r} + \text{ terms in } U'_+ \otimes U'_-,$$

where $U'_\pm$ *is the augmentation ideal of the subalgebra* $U_\pm$ *of* $U_q(\hat{\mathfrak{g}})$ *generated by the elements* $x^\pm_{j,m}$ *for* $j \in I, m \in \mathbb{Z}$.

The defining relations of $\mathcal{A}_q(\mathfrak{g})$ are clearly $\mathbb{Z}$-graded. Indeed, it is easy to verify that, for any $a \in \mathbb{C}^*$, there is a Hopf algebra automorphism τ_a of $U_q(\hat{\mathfrak{g}})$ defined by

$$\tau_a(x^\pm_{i,r}) = a^r x^\pm_{i,r}, \quad \tau_a(\Phi^\pm_{i,r}) = a^r \Phi^\pm_{i,r},$$
$$\tau_a(c^{1/2}) = c^{1/2}, \quad \tau_a(k_i) = k_i$$

for all $i \in I, r \in \mathbb{Z}$.

1.4 Representation theory: finite case

If V is a representation of $U_q(\mathfrak{g})$ then $\lambda \in P$ is a weight of V if the weight space

$$V_\lambda = \{v \in V | k_i.v = q_i^{\lambda(i)} v \ \forall i \in I\}$$

is non-zero. Denote the set of weights of V by $P(V)$. We say that V is of type 1 if

$$V = \bigoplus_{\lambda \in P(V)} V_\lambda.$$

A vector $v \in V_\lambda$ is a highest weight vector if $x_i^+.v = 0 \ \forall i \in I$. If $V = U_q(\mathfrak{g}).v$, then V is said to be a highest weight representation with highest weight λ. There is, up to isomorphism, a unique irreducible highest weight representation of any given highest weight λ, denoted by $V_q(\lambda)$.

We shall need the following well-known properties of representations of $U_q(\mathfrak{g})$ [7, Section 10.1]:

(a) every finite-dimensional representation of $U_q(\mathfrak{g})$ is completely reducible;

(b) every finite-dimensional irreducible representation of $U_q(\mathfrak{g})$ of type 1 is highest weight;

(c) the representation $V_q(\lambda)$ is finite dimensional if and only if $\lambda \in P^+$;

(d) if $\lambda \in P^+$, then $V_q(\lambda)$ has the same character as the irreducible representation $V(\lambda)$ of $\mathfrak{g}$ of the same highest weight, i.e.

$$\dim V_q(\lambda)_\mu = \dim V(\lambda)_\mu \quad \forall \mu \in P;$$

(e) the multiplicity $m_\nu(V_q(\lambda)\otimes V_q(\mu))$ of $V_q(\nu)$ in the tensor product $V_q(\lambda)\otimes V_q(\mu)$ $(\lambda,\mu,\nu\in P^+)$ is the same as in the tensor product of the irreducible representations of $\mathfrak{g}$ of the same highest weights.

1.5 Representation theory: affine case

A representation V of $U_q(\hat{\mathfrak{g}})$ is said to be of type 1 if $c^{1/2}$ acts as the identity on V and if V is of type 1 as a representation of $U_q(\mathfrak{g})$. A vector $v\in V$ is an L-highest weight vector if

$$x^+_{i,r}.v=0,\quad \Phi^{\pm}_{i,r}.v=\phi^{\pm}_{i,r}v,\quad \forall\, r\in\mathbb{Z}, i\in I,$$

for some complex numbers $\phi^{\pm}_{i,r}$. Note that $\phi^{+}_{i,r}=0$ if $r<0$, $\phi^{-}_{i,r}=0$ if $r>0$ and $\phi^{+}_{i,0}\phi^{-}_{i,0}=1$. A type 1 representation V is said to be an L-highest weight representation if $V=U_q(\hat{\mathfrak{g}}).v$ for some highest weight vector v and the pair of $(I\times\mathbb{Z})$-tuples $(\phi^{\pm}_{i,r})_{i\in I,r\in\mathbb{Z}}$ is said to be its L-highest weight.

Let $\mathcal{P}$ be the set of all I-tuples $(P_i)_{i\in I}$ of polynomials $P_i\in\mathbb{C}[u]$ with constant term 1.

Theorem 1.3 [6, Theorem 3.3]

1) *Every finite-dimensional irreducible representation of $U_q(\hat{\mathfrak{g}})$ of type 1 is L-highest weight.*

2) *The irreducible representation V of $U_q(\hat{\mathfrak{g}})$ of type 1 and L-highest weight $(\phi^{\pm}_{i,r})_{i\in I,r\in\mathbb{Z}}$ is finite dimensional iff there exists $\mathbf{P}=(P_i)_{i\in I}\in\mathcal{P}$ such that*

$$\sum_{r=0}^{\infty}\phi^{\pm}_{i,\pm r}u^{\pm r}=q_i^{deg(P_i)}\frac{P_i(q_i^{-2}u)}{P_i(u)}$$

as an element of $\mathbb{C}[[u^{\pm 1}]]$.

We denote the finite-dimensional irreducible representation of $U_q(\hat{\mathfrak{g}})$ associated to $\mathbf{P}$ by $V(\mathbf{P})$ and call $\mathbf{P}$ its highest weight, by abuse of notation. The highest weight of $V(\mathbf{P})$ considered as a $U_q(\mathfrak{g})$-module is

$$\lambda=\sum_{i\in I}(\mathrm{deg}P_i)\lambda_i.$$

Proposition 1.4 [6, Corollary 3.5] *If $\mathbf{P}=(P_i)_{i\in I}$ and $\mathbf{Q}=(Q_i)_{i\in I}$ are members of $\mathcal{P}$, denote by $\mathbf{P}\otimes\mathbf{Q}\in\mathcal{P}$ the I-tuple $(P_iQ_i)_{i\in I}$. Then $V(\mathbf{P}\otimes\mathbf{Q})$ is isomorphic, as a representation of $U_q(\hat{\mathfrak{g}})$, to a quotient of the subrepresentaion of $V(\mathbf{P})\otimes V(\mathbf{Q})$*

generated by the tensor product of the L-highest weight vectors in $V(\mathbf{P})$ and $V(\mathbf{Q})$.

Since every polynomial is a product of linear polynomials, we define a representation $V(\mathbf{P})$ of $U_q(\hat{\mathfrak{g}})$ to be fundamental if, for some $i \in I$,

$$P_j = 1 \quad \text{if } j \neq i, \quad \deg(P_i) = 1.$$

With this definition, we have the following corollary of Proposition 1.4:

Corollary 1.5 [6, Corollary 3.6] *For any $\mathbf{P} \in \mathcal{P}$, the representation $V(\mathbf{P})$ of $U_q(\hat{\mathfrak{g}})$ is isomorphic to a subquotient of a tensor product of fundamental representations.*

The left dual ${}^tV(\mathbf{P})$ of $V(\mathbf{P})$ is the representation of $U_q(\hat{\mathfrak{g}})$ on the vector space dual of $V(\mathbf{P})$ given by

$$\langle a.f, v\rangle = \langle f, S(a).v\rangle\,, \quad \forall\ a \in\ U_q(\hat{\mathfrak{g}}), v \in V(\mathbf{P}), f \in\ {}^tV(\mathbf{P}),$$

where S is the antipode of $U_q(\hat{\mathfrak{g}})$ and $\langle\ ,\ \rangle$ is the natural pairing between $V(\mathbf{P})$ and its dual. The right dual $V(\mathbf{P})^t$ is defined in the same way, but replacing S by S^{-1}.

Lemma 1.6 [4, Lemma 6.2] *If $\mathfrak{g}$ is of type C_2, with α_1 being the short root, then for any $a \neq 0$,*

$$V(1, 1 - uaq)^t \cong V(1, 1 - uaq^{-6}), \qquad {}^tV(1, 1 - uaq) \cong V(1, 1 - uaq^{6}).$$

The following result is easy to check.

Lemma 1.7 *For any $\mathbf{P} \in \mathcal{P}, a \in \mathbb{C}^*$, let $\tau_a^*(V(\mathbf{P}))$ be the pull-back of $V(\mathbf{P})$ by the automorphism τ_a. Then,*

$$\tau_a^*(V(\mathbf{P})) \cong V(\mathbf{P}^a),$$

where $\mathbf{P}^a = (P_i^a)_{i\in I}$ and $P_i^a(u) = P_i(au)$.

1.6 Minimal affinizations

Let $\mathbf{P} = (P_i)_{i\in I} \in \mathcal{P}$, and let $0 \neq v_{\mathbf{P}}$ be an L-highest weight vector in $V(\mathbf{P})$. If $\lambda \in P^+$ is defined by

$$\lambda(i) = \deg(P_i) \quad \forall\, i \in I,$$

then

$$k_i.v_{\mathbf{P}} = q_i^{\lambda(i)} v_{\mathbf{P}}$$

so

$$V(\mathbf{P})_\lambda = \mathbb{C}v_{\mathbf{P}} \ \text{ and } \ V(\mathbf{P}) = \bigoplus_{\nu\in Q^+} V(\mathbf{P})_{\lambda-\nu}.$$

Since $V(\mathbf{P})$ is finite dimensional it is completely reducible as a representation of $U_q(\mathfrak{g})$. Hence,

$$V(\mathbf{P}) \cong V_q(\lambda) \oplus \bigoplus_{\mu\in P^+} V_q(\mu)^{\oplus m_\mu}$$

as a representation of $U_q(\mathfrak{g})$, where the multiplicities $m_\mu \in \mathbb{N}$ are zero unless $\mu < \lambda$.

Thus, $V(\mathbf{P})$ gives a way of extending the action of $U_q(\mathfrak{g})$ on $V_q(\lambda)$ to an action of $U_q(\hat{\mathfrak{g}})$, by enlarging $V_q(\lambda)$ by the addition of representations of $U_q(\mathfrak{g})$ of smaller highest weight. For this reason, $V(\mathbf{P})$ is called an affinization of λ, or of $V_q(\lambda)$. Two affinizations of λ are said to be equivalent if they are isomorphic as representations of $U_q(\mathfrak{g})$. If V is an affinization of $\lambda \in P^+$, its equivalence class is denoted by $[V]$ and we write A^λ for the set of equivalence classes of affinizations of λ.

We define a partial ordering on the set of equivalence classes of affinizations as follows. If $\lambda \in P^+, [V], [W] \in A^\lambda$, then $[V] \leq [W]$ if and only if, for all $\mu \in P^+$, either

$$m_\mu(V) \leq m_\mu(W),$$

or

$$\text{there exists } \nu \geq \mu \text{ with } m_\nu(V) < m_\nu(W).$$

An affinization V of λ is minimal if $[V]$ is a minimal element of A^λ for the partial order defined above, i.e. if $[W] \in A^\lambda$ and $[W] \leq [V]$ implies $[V] = [W]$.

To state the main general result about minimal affinizations we shall use, we need one more definition. Namely, the q-segment of centre $a \in \mathbb{C}^*$ and length $r \in \mathbb{N}$ is defined to be the r-tuple of complex numbers

$$(aq^{-r+1}, aq^{-r+3}, \ldots, aq^{r-1}).$$

Two q-segments are said to be in *special position* if their union is a q-segment that properly contains both of them; they are said to be in general position otherwise. It is an elementary combinatorial fact that, if S is any finite set of non-zero complex number with multiplicities; then S can be written uniquely as the union of q-segments $S_1, \ldots, S_N$, any two of which are in general position.

In the case $\mathfrak{g} = sl_2(\mathbb{C})$, it is known ([7, Section 12.2]) that $V(P)$ is a minimal affinization if and only if the roots of the polynomial P form a q-segment, and that $V(P)$ is then irreducible as a $U_q(\mathfrak{g})$-module. The situation in rank 2 is as follows:

Proposition 1.8 [2, Theorem 5.1] *Let $\mathfrak{g}$ be of rank 2, $r_1, r_2 \geq 0$ and $\lambda = r_1\lambda_1 + r_2\lambda_2 \in P^+$. Then, A^λ has a unique minimal element. This element is represented by $V(\mathbf{P})$ ($\mathbf{P} \in \mathcal{P}$) if and only if the following two conditions are satisfied:*

(a) for all $i = 1, 2$, either $P_i = 1$ or the roots of P_i form a q_i-segment with center a_i and length r_i, say, and

(b) if $P_1 \neq 1$ and $P_2 \neq 1$, then

$$\frac{a_1}{a_2} = q^{d_1r_1+d_2r_2+2d_2-1} \quad \text{or} \quad \frac{a_1}{a_2} = q^{-(d_1r_1+d_2r_2+2d_1-1)}.$$

The following results describe the structure of minimal affinizations of $U_q(\mathfrak{g})$-modules when $\mathfrak{g}$ is of type A_2 or C_2.

Proposition 1.9 [5, Section 3] *Let $\mathfrak{g}$ be of type $A_2, \lambda \in P^+$ and let $V(\mathbf{P})$ be a minimal affinization of λ. Then, $V(\mathbf{P}) \cong V_q(\lambda)$ as a representation of $U_q(\mathfrak{g})$.*

For any real number b, $\operatorname{int} b$ denotes the greatest integer less than or equal to b.

Proposition 1.10 [2, Theorem 6.1] *Let $\mathfrak{g}$ be of type $C_2, \lambda \in P^+$ and let $V(\mathbf{P})$ be a minimal affinization of λ. Then, as a representation of $U_q(\mathfrak{g})$,*

$$V(\mathbf{P}) \cong \bigoplus_{r=0}^{int(\frac{1}{2}\lambda(2))} V_q(\lambda - 2r\lambda_2).$$

In the G_2 case we have less complete information.

Proposition 1.11 *Let $\mathfrak{g}$ be of type G_2 and let $m \in \mathbb{N}$. Let $V(\mathbf{P})$ (resp. $V(\mathbf{Q})$) be a minimal affinization of $m\lambda_1$ (resp. λ_2). Then, as $U_q(\mathfrak{g})$-modules,*

$$V(\mathbf{P}) \cong \bigoplus_{k=0}^{m} V_q(k\lambda_1), \qquad V(\mathbf{Q}) \cong V_q(\lambda_2).$$

(*The nodes of the Dynkin diagram of $\mathfrak{g}$ are numbered so that α_1 is the long root*).

Proof. See [3, p. 645] and [2, p. 125].

1.7 Quantum characters

Many basic questions remain unanswered about the structure of the finite dimensional representations of quantum affine algebras. A useful tool for studying these questions is the theory of q-characters developed in [10], [11].

Recall that if V is a finite-dimensional $U_q(\mathfrak{g})$-module, its character is the element of the ring $\mathbb{Z}[y_i^{\pm 1}]_{i\in I}$ defined by

$$\chi(V) = \sum_{\lambda \in P(V)} \dim V_\lambda \prod_{i\in I} y_i^{\lambda(i)}.$$

If Rep A denotes the Grothendieck ring of finite-dimensional modules for an algebra A over $\mathbb{C}$, then χ defines a ring homomorphism

$$\chi : \operatorname{Rep} U_q(\mathfrak{g}) \to \mathbb{Z}[y_i^{\pm 1}]_{i\in I}.$$

Turning now to the $U_q(\hat{\mathfrak{g}})$ case, note that it follows from the defining relations of $U_q(\hat{\mathfrak{g}})$ that the $\Phi_{i,r}^{\pm}$ ($i \in I, r \in \mathbb{Z}$) commute with each other on any finite-dimensional $U_q(\hat{\mathfrak{g}})$-module V of type 1. Hence, V can be decomposed into generalized eigenspaces

$$V = \bigoplus V_{(\gamma_{i,r}^{\pm})},$$

indexed by families $\gamma = (\gamma_{i,r}^{\pm})$ of complex numbers, where

$$V_{(\gamma_{i,r}^{\pm})} = \{v \in V \mid \forall\ i \in I,\ r \in \mathbb{Z},\ \exists p \in \mathbb{N} \text{ such that } (\Phi_{i,r}^{\pm} - \gamma_{i,r}^{\pm})^p.v = 0\}.$$

Given such a family γ, form the generating functions $\gamma_i^{\pm}(u) = \sum_{r=1}^{\infty} \gamma_{i,\pm r}^{\pm} u^{\pm r}$. By [11, Proposition 1], the $\gamma_i^{\pm}(u)$ have the form

$$\gamma_i^{\pm}(u) = q_i^{\deg Q_i - \deg R_i} \frac{Q_i(q_i^{-1}u^{-1})R_i(q_i u^{-1})}{Q_i(q_i u^{-1})R_i(q_i^{-1}u)},$$

where Q_i, R_i ($i \in I$) are polynomials in u with constant term 1. Write

$$Q_i(u) = \prod_{r=1}^{m_i}(1 - a_r u), \quad R_i(u) = \prod_{s=1}^{n_i}(1 - b_s u).$$

Definition 1.12 *Let $Y_{i,a}$, for $i \in I, a \in \mathbb{C}^*$, be invertible indeterminates. With the above notation, the q-character $\chi_q(V)$ is the element of the ring $\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^*}$ given by*

$$\chi_q(V) = \sum_\gamma \dim V_\gamma \prod_{i\in I}\prod_{r=1}^{m_i} Y_{i,a_r} \prod_{s=1}^{n_i} Y_{i,b_s}^{-1}.$$

Assign the weight λ_i to $Y_{i,a}$, and compute the weight of a monomial

$$\prod_{i\in I}\prod_{r=1}^{m_i} Y_{i,a_r} \prod_{s=1}^{n_i} Y_{i,b_s}^{-1}$$

by adding the weights of the factors. Such a monomial is said to be dominant if $n_i = 0$ for all $i \in I$.

To state the main results about χ_q we shall need, denote by

$$\beta : \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^*} \to \mathbb{Z}[y_i^{\pm 1}]_{i\in I}$$

the ring homomorphism taking $Y_{i,a}$ to y_i for all $i \in I, a \in \mathbb{C}^*$, and by

$$\mathrm{res} : \mathrm{Rep}\, U_q(\hat{\mathfrak{g}}) \to \mathrm{Rep}\, U_q(\mathfrak{g})$$

the restriction homomorphism. More generally, given $J \subset I$, denote by $U_q(\mathfrak{g})_J$ the subalgebra of $U_q(\mathfrak{g})$ generated by the $k_i^{\pm 1}, x_i^{\pm}$ for $i \in J$, and by $U_q(\hat{\mathfrak{g}})_J$ the subalgebra of $U_q(\hat{\mathfrak{g}})$ generated by $k_i^{\pm 1}, h_{i,r}, x_{i,r}^{\pm}$ for $i \in J, r \in \mathbb{Z}$. Let

$$\beta_J : \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^*} \to \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^*}$$

be the homomorphism such that $\beta_J(Y_{i,a}) = Y_{i,a}$ if $i \in J$, and $\beta_J(Y_{i,a}) = 1$ if $i \notin J$; let $\mathrm{res}_J : U_q(\hat{\mathfrak{g}})_J \to U_q(\mathfrak{g})_J$ be the restriction homomorphism; and define

$$\chi_{q,J} : \mathrm{Rep}\ U_q(\hat{\mathfrak{g}})_J \to \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in J, a\in\mathbb{C}^*}$$

in the obvious way.

Theorem 1.13 [11, Theorem 3]

1) χ_q *defines an injective homomorphism*

$$\chi_q : \mathit{Rep}\ U_q(\hat{\mathfrak{g}}) \to \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^*}.$$

2) *The diagram*

$$\begin{array}{ccc} Rep\ U_q(\hat{\mathfrak{g}}) & \xrightarrow{\chi_q} & \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^*} \\ res\big\downarrow & & \big\downarrow\beta \\ Rep\, U_q(\mathfrak{g}) & \xrightarrow{\chi} & \mathbb{Z}[y_i^{\pm 1}]_{i\in I} \end{array}$$

commutes.

3) *For any $J \subset I$, the diagram*

$$\begin{array}{ccc} Rep\ U_q(\hat{\mathfrak{g}}) & \xrightarrow{\chi_q} & \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^*} \\ res_J\big\downarrow & & \big\downarrow\beta_J \\ Rep\ U_q(\hat{\mathfrak{g}})_J & \xrightarrow{\chi_{q,J}} & \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in J, a\in\mathbb{C}^*} \end{array}$$

commutes.

In [11] the q-characters of the first fundamental representation (with standard numbering) are given. From these examples, and using the non-trivial diagram automorphism of A_2, we have in the type A_2 case:

$$\chi_q(V(1-aqu,0)) = Y_{1,aq} + Y_{2,aq^2}Y_{1,aq^3}^{-1} + Y_{2,aq^4}^{-1},$$
$$\chi_q(V(0,1-aqu)) = Y_{2,a} + Y_{1,aq}Y_{2,aq^2}^{-1} + Y_{1,aq^3}^{-1}.$$

Similarly, making use of the natural isomorphism between types B_2 and C_2, we have in type C_2:

$$\chi_q(V(1-aqu,0)) = Y_{1,a} + Y_{2,aq}Y_{2,aq^3}Y_{1,aq^4}^{-1} + Y_{2,aq}Y_{2,aq^5}^{-1} + Y_{1,aq^2}Y_{2,aq^3}^{-1}Y_{2,aq^5}^{-1} + Y_{1,aq^7}^{-1},$$
$$\chi_q(V(0,1-aqu)) = Y_{2,a} + Y_{2,aq^4}Y_{1,aq^5}^{-1} + Y_{1,aq}Y_{2,aq^2}^{-1} + Y_{2,aq^6}^{-1}.$$

These special cases will be used extensively in the following chapters.

2. Computations: the A_2 case

In this section, $\mathfrak{g}$ is of type A_2. We denote by $V_a(m,n)$ the minimal affinization of $m\lambda_1 + n\lambda_2$ in which the inverse roots of the polynomial P_2 form a q-segment with centre a (if $n=0$ the parameter a will denote the centre of the inverse roots of P_1).

2.1 Computation of $W_{aq}(0,n)$

Proposition 2.1 *There exists an exact sequence of $U_q(\hat{\mathfrak{g}})$-modules*

$$0 \to V_{aq^{-3}}(0,n-3) \to V_{aq^{-2}}(0,n-2) \otimes V_{aq^{n-2}}(1,0)$$
$$\downarrow$$
$$V_{aq^{-1}}(0,n-1) \otimes V_{aq^{n-1}}(0,1) \to V_a(0,n) \to 0.$$

Proof. By Proposition 1.8, $V_a(0,n)$ has polynomials

$$P_1(u) = 1, \qquad P_2(u) = \prod_{j=1}^{n}(1 - uaq^{-n-1+2j}).$$

From Proposition 1.4, $V_a(0,n)$ is a quotient of $V_c(0,n-1) \otimes V_b(0,1)$ for some $b, c \in C^*$. For this to happen the polynomials of $V_c(0,n-1), V_b(0,1)$ must multiply to P_1 and P_2. Now $V_c(0,n-1)$ has polynomials

$$Q_1(u) = 1, \quad Q_2(u) = \prod_{j=1}^{n-1}(1 - ucq^{-n+2j}),$$

and $V_b(0,1)$ has polynomials

$$R_1(u) = 1, \quad R_2(u) = 1 - ub.$$

So we must have $R_1Q_1 = P_1$, which happens automatically, and $R_2Q_2 = P_2$. It is necessary to set $b = aq^{n-1}, c = aq^{-1}$, and so

$$R_2(u) = 1 - uaq^{n-1}.$$

Consider now the kernel K_1 of the quotient map $V_c(0,n-1) \otimes V_b(0,1) \to V_a(0,n)$. As $U_q(\mathfrak{g})$-modules,

$$V_c(0,n-1) \cong V_q((n-1)\lambda_2),\ V_b(0,1) \cong V_q(\lambda_2),\ V_a(0,n) \cong V_q(n\lambda_2).$$

Now,

$$V_q((n-1)\lambda_2) \otimes V_q(\lambda_2) \cong V_q(n\lambda_2) \oplus V_q(\lambda_1 + (n-2)\lambda_2).$$

So $K_1 \cong V_q(\lambda_1 + (n-2)\lambda_2)$ as a $U_q(\mathfrak{g})$-module; in particular, K_1 is irreducible as a $U_q(\hat{\mathfrak{g}})$-module. Now, for any L-highest weight vector u in a $U_q(\hat{\mathfrak{g}})$-module, and any $i \in I, k \in \mathbb{Z}$, $x_{i,k}^-.u$ is a scalar multiple of $x_i^-.u$. It follows that K_1 is spanned by the vectors

$$x_2^-.v \otimes w, v \otimes x_2^-.w,$$

with v highest weight in $V_c(0,n-1)$ and w highest weight in $V_b(0,1)$. Thus, any non-zero L-highest weight vector $z \in K_1$ is of the form

$$z = Ax_2^-.v \otimes w + Bv \otimes x_2^-.w$$

for some $A, B \in \mathbb{C}$. Since K_1 is irreducible as a $U_q(\hat{\mathfrak{g}})$-module, it is generated by z as a $U_q(\hat{\mathfrak{g}})$-module. Using Lemma 1.2, for any $k > 0$,

$$\begin{aligned} h_{i,k}.z &= \Delta(h_{i,k}).z \\ &= A(h_{i,k}x_2^-.v \otimes w + x_2^-.v \otimes h_{i,k}.w + \text{ a term in } U'_+x_2^-.v \otimes U'_-.w) \\ &+ B(h_{i,k}.v \otimes x_2^-.w + v \otimes h_{i,k}x_2^-.w + \text{ a term in } U'_+.v \otimes U'_-x_2^-.w). \end{aligned}$$

Since v is L-highest weight, it follows that $U'_+.v \otimes U'_-x_2^-.w$ is zero and $U'_+x_2^-.v \otimes U'_-.w$ is a scalar multiple of $v \otimes x_2^-.w$. So

$$h_{i,k}.z = A(h_{i,k}x_2^-.v \otimes w + x_2^-.v \otimes h_{i,k}.w) + \text{ scalar multiples of } v \otimes x_2^-.w.$$

Since we know that z is an eigenvector of $h_{i,k}$, it follows that to calculate $h_{i,k}.z$ it suffices to calculate $h_{i,k}x_2^-.v \otimes w + x_2^-.v \otimes h_{i,k}.w$.

Using the relations from Theorem 1.1, we have

$$\begin{aligned} h_{i,k}.z &= A(h_{i,k}x_2^-.v \otimes w + x_2^-.v \otimes h_{i,k}.w) \\ &= A\left(-\frac{1}{k}[ka_{i,2}]_q x_{2,k}^-.v \otimes w + x_2^- h_{i,k}.v \otimes w + x_2^-.v \otimes h_{i,k}.w\right). \end{aligned} \tag{1}$$

To calculate this it is necessary to use Theorems 1.1 and 1.3, which give

$$\sum_{k=1}^{\infty} u^k h_{i,k}.v = \frac{1}{q-q^{-1}} \mathrm{Log}\left(\frac{Q_i(q^{-2}u)}{Q_i(u)}\right) v.$$

Since $Q_1(u) = 1 = R_1(u)$, this gives $h_{1,k}.v = 0$ and $h_{1,k}.w = 0$. Now

$$\frac{Q_2(uq^{-2})}{Q_2(u)} = \frac{1 - uaq^{-n-1}}{1 - uaq^{n-3}}$$

so

$$\begin{aligned} \sum_{k=1}^{\infty} u^k h_{2,k}.v &= \frac{1}{q-q^{-1}} \mathrm{Log}\left(\frac{1 - uaq^{-n-1}}{1 - uaq^{n-3}}\right) v \\ &= \frac{1}{q-q^{-1}} (\mathrm{Log}(1 - uaq^{-n-1}) - \mathrm{Log}(1 - uaq^{n-3}))v. \end{aligned}$$

Hence, comparing coefficients of u^k,

$$h_{2,k}.v = \frac{a^k}{k(q-q^{-1})}\left(q^{k(n-3)} - q^{-k(n+1)}\right) v. \tag{2}$$

Using the fact that

$$\frac{R_2(uq^{-2})}{R_2(u)} = \frac{1 - uaq^{n-3}}{1 - uaq^{n-1}},$$

similar calculations give

$$h_{2,k}.w = \frac{a^k}{k(q-q^{-1})}\left(q^{k(n-1)} - q^{k(n-3)}\right) w.$$

We must now calculate the term $x^-_{2,k}.v$. For some $C_k \in \mathbb{C}$ $(k > 0)$, we have

$$x^-_{2,k}.v = C_k x^-_2.v.$$

Applying x^+_2 gives

$$\frac{\Phi^+_{2,k}}{q - q^{-1}}.v = C_k \frac{q^{n-1} - q^{-n+1}}{q - q^{-1}} v.$$

By Theorem 1.3,

$$\begin{aligned}\sum_{k=0}^{\infty} u^k \Phi^+_{2,k} v &= q^{\deg Q_2} \frac{Q_2(uq^{-2})}{Q_2(u)} v \\ &= q^{n-1}(1 - uaq^{-n-1})(1 - uaq^{n-3})^{-1} v,\end{aligned}$$

and comparing coefficients of u^k we get

$$\Phi^+_{2,k}.v = a^k q^{k(n+1)}(q^{n-1} - q^{-n+1})v,$$

so

$$\frac{a^k q^{k(n+1)}(q^{n-1} - q^{-n+1})}{(q - q^{-1})} v = C_k \frac{q^{n-1} - q^{-n+1}}{q - q^{-1}} v$$

hence

$$C_k = a^k q^{k(n-3)}.$$

Finally, we have

$$x^-_{2,k}.v = a^k q^{k(n-3)} x^-_2.v.$$

Now, for $i = 1$ the term in braces on the right-hand side of equation (1) becomes

$$\frac{[k]_q}{k} x^-_{2,k}.v \otimes w + x^-_2 h_{1,k}.v \otimes w + x^-_2.v \otimes h_{1,k}.w$$

so

$$h_{1,k}.z = \frac{a^k}{k(q - q^{-1})} \left(q^{k(n-2)} - q^{k(n-4)}\right) z,$$

while for $i = 2$ it becomes

$$-\frac{[2k]_q}{k} x^-_{2,k}.v \otimes w + x^-_2 h_{2,k}.v \otimes w + x^-_2.v \otimes h_{2,k}.w$$

so

$$\begin{aligned}h_{2,k}.z &= \frac{a^k}{k} \left(q^{k(n-5)} - q^{k(n-1)} + q^{k(n-1)} - q^{k(n-3)} + q^{k(n-3)} - q^{-k(n+1)}\right) z \\ &= \frac{a^k}{k(q - q^{-1})}(q^{k(n-5)} - q^{-k(n+1)}) z.\end{aligned}$$

These expressions for $h_{1,k}.z$ and $h_{2,k}.z$ are enough to determine the defining polynomials S_1, S_2 of K_1 via

$$\sum_{k=1}^{\infty} u^k h_{i,k}.z = \frac{1}{q-q^{-1}} \mathrm{Log}\left(\frac{S_i(q^{-2}u)}{S_i(u)}\right) z.$$

This gives

$$S_1(u) = 1 - uaq^{n-2}, \quad S_2(u) = \prod_{j=1}^{n-2} (1 - uaq^{-n-1+2j}),$$

and we have an exact sequence

$$0 \to V(S_1, S_2) \to V_{aq^{-1}}(0, n-1) \otimes V_{aq^{n-1}}(0,1) \to V_a(0,n) \to 0.$$

The next step is to get K_1 as a quotient of the tensor product of evaluation representations

$$V_d(1,0) \otimes V_e(0, n-2).$$

Now, $V_d(1,0)$ has polynomials

$$Q_1(u) = 1 - ud, \quad Q_2(u) = 1,$$

and $V_e(0, n-1)$ has polynomials

$$R_1(u) = 1, \quad R_2(u) = \prod_{j=1}^{n-2} (1 - ueq^{-n+1+2j}),$$

so by Proposition 1.4, for K_1 to be in the quotient of this tensor product we must have $S_1 = R_1Q_1, S_2 = R_2Q_2$, which gives $d = aq^{n-2}$ and $e = aq^{-2}$. Thus, we have an exact sequence

$$0 \to K_2 \to V_{aq^{n-2}}(1,0) \otimes V_{aq^{-2}}(0, n-2) \to K_1 \to 0, \tag{3}$$

say.

Let $v' \neq 0$ be an L-highest weight vector in $V_{aq^{-2}}(0, n-2)$ and $w' \neq 0$ an L-highest weight vector in $V_{aq^{n-2}}(1,0)$. Using the $U_q(\mathfrak{g})$-structure of these modules and the fact that

$$x_1^-.v' = x_2^-.w' = 0,$$

an L-highest weight vector in K_2 must have the form

$$z' = A'x_1^-x_2^-.v' \otimes w' + B'v' \otimes x_2^-x_1^-.w' + C'x_2^-.v' \otimes x_1^-.w'$$

for some $A', B', C' \in \mathbb{C}$. As in the proof of Proposition 2.7 below, one sees that K_2 is generated by z' as a $U_q(\hat{\mathfrak{g}})$-module. Let the L-highest weight of z' be given by the polynomials T_1, T_2. Then,

$$\begin{aligned} h_{i,k}.z' &= \Delta(h_{i,k}).z' \\ &= A'(h_{i,k}x_1^- x_2^-.v' \otimes w' + x_1^- x_2^-.v' \otimes h_{i,k}.w' + \text{ a term in } U'_+ x_1^- x_2^-.v' \otimes U'_-.w') \\ &+ B'(h_{i,k}.v' \otimes x_2^- x_1^-.w' + v' \otimes h_{i,k} x_2^- x_1^-.w') \\ &+ C'(h_{i,k}x_2^-.v' \otimes x_1^-.w' + x_2^-.v' \otimes h_{i,k}x_1^-.w' + \text{ a term in } U'_+ x_2^-.v' \otimes U'_- x_1^-.w'). \end{aligned}$$

Note that $U'_+ x_1^- x_2^-.v' \otimes U'_-.w'$ gives unknown scalar multiples of $x_2^-.v' \otimes x_1^-.w'$ and $v' \otimes x_2^- x_1^-.w'$, and $U'_+ x_2^-.v' \otimes U'_- x_1^-.w'$ gives unknown scalar multiples of $v' \otimes x_2^- x_1^-.w'$. Hence, to calculate $h_{i,k}.z'$ we consider the terms involving $x_1^- x_2^-.v' \otimes w'$. Hence, we must calculate

$$h_{i,k}x_1^- x_2^-.v' \otimes w' + x_1^- x_2^-.v' \otimes h_{i,k}.w'. \tag{4}$$

Now,

$$\begin{aligned} \frac{Q_1(q^{-2}u)}{Q_1(u)} &= \frac{1 - uaq^{n-4}}{1 - uaq^{n-2}}, \quad \frac{Q_2(q^{-2}u)}{Q_2(u)} = 1, \\ \frac{R_1(q^{-2}u)}{R_1(u)} &= 1, \quad \frac{R_2(q^{-2}u)}{R_2(u)} = \frac{1 - uaq^{-n-1}}{1 - uaq^{n-5}}. \end{aligned}$$

which gives

$$\begin{aligned} &h_{1,k}.v' = 0, \quad h_{1,k}.w' = \frac{a^k}{k(q - q^{-1})}(q^{k(n-2)} - q^{k(n-4)})w', \\ &h_{2,k}.v' = \frac{a^k}{k(q - q^{-1})}(q^{k(n-5)} - q^{k(-n-1)})v', \quad h_{2,k}.w' = 0, \\ &x_{2,k}^-.v' = a^k q^{k(n-5)} x_2^-.v', \quad x_{1,k}^-.w' = a^k q^{k(n-2)} x_1^-.w'. \end{aligned}$$

To calculate (4), it is also necessary to find an expression for $x_{1,k}^- x_2^-.v'$. For some $D'_k \in \mathbb{C}$,

$$x_{1,k}^- x_2^-.v' = D'_k x_1^- x_2^-.v'.$$

Applying x_1^+ gives

$$\frac{\Phi_{1,k}^+}{q - q^{-1}} x_2^-.v' = D'_k x_2^-.v'.$$

By Theorem 3.1,

$$\begin{aligned}\sum_{k=0}^{\infty} u^k \Phi^+_{1,k} x_2^- .v' &= k_1 \exp\left((q-q^{-1}) \sum_{s=1}^{\infty} h_{1,s} u^s \right) x_2^- .v' \\ &= q \exp\left(\sum_{s=1}^{\infty} \frac{u^s a^s}{s} (q^{s(n-4)} - q^{s(n-6)}) \right) x_2^- .v' \\ &= q \frac{(1-auq^{n-6})}{(1-auq^{n-4})} x_2^- .v',\end{aligned}$$

so

$$\Phi^+_{1,k} x_2^- .v' = a^k q^{k(n-4)} \ (q-q^{-1}) x_2^- .v'.$$

This gives $D'_k = a^k q^{k(n-4)}$, and hence

$$x^-_{1,k} x_2^- .v' = a^k q^{k(n-4)} x_1^- x_2^- .v'.$$

For $i=1$, (4) now becomes

$$-\frac{[2k]_q}{k} x^-_{1,k} x^-_2 .v' \otimes w' + x_1^- x_2^- h_{1,k} .v' \otimes w' + \frac{[k]_q}{k} x_1^- x^-_{2,k} .v' \otimes w' + x_1^- x_2^- .v' \otimes h_{1,k} .w',$$

which gives

$$\begin{aligned}h_{1,k}.z' &= \frac{a^k}{k(q-q^{-1})} (q^{k(n-6)} - q^{k(n-2)} + q^{k(n-4)} - q^{k(n-6)} + q^{k(n-2)} - q^{k(n-4)}) z' \\ &= 0.\end{aligned}$$

For $i=2$, (4) becomes

$$\frac{[k]_q}{k} x^-_{1,k} x^-_2 .v' \otimes w' + x_1^- x_2^- h_{2,k} .v' \otimes w' - \frac{[2k]_q}{k} x_1^- x^-_{2,k} .v' \otimes w' + x_1^- x_2^- .v' \otimes h_{2,k} .w'$$

which gives

$$\begin{aligned}h_{2,k}.z' &= \frac{a^k}{k(q-q^{-1})} (q^{k(n-3)} - q^{k(n-5)} + q^{k(n-5)} - q^{k(-n-1)} + q^{k(n-7)} - q^{k(n-3)}) z' \\ &= \frac{a^k}{k(q-q^{-1})} (q^{k(n-7)} - q^{k(-n-1)}) z'.\end{aligned}$$

These values for $h_{1,k}.z'$ and $h_{2,k}.z'$ give the polynomials for K_2 as

$$T_1(u) = 1, \quad T_2(u) = \prod_{j=1}^{n-3} (1 - uaq^{-n-1+2j}) = \prod_{j=1}^{n-3} (1 - uaq^{-3} q^{-n+2+2j}).$$

By Proposition 1.8, K_2 is isomorphic to the minimal affinization $V_{aq^{-3}}(0, n-3)$, and we have an exact sequence

$$0 \to V_{aq^{-3}}(0, n-3) \to V_{aq^{-2}}(0, n-2) \otimes V_{aq^{n-3}}(1, 0) \to K_1 \to 0.$$

Putting this sequence together with sequence (3) proves Proposition 2.1.

Denote by $W_a(m,n)$ the q-character $\chi_q(V_a(m,n))$ of the $U_q(\hat{\mathfrak{g}})$-module $V_a(m,n)$. The following corollary is an immediate consequence of Proposition 2.1.

Corollary 2.2 *For* $a \in \mathbb{C}^*, n \geq 0$,

$$W_a(0,n) = W_{aq^{-1}}(0,n-1)W_{aq^{n-1}}(0,1) - W_{aq^{-2}}(0,n-2)W_{aq^{n-2}}(1,0) + W_{aq^{-3}}(0,n-3).$$

(*Terms of the form* $W_b(0,m)$ *with* $m < 0$ *are understood to be zero.*)

This corollary enables us to compute $W_a(0,n)$ by induction on n. The proof is straightforward, if lengthy.

Proposition 2.3 *For* $a \in \mathbb{C}^*, n \geq 0$, *the q-character* $W_{aq}(0,n)$ *is given by*

$$\sum_{i=1}^{n+1} \prod_{j=1}^{n+1-i} Y_{2,aq^{2j-(n+1)}} \sum_{k=1}^{i} \prod_{r=1}^{i-k} Y_{1,aq^{n-2(r+k-2)}} Y^{-1}_{2,aq^{n+1-2(r+k-2)}} \prod_{s=1}^{k-1} Y^{-1}_{1,aq^{n-2(s-2)}}.$$

2.2 Computation of $W_{aq}(1,n)$

The proof of the following proposition is similar to that of Proposition 2.1.

Proposition 2.4 *For* $a \in \mathbb{C}^*, n \geq 1$, *there exists an exact sequence of* $U_q(\hat{\mathfrak{g}})$-*modules*

$$0 \to V_{aq}(0,n-1) \to V_{aq^{-n-2}}(1,0) \otimes V_a(0,n) \to V_a(1,n) \to 0.$$

The following result is immediate from Proposition 2.4.

Corollary 2.5 *For* $a \in \mathbb{C}^*, n \geq 0$,

$$W_a(1,n) = W_a(0,n)W_{aq^{-n-2}}(1,0) - W_{aq}(0,n-1).$$

(*Here,* $W_{aq}(0,n)$ *is interpreted as zero if* $n < 0$).

This leads to

Proposition 2.6 *For* $a \in \mathbb{C}^*, n \geq 0,$

$$W_{aq}(1,n) = \left(Y_{1,aq^{-n-2}} + Y_{2,aq^{-n-1}}Y^{-1}_{1,aq^{-n}}\right) W_{aq}(0,n)$$
$$+ Y^{-1}_{2,aq^{-n+1}} \sum_{k=1}^{n+1} \prod_{r=1}^{n+1-k} Y_{1,aq^{n-2(r+k-2)}} Y^{-1}_{2,aq^{n+1-2(r+k-2)}} \prod_{s=1}^{k-1} Y^{-1}_{1,aq^{n-2(s-2)}}.$$

2.3 Computation of $W_{aq}(m,n)$

Proposition 2.7 *For* $m, n \geq 0, a \in \mathbb{C}^*$*, there exists an exact sequence of* $U_q(\hat{\mathfrak{g}})$*-modules*

$$0 \to V_a(m-3,n) \to V_{aq^{-2m-n+1}}(0,1) \otimes V_a(m-2,n)$$
$$\downarrow$$
$$V_{aq^{-2m-n}}(1,0) \otimes V_a(m-1,n) \to V_a(m,n) \to 0.$$

Proof. We include some of the details of this proof as it involves a new feature (a simpler version of which was actually used in the proof of Proposition 2.1). First, we find as in the proof of Proposition 2.1 that $V_a(m,n)$ is a quotient of the tensor product of evaluation representations

$$V_c(1,0) \otimes V_d(m-1,n),$$

where the polynomials for $V_d(m-1,n)$ are

$$R_1(u) = \prod_{j=1}^{m-1}(1 - uaq^{-2m-n+2j}), \quad R_2(u) = \prod_{j=1}^{n}(1 - uaq^{-n-1+2j})$$

and those of $V_c(1,0)$ are

$$Q_1(u) = 1 - uaq^{-2m-n}, \quad Q_2(u) = 1.$$

Let $v \neq 0$ be an L-highest weight vector in $V_{aq^{-2m-n)}}(1,0)$ and $w \neq 0$ an L-highest weight vector in $V_a(m-1,n)$. As $U_q(\mathfrak{g})$-modules

$$V_a(m,n) \cong V_q(m\lambda_1 + n\lambda_2)$$
$$V_c(1,0) \cong V_q(\lambda_1)$$
$$V_d(m-1,n) \cong V_q((m-1)\lambda_1 + n\lambda_2).$$

We have

$$V_q(\lambda_1) \otimes V_q((m-1)\lambda_1 + n\lambda_2) \cong$$
$$V_q((m-2)\lambda_1 + (n+1)\lambda_2) \oplus V_q((m-1)\lambda_1 + (n-1)\lambda_2) \oplus V_q(m\lambda_1 + n\lambda_2).$$

So the kernel K of the quotient map

$$V_c(1,0)\otimes V_d(m-1,n)\to V_a(m,n)$$

is isomorphic to $V_q((m-2)\lambda_1+(n+1)\lambda_2)\oplus V_q((m-1)\lambda_1+(n-1)\lambda_2)$ as a $U_q(\mathfrak{g})$-module. Let z be a non-zero vector in K of weight $(m-2)\lambda_1+(n+1)\lambda_2$. As this is the maximal weight in K, z is L-highest weight. Then z generates K as a $U_q(\hat{\mathfrak{g}})$-module. Indeed, let

$$J=U_q(\hat{\mathfrak{g}}).z\subseteq K.$$

As a $U_q(\mathfrak{g})$-module, either

$$J\cong V_q((m-2)\lambda_1+(n+1)\lambda_2)$$

or

$$J\cong V_q((m-2)\lambda_1+(n+1)\lambda_2)\oplus V_q((m-1)\lambda_1+(n-1)\lambda_2).$$

In the former case, J would be an evaluation representation; but the calculations below will show the L-highest weight of z is not that of an evaluation representation. This contradiction shows that $J=K$, i.e. that K is generated by z. (A similar argument is also applicable to the C_2 case in the next section.)

Using the $U_q(\mathfrak{g})$-structure of these modules given above,

$$z=Ax_1^-.v\otimes w+Bv\otimes x_1^-.w$$

for some $A,B\in\mathbb{C}$. Then

$$\begin{aligned}h_{i,k}.z&=\Delta(h_{i,k}).z\\&=A(h_{i,k}x_1^-.v\otimes w+x_1^-.v\otimes h_{i,k}.w+\text{ a term in }U'_+x_1^-.v\otimes U'_-.w)\\&+B(h_{i,k}.v\otimes x_1^-.w+v\otimes h_{i,k}x_1^-.w).\end{aligned}$$

But, $U'_+x_1^-.v\otimes U'_-.w$ gives unknown scalar multiples of $v\otimes x_1^-.w$, so in order to calculate $h_{i,k}.z$ we calculate

$$h_{i,k}x_1^-.v\otimes w+x_1^-.v\otimes h_{i,k}.w. \tag{5}$$

Now,

$$\begin{aligned}\frac{Q_1(q^{-2}u)}{Q_1(u)}&=\frac{1-uaq^{-2m-n-2}}{1-uaq^{-n-2m}},\quad\frac{Q_2(q^{-2}u)}{Q_2(u)}=1,\\\frac{R_1(q^{-2}u)}{R_1(u)}&=\frac{1-uaq^{-2m-n}}{1-uaq^{-n-2}},\quad\frac{R_2(q^{-2}u)}{R_2(u)}=\frac{1-uaq^{-n-1}}{1-uaq^{n-1}},\end{aligned}$$

which gives

$$h_{1,k}.w = \frac{a^k}{k(q-q^{-1})}(q^{k(-n-2)} - q^{k(-2m-n)})w,$$
$$h_{1,k}.v = \frac{a^k}{k(q-q^{-1})}(q^{k(-2m-n)} - q^{k(-2m-n-2)})v,$$
$$h_{2,k}.w = \frac{a^k}{k(q-q^{-1})}(q^{k(n-1)} - q^{k(-n-1)})w, \quad h_{2,k}.v = 0,$$
$$x^-_{1,k}.v = a^k q^{k(-n-2m)} x^-_1.v.$$

For $i = 1$, (5) becomes

$$-\frac{[2k]_q}{k} x^-_{1,k}.v \otimes w + x^-_1 h_{1,k}.v \otimes w + x^-_1.v \otimes h_{1,k}.w$$

which gives

$$h_{1,k}.z = \frac{a^k}{k(q-q^{-1})}(q^{k(-n-2)} - q^{k(-2m-n+2)})z.$$

For $i = 2$, (5) becomes

$$\frac{[k]_q}{k} x^-_{1,k}.v \otimes w + x^-_1 h_{2,k}.v \otimes w + x^-_1.v \otimes h_{2,k}.w,$$

which gives

$$h_{2,k}.z = \frac{a^k}{k(q-q^{-1})}(q^{k(-2m-n+1)} - q^{k(-2m-n-1)} + q^{k(n-1)} - q^{k(-n-1)})z.$$

These values for $h_{1,k}.z$ and $h_{2,k}.z$ give the polynomials for K as

$$S_1(u) = \prod_{j=2}^{m-1}(1 - uaq^{-2m-n+2j}) = \prod_{j=1}^{m-2}(1 - uaq^{-2m-n+2+2j}),$$
$$S_2(u) = (1 - ua^{-2m-n+1})\prod_{j=1}^{n}(1 - uaq^{-n-1+2j}).$$

This implies that K is *not* an evaluation representation, as the roots of S_2 do not form a q-segment.

Now the evaluation representation $V_a(m-2,n)$ has polynomials

$$T_1(u) = \prod_{j=1}^{m-2}(1 - uaq^{-2m-n+2j}), \quad T_2(u) = \prod_{j=1}^{n}(1 - uaq^{-n-1+2j}),$$

so K is a quotient of the tensor product

$$V_{aq^{-2m-n+1}}(0,1) \otimes V_a(m-2,n),$$

where $V_{aq^{-2m-n+1}}(0,1)$ has polynomials

$$Q_1(u) = 1, \quad Q_2(u) = 1 - uaq^{-2m-n+1}.$$

Let K_1 be the kernel of this quotient map. Let $v' \neq 0$ be an L-highest weight vector in $V_{aq^{-2m-n+1)}}(0,1)$ and $w' \neq 0$ an L-highest weight vector in $V_a(m-2,n)$. Using the $U_q(\mathfrak{g})$-structure of these modules and the fact that $x_1^-.v' = 0$, a highest weight vector in K_1 must have the form

$$z' = A'x_1^-x_2^-.v' \otimes w' + B'v \otimes x_1^-x_2^-.w' + C'x_2^-.v \otimes x_1^-.w' + D'x_1^-x_2^-.v' \otimes x_1^-x_2^-.w'$$

for some $A', B', C', D' \in \mathbb{C}$. So, in order to calculate $h_{i,k}.z$, we calculate

$$h_{i,k}x_1^-x_2^-.v' \otimes w' + x_1^-x_2^-.v' \otimes h_{i,k}.w'. \tag{6}$$

Now, if (U_1, U_2) are the polynomials for $V_{aq^{-2m-n+1}}(0,1)$, we have

$$\frac{U_1(q^{-2}u)}{U_1(u)} = 1, \quad \frac{U_2(q^{-2}u)}{U_2(u)} = \frac{1-uaq^{-2m-n-1}}{1-uaq^{-2m-n+1}},$$
$$\frac{T_1(q^{-2}u)}{T_1(u)} = \frac{1-uaq^{-2m-n+2}}{1-uaq^{-n-2}}, \quad \frac{T_2(q^{-2}u)}{T_2(u)} = \frac{1-uaq^{-n-1}}{1-uaq^{n-1}},$$

which gives

$$h_{1,k}.v' = 0, \quad h_{1,k}.w' = \frac{a^k}{k(q-q^{-1})}(q^{k(-n-2)} - q^{k(-2m-n+2)})w',$$
$$h_{2,k}.w' = \frac{a^k}{k(q-q^{-1})}(q^{k(n-1)} - q^{k(-n-1)})w',$$
$$h_{2,k}.v' = \frac{a^k}{k(q-q^{-1})}(q^{k(-2m-n+1)} - q^{k(-2m-n-1)})v',$$
$$x_{2,k}^-.v' = a^k q^{k(-n-2m+1)}x_2^-.v'.$$

To calculate (6) it is also necessary to find an expression for $x_{1,k}^-x_2^-.v'$. For some scalars E_k' $(k > 0)$, we have

$$x_{1,k}^-x_2^-.v' = E_k'x_1^-x_2^-.v'.$$

Applying x_1^+ gives

$$\frac{\Phi_{1,k}^+}{q-q^{-1}}x_2^-.v' = E_k'x_2^-.v'.$$

By Theorem 1.3,

$$\begin{aligned}\sum_{k=0}^{\infty} u^k \Phi_{1,k}^+ x_2^-.v' &= k_1 \exp\left((q-q^{-1})\sum_{s=1}^{\infty} h_{1,s}u^s\right) x_2^-.v' \\ &= q\exp\left((q-q^{-1})\sum_{s=1}^{\infty} u^s \frac{[s]_q}{s}\right) x_2^-.v' \\ &= q\frac{(1-auq^{-2m-n})}{(1-auq^{-2m-n+2})}x_2^-.v'\end{aligned}$$

so

$$\Phi^+_{1,k} x^-_2 .v' = a^k q^{k(-2m-n+2)} x^-_2 .v'.$$

This gives

$$E'_k = a^k q^{k(-2m-n+2)}$$

and hence

$$x^-_{1,k} x^-_2 .v' = a^k q^{k(-2m-n+2)} x^-_1 x^-_2 .v'.$$

For $i = 1$, (6) now becomes

$$-\frac{[2k]_q}{k} x^-_{1,k} x^-_2 .v' \otimes w' + x^-_1 x^-_2 h_{1,k}.v' \otimes w' + \frac{[k]_q}{k} x^-_1 x^-_{2,k}.v' \otimes w' + x^-_1 x^-_2 .v' \otimes h_{1,k}.w',$$

which gives

$$h_{1,k}.z' = \frac{a^k}{k(q-q^{-1})}(q^{k(-n-2)} - q^{k(-2m-n+4)})z'.$$

For $i = 2$, (6) becomes

$$\frac{[k]_q}{k} x^-_{1,k} x^-_2 .v' \otimes w' + x^-_1 x^-_2 h_{2,k}.v' \otimes w' - \frac{[2k]_q}{k} x^-_1 x^-_{2,k}.v' \otimes w' + x^-_1 x^-_2 .v' \otimes h_{2,k}.w',$$

which gives

$$h_{2,k}.z' = \frac{a^k}{k(q-q^{-1})}(q^{k(n-1)} - q^{k(-n-1)})z'.$$

These values for $h_{1,k}.z'$ and $h_{2,k}.z'$ give the polynomials for K_1 as

$$V_1(u) = \prod_{j=2}^{m-2}(1 - uaq^{-2m-n+2+2j}) = \prod_{j=1}^{m-3}(1 - uaq^{-2m-n+4+2j}),$$

$$V_2(u) = \prod_{j=1}^{n}(1 - uaq^{-n-1+2j}).$$

The roots of V_1 (resp. V_2) form a q-segment with centre $a^{-1}q^{n+m-2}$ (resp. a^{-1}), and since

$$\frac{a^{-1}}{a^{-1}q^{n+m-2}} = q^{-n-m+2},$$

K_1 is an evaluation representation by Proposition 1.8.

We now have two exact sequences

$$0 \to K \to V_{aq^{-2m-n}}(1,0) \otimes V_a(m-1,n) \to V_a(m,n) \to 0$$

$$0 \to V_a(m-3,n) \to V_{aq^{-2m-n+1}}(0,1) \otimes V_a(m-2,n) \to K \to 0,$$

and putting these together completes the proof of Proposition 2.7.

Corollary 2.8 *Let $a \in \mathbb{C}^*, m, n \geq 0$. Then*

$$W_a(m,n) = W_a(m-1,n)W_{aq^{-2m-n}}(1,0) - W_a(m-2,n)W_{aq^{-2m-n+1}}(0,1) + W_a(m-3,n).$$

This leads to

Theorem 2.9 *Let* $a \in \mathbb{C}^*, m, n \geq 0$. *Then,* $W_{aq}(m,n) = A_{a,n} \sum_{i=1}^{m} \prod_{j=1}^{m-i} Y_{1,aq^{2j-n-2(m+1)}}$
$\sum_{k=1}^{i} \prod_{r=1}^{i-k} Y_{2,aq^{-n-3-2(r+k-2)}} Y^{-1}_{1,aq^{-n-2-2(r+k-2)}} \prod_{s=1}^{k-1} Y^{-1}_{2,aq^{-n-1-2(s-1)}}$
$+ W_{aq}(0,n) \sum_{i=1}^{m+1} \prod_{j=1}^{m+1-i} Y_{1,aq^{2j-n-2(m+1)}} \prod_{r=1}^{i-1} Y_{2,aq^{-n-1-2(r-1)}} Y^{-1}_{1,aq^{-n-2(r-1)}}$, *where*

$$A_{a,n} = Y^{-1}_{2,aq^{-n+1}} \sum_{k=1}^{n+1} \prod_{r=1}^{n+1-k} Y_{1,aq^{n-2(r+k-2)}} Y^{-1}_{2,aq^{n+1-2(r+k-2)}} \prod_{s=1}^{k-1} Y^{-1}_{1,aq^{n-2(s-2)}}.$$

3. The C_2 case

In this section, $\mathfrak{g}$ denotes a complex simple Lie algebra of type C_2. We take

$$A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \quad d_1 = 2, \ d_2 = 1.$$

If $m > 0, n \geq 0$, we denote by $V_a(m,n)$ the irreducible representation of $U_q(\hat{\mathfrak{g}})$ having polynomials (P_1, P_2) with $\deg P_1 = m > 0$ and $\deg P_2 = n$ and in which the inverse roots of P_1 form a q-segment with centre a. If $m = 0, n > 0$ the parameter a will denote the centre of the q^2-segment formed by the inverse roots of P_2. As the proofs are similar to those in the A_2 case, we omit many of the details.

3.1 Some exact sequences

The proof of the following proposition is similar to that of Proposition 2.1.

Proposition 3.1 *Let* $a \in \mathbb{C}^*, r \geq 1$. *There exist exact sequences of* $U_q(\hat{\mathfrak{g}})$*-modules*

$$0 \to V_{aq^4}(r-1,2) \to V_{aq^{-2r}}(1,0) \otimes V_{aq^2}(r,0) \to V_a(r+1,0) \to 0,$$

$$0 \to V_{aq^2}(r-2,0) \to V_a(r-1,0) \otimes V_{aq^{-2r-2}}(1,0)$$
$$\downarrow$$
$$V_{aq^{-2r-4}}(0,1) \otimes V_a(r-1,1) \to V_a(r-1,2) \to 0,$$

$$0 \to V_{aq^2}(r-2,1) \to V_{aq^{-2r-2}}(0,1) \otimes V_a(r-1,0) \to V_a(r-1,1) \to 0.$$

Corollary 3.2 *Let* $a \in \mathbb{C}^*, r \geq 0$. *Then,*

$$W_a(r+1,0) = W_{aq^2}(r,0)W_{aq^{-2r}}(1,0) - W_{aq^4}(r-1,2),$$

$$W_a(r-1,2) = W_a(r-1,1)W_{aq^{-2r-4}}(0,1) - W_a(r-1,0)W_{aq^{-2r-2}}(1,0) + W_{aq^2}(r-2,0),$$

$$W_a(r-1,1) = W_a(r-1,0)W_{aq^{-2r-2}}(0,1) - W_{aq^2}(r-2,1).$$

(*Terms involving* $W_a(r,s)$ *with* $r < 0$ *are understood to be zero.*)

Corollary 3.2 gives an inductive scheme for calculating $W_a(r+1,0), W_a(r-1,2)$ and $W_a(r+1,1)$, starting with the formula for $W_a(1,0)$ which is given in Section 1.7. In the next section, we shall obtain a similar scheme for calculating $W_a(r,s)$ for arbitrary $r, s \geq 0$.

3.2 Computation of $W_a(r,s)$

Proposition 3.3 *For* $a \in \mathbb{C}^*, r, s \geq 0$, *there is an exact sequence of* $U_q(\hat{\mathfrak{g}})$*-modules*

$$0 \to V_a(r,s-5) \to V_a(r,s-4) \otimes V_{aq^{-2r-2s}}(0,1)$$
$$\downarrow$$
$$V_{aq^{-2r-2s}}(1,0) \otimes V_a(r,s-3)$$
$$\downarrow$$
$$V_{aq^{-2r-2s-2}}(1,0) \otimes V_a(r,s-1)$$
$$\downarrow$$
$$V_{aq^{-2r-2s-4}}(0,1) \otimes V_a(r,s) \to V_a(r,s+1) \to 0.$$

(*Terms of the form* $V_b(r,s)$ *with* r *or* $s < 0$ *are understood to be zero.*)

Proof. As above, we find that the minimal affinization $V_a(r,s+1)$ is a quotient of the tensor product of evaluation representations

$$V_{aq^{-2r-2s-4}}(0,1) \otimes V_a(r,s).$$

Let $v \neq 0$ be an L-highest weight vector in $V_{aq^{-2r-2s-4}}(0,1)$ and $w \neq 0$ an L-highest weight vector in $V_a(r,s)$. Using the $U_q(\mathfrak{g})$-structure of these modules, a highest weight vector in the kernel K of the quotient map

$$V_{aq^{-2r-2s-4}}(0,1) \otimes V_a(r,s) \to V_a(r,s+1)$$

has the form

$$z = Ax_2^-.v \otimes w + Bv \otimes x_2^-.w$$

for some $A, B \in \mathbb{C}$. Arguing as in the proof of Proposition 2.7, we find that the polynomials associated to z are

$$S_1(u) = (1 - uaq^{-2r-2s-2}) \prod_{j=1}^{r}(1 - uaq^{2r+2-4j}), \quad S_2(u) = \prod_{j=1}^{s-1}(1 - uaq^{-2r-2-2j}).$$

This implies that K is *not* a minimal affinization as the roots of S_1 do not form a q^2-segment. But K appears in the quotient of the tensor product of

$$V_{aq^{-2r-2s-2}}(1,0) \otimes V_a(r, s-1).$$

where $V_{aq^{-2r-2}}(1,0)$ has polynomials

$$T_1(u) = 1 - uaq^{-2r-2-2s}, \quad T_2(u) = 1,$$

and $V_a(r, s-1)$ has polynomials

$$U_1(u) = \prod_{j=1}^{r}(1 - uaq^{2r+2-4j}), \quad U_2(u) = \prod_{j=1}^{s-1}(1 - uaq^{-2r-2-2j}).$$

Let K_1 be the kernel of the quotient map

$$V_{aq^{-2r-2s-2}}(1,0) \otimes V_a(r, s-1) \to K.$$

Arguing as above, we find that a $U_q(\hat{\mathfrak{g}})$-highest weight vector in K_1 must have polynomials

$$S_1^{'}(u) = (1 - uaq^{-2r-2s}) \prod_{j=1}^{r}(1 - uaq^{2r+2-4j}), \quad S_2^{'}(u) = \prod_{j=1}^{s-3}(1 - uaq^{-2r-2-2j}).$$

This implies that K_1 is *not* a minimal affinization as the roots of $S_1(u)$ do not form a q^2-segment. But K_1 appears as a quotient of the tensor product

$$V_{aq^{-2r-2s}}(1,0) \otimes V_a(r, s-3),$$

where $V_{aq^{-2r-2s}}(1,0)$ has polynomials

$$T_1'(u) = 1 - uaq^{-2r-2}, \quad T_2'(u) = 1,$$

and $V_a(r, s-1)$ has polynomials

$$U_1'(u) = \prod_{j=1}^{r}(1 - uaq^{2r+2-4j}), \quad U_2'(u) = \prod_{j=1}^{s-3}(1 - uaq^{-2r-2-2j}).$$

Let K_2 be the kernel of the quotient map

$$V_{aq^{-2r-2s}}(1,0) \otimes V_a(r,s-3) \to K_1.$$

We find that a $U_q(\hat{\mathfrak{g}})$-highest weight vector in K_2 must have polynomials

$$T_1''(u) = \prod_{j=1}^{r}(1-uaq^{2r+2-4j}), \quad T_2''(u) = (1-uaq^{-2r-2s})\prod_{j=1}^{s-4}(1-uaq^{-2r-2-2j}).$$

This implies that K_2 is *not* a minimal affinization as the roots of $S_1(u)$ do not form a q^2-segment. But K_2 appears as a quotient of the tensor product

$$V_{aq^{-2r-2s}}(0,1) \otimes V_a(r,s-4),$$

where $V_{aq^{-2r-2s}}(0,1)$ has polynomials

$$W_1(u) = 1, \quad W_2(u) = 1-uaq^{-2r-2s}$$

and $V_a(r,s-4)$ has polynomials

$$W_1'(u) = \prod_{j=1}^{r}(1-uaq^{2r-4j+2}), \quad W_2'(u) = \prod_{j=1}^{s-4}(1-uaq^{-2r-2j-2}).$$

Let K_3 be the kernel of this quotient map. Dualizing the inclusion map

$$K_3 \to V_{aq^{-2r-2s}}(0,1) \otimes V_a(r,s-4)$$

gives a non-zero homomorphism of $U_q(\hat{\mathfrak{g}})$-modules

$$V_a(r,s-4) \to K_3 \otimes V^t_{aq^{-2r-2s}}(0,1).$$

By Lemma 1.6,

$$V^t_{aq^{-2r-2}}(0,1) \cong V_{aq^{-2r-2s+6}}(0,1),$$

so we have a non-zero $U_q(\hat{\mathfrak{g}})$-module homomorphism

$$V_a(r,s-4) \to K_3 \otimes V_{aq^{-2r-2s+6}}(0,1).$$

If K_3 has polynomials (U_1'', U_2''),

$$(1-uaq^{-2r-2s+6})U_2''(u) = \prod_{j=1}^{s-4}(1-uaq^{-2r-2-2j}).$$

So K_3 has polynomials

$$U_1''(u) = \prod_{j=1}^{r}(1-uaq^{2r+2-4j}), \quad U_2''(u) = \prod_{j=1}^{s-5}(1-uaq^{-2r-2-2j}).$$

Hence, the inverse roots of U_1'' form a q^2-segment with centre a, and those of U_2'' form a q-segment with centre $aq^{-2r-s+2}$. By Proposition 1.8,

$$K_3 \cong V_a(r, s-5).$$

Summarizing, we have the following exact sequences of $U_q(\hat{\mathfrak{g}})$-modules:

$$0 \to K \to V_{aq^{-2r-2s-4}}(0,1) \otimes V_a(r,s) \to V_a(r,s+1) \to 0,$$
$$0 \to K_1 \to V_{aq^{-2r-2}}(1,0) \otimes V_a(r,s-1) \to K \to 0,$$
$$0 \to K_2 \to V_{aq^{-2r-2s}}(1,0) \otimes V_a(r,s-3) \to K_1 \to 0,$$
$$0 \to V_a(r,s-5) \to V_a(r,s-4) \otimes V_{aq^{-2r-2s}}(0,1) \to K_2 \to 0,$$

and the proposition follows.

Corollary 3.4 *For $a \in \mathbb{C}^*, r, s \geq 0$, we have*

$$W_a(r,s+1) = W_{aq^{-2r-2s-4}}(0,1)W_a(r,s) + W_{aq^{-2r-2s}}(1,0)W_a(r,s-3) - W_{aq^{-2r-2s-2}}(1,0)W_a(r,s-1) - W_a(r,s-5).$$

4. The G$_2$ Case

In this section $\mathfrak{g}$ denotes a complex simple Lie algebra of type G_2. We take

$$A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \quad d_1 = 3,\ d_2 = 1.$$

We have

$$\alpha_1 = 2\lambda_1 - 3\lambda_2, \quad \alpha_2 = -\lambda_1 + 2\lambda_2 \quad \theta = 2\alpha_1 + 3\alpha_2 = \lambda_1.$$

4.1 A special notation

We shall attempt to understand the $U_q(\mathfrak{g})$-structure of certain minimal affinizations of $U_q(\hat{\mathfrak{g}})$. Since the necessary calculations are rather complex, we have simplified their presentation by employing the following ‘approximation’ notation: if $v_1, \ldots, v_M, v_1', \ldots, v_N'$ are vectors in a vector space V, we write

$$v_1 + \cdots + v_M \approx v_1' + \cdots + v_N'$$

to mean that there exist $t_1, \ldots, t_M, t'_1, \ldots, t'_N \in C^*$ such that

$$\sum_{r=1}^{M} t_r v_r = \sum_{s=1}^{N} t'_s v'_s.$$

For example, we shall make extensive use of the defining relations

$$x^{\pm}_{i,r+1} x^{\pm}_{j,s} - q^{\pm a_{ij}} x^{\pm}_{j,s} x^{\pm}_{i,r+1} = q^{\pm a_{ij}} x^{\pm}_{i,r} x^{\pm}_{j,s+1} - x^{\pm}_{j,s+1} x^{\pm}_{i,r}, \tag{7}$$

$$\sum_{\pi \in \sum_m} \sum_{k=0}^{m} (-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_{q_i} x^{\pm}_{i,r_{\pi(1)}} \ldots x^{\pm}_{i,r_{\pi(k)}} x^{\pm}_{j,s} x^{\pm}_{i,r_{\pi(k+1)}} \ldots x^{\pm}_{i,r_{\pi(m)}} = 0 \quad \text{if } i \neq j. \tag{8}$$

Using the approximation notation, (7) becomes

$$x^{\pm}_{i,r+1} x^{\pm}_{j,s} - x^{\pm}_{j,s} x^{\pm}_{i,r+1} \approx x^{\pm}_{i,r} x^{\pm}_{j,s+1} - x^{\pm}_{j,s+1} x^{\pm}_{i,r}. \tag{9}$$

In particular,

$$x^{\pm}_{i,\pm 1} x_i \approx x_i x^{\pm}_{i,\pm 1}. \tag{10}$$

Similarly, we have the following special cases of (8):

$$(x^+_1)^2 x^+_2 + x^+_1 x^+_2 x^+_1 + x^+_2 (x^+_1)^2 \approx 0, \tag{11}$$

$$(x^+_2)^4 x^+_1 + (x^+_2)^3 x^+_1 x^+_2 + (x^+_2)^2 x^+_1 (x^+_2)^2 + x^+_2 x^+_1 (x^+_2)^3 + x^+_1 (x^+_2)^4 \approx 0. \tag{12}$$

4.2 Multiplicities

Let $\lambda = r\lambda_1 + \lambda_2 \in P^+$, with $r \geq 0$, and let $V(\mathbf{P})$ be a minimal affinization of λ. Choose $\mathbf{Q}$ and $\mathbf{R}$ such that $V(\mathbf{Q})$ is a minimal affinization of $r\lambda_1$, $V(\mathbf{R})$ is a minimal affinization of λ_2 and $V(\mathbf{P})$ is isomorphic to a quotient of $V(\mathbf{Q}) \otimes V(\mathbf{R})$.

Lemma 4.1 *As $U_q(\mathfrak{g})$-modules, we have*
$V_q(r\lambda_1) \otimes V_q(\lambda_2) \cong V_q(r\lambda_1 + \lambda_2) \oplus V_q(r\lambda_1 + \lambda_2 - \theta) \oplus V_q(r\lambda_1 + \lambda_2 - (\alpha_1 + \alpha_2))$.

Proof. In view of the results of Section 1.4, it suffices to prove the analogous classical result, which we leave to the reader.

Proposition 4.2 *As $U_q(\mathfrak{g})$-modules, we have*

$$V(\mathbf{Q}) \otimes V(\mathbf{R}) \cong V_q(\lambda) \oplus \bigoplus_{j=1}^{r} \left[2V_q(\lambda - j\theta) \oplus V_q(\lambda - (\alpha_1 + \alpha_2) - (j-1)\theta) \right].$$

Proof. By Proposition 1.11, as $U_q(\mathfrak{g})$-modules,

$$V(\mathbf{Q}) \cong \bigoplus_{j=0}^{r} V_q((r-j)\lambda_1), \quad V(\mathbf{R}) \cong V_q(\lambda_2).$$

The result now follows from Lemma 4.1.

For any $U_q(\mathfrak{g})$-module V, set

$$V_\lambda^+ = \left\{ v \in V_\lambda | x_i^+ . v = 0 \text{ for all } i \in I \right\}, \quad m_\lambda(V) = \dim V_\lambda^+,$$

so that $m_\lambda(V)$ is the multiplicity of $V_q(\lambda)$ in V.

Proposition 4.3 *Let $\mu \in P^+$. Then,*

(i) $m_\mu(V(\mathbf{P})) \leq 1$ *if μ is of the form $\lambda - n\theta - (\alpha_1 + \alpha_2)$ for some $n \in \mathbb{N}$;*

(ii) $m_\mu(V(\mathbf{P})) \leq 2$ *if μ is of the form $\lambda - n\theta$ for some $n \in \mathbb{N}$;*

(iii) $m_\mu(V(\mathbf{P})) = 0$ *if μ is not of the form $\lambda - n\theta - (\alpha_1 + \alpha_2)$ or $\lambda - n\theta$ for any $n \in \mathbb{N}$;*

(iv) $m_{\lambda - n\theta}(V(\mathbf{P})) \geq 1$ *for $0 \leq n \leq r$.*

Proof. Parts (i), (ii) and (iii) follow from Proposition 4.2 and the fact that $V(\mathbf{P})$ is isomorphic to a quotient of $V(\mathbf{Q}) \otimes V(\mathbf{R})$.

To show part (iv), let the roots of the polynomial P_1 be $p_1, \ldots, p_r$. Each fundamental representation $V(1-up_j, 1)$ can be identified with $V_q(\lambda_1)$ as a $U_q(\mathfrak{g})$-module; let $v_{\lambda_1} \in V_q(\lambda_1)$ be a non-zero $U_q(\mathfrak{g})$-highest weight vector. Then, for $1 \leq j \leq r$, there exists a non-zero vector $w_j \in V(1-up_j, 1)_0^+$ such that $x_0^- . w_j = v_{\lambda_1}$. This follows since $V(1-up_j, 1) \cong V(\lambda_1) \oplus \mathbb{C}$ as a $U_q(\mathfrak{g})$-module. For $1 \leq j \leq r$, consider

$$w = w_1 \otimes w_2 \ldots \otimes w_j \otimes v_{\lambda_1}^{\otimes r-j} \otimes v_{\lambda_2},$$

so that

$$w \in V(1-up_1, 1) \otimes \ldots \otimes V(1-up_r, 1) \otimes V(0, P_2).$$

Clearly, $x_1^+ . w = x_2^+ . w = 0$. A lengthy but straightforward computation gives

$$(x_0^-)^j . w = q^{3j(j-1)/2} [j]_{q^3} v_{\lambda_1}^{\otimes^r} \otimes v_{\lambda_2},$$

which is non-zero. Since $V(\mathbf{P})$ is isomorphic to a quotient of $V(\mathbf{Q}) \otimes V(\mathbf{R})$ and $V(\mathbf{Q})$ to a quotient of $V(1-up_1, 1) \otimes \ldots \otimes V(1-up_r, 1)$, there exists a surjective homomorphism of $U_q(\hat{\mathfrak{g}})$-modules

$$\pi : V(1-up_1,1)\otimes\ldots\otimes V(1-up_r,1)\otimes V(0,P_2)\to V(\mathbf{P})$$

such that

$$\pi(v_{\lambda_1}^{\otimes r}\otimes v_{\lambda_2}) = v_{\mathbf{P}},$$

where $v_{\mathbf{P}}\neq 0$ is an L-highest weight vector in $V(\mathbf{P})$. Hence,

$$\pi((x_0^-)^j.w)\neq 0,$$

and so $\pi(w)$ is a non-zero element of $V(\mathbf{P})^+_{\lambda-j\theta}$.

To prove the next proposition the following two lemmas are needed. Recall that $V(\mathbf{P})$ is an affinization of $\lambda\in P^+$.

Lemma 4.4 *Suppose there exists a non-zero vector $v\in V(\mathbf{P})^+_\mu$ such that*

$$x^+_{i,1}.v = x^+_{i,-1}.v = 0 \quad (i=1,2).$$

Assume that $m_{\mu+\alpha_i}(V(\mathbf{P}))=0$ $(i=1,2)$. Then, $\lambda=\mu$.

For a proof of this lemma see [2, Lemma 7.2].

Lemma 4.5 *Let $\mu\in P^+$ be such that $m_{\mu+\nu}(V(\mathbf{P}))=0$ if ν is not of the form $s\theta$ for any $s\in\mathbb{N}$. Then, $x_1^+(x_2^+)^3x^+_{1,\pm1}$ maps $V(\mathbf{P})^+_\mu$ to $V(\mathbf{P})^+_{\mu+\theta}$ and, if $v\in V(\mathbf{P})^+_\mu$ is such that $x^+_{1,\pm1}.v\neq 0$, then $x_1^+(x_2^+)^3x^+_{1,\pm1}.v\neq 0$.*

Proof. It is obvious for weight reasons that $x_1^+(x_2^+)^3x^+_{1,\pm1}$ maps $V(\mathbf{P})^+_\mu$ to $V(\mathbf{P})_{\mu+\theta}$. Let $v\in V(\mathbf{P})^+_\mu$ be such that

$$x_i^+x_1^+(x_2^+)^3x^+_{1,\pm1}.v = 0 \quad (i=1,2).$$

For $i=1$ we consider

$$(x_1^+)^2(x_2^+)^3x^+_{1,\pm1}.$$

By (11), we have

$$(x_1^+)^2x_2^+ \approx x_1^+x_2^+x_1^+ + x_2^+(x_1^+)^2,$$

so

$$\begin{aligned}(x_1^+)^2(x_2^+)^3x^+_{1,\pm1} &\approx x_1^+x_2^+x_1^+(x_2^+)^2x^+_{1,\pm1} + x_2^+(x_1^+)^2(x_2^+)^2x^+_{1,\pm1}\\ &\approx x_1^+x_2^+x_1^+(x_2^+)^2x^+_{1,\pm1} + (x_2^+x_1^+)^2x_2^+x^+_{1,\pm1} + (x_2^+)^2(x_1^+)^2x_2^+x^+_{1,\pm1},\end{aligned}$$

by (11) again. But, by (9),

$$x_2^+ x_{1,\pm1}^+ \approx (x_{1,\pm1}^+ x_2^+ + x_{2,\mp1}^+ x_1^+ + x_1^+ x_{2,\mp1}^+),$$

so we have

$$(x_1^+)^2(x_2^+)^3 x_{1,\pm1}^+ \approx ((x_1^+ x_2^+)^2 + (x_2^+ x_1^+)^2 + (x_2^+)^2(x_1^+)^2)(x_{1,\pm1}^+ x_2^+ + x_{2,\mp1}^+ x_1^+ + x_1^+ x_{2,\mp1}^+).$$

Now, by (11),

$$x_1^+ x_2^+ x_1^+ x_2^+ x_1^+ x_{2,\mp1}^+ \approx x_1^+ x_2^+ (x_1^+)^2 x_2^+ x_{2,\mp1}^+ + x_1^+ (x_2^+)^2 (x_1^+)^2 x_{2,\mp1}^+,$$

so

$$(x_1^+)^2(x_2^+)^3 x_{1,\pm1}^+ \in \ U_q(\hat{\mathfrak{g}}) x_2^+ + \ U_q(\hat{\mathfrak{g}}) x_1^+ + \ U_q(\hat{\mathfrak{g}})(x_1^+)^2 x_{2,\mp1}^+ + \ U_q(\hat{\mathfrak{g}}).x_2^+ x_{2,\mp1}^+.$$

But, by (11) again,

$$(x_1^+)^2 x_{2,\mp1}^+ \in \ U_q(\hat{\mathfrak{g}}) x_1^+, \quad x_2^+ x_{2,\mp1}^+ \in \ U_q(\hat{\mathfrak{g}}) x_2^+,$$

so

$$(x_1^+)^2(x_2^+)^3 x_{1,\pm1}^+ \in \ U_q(\hat{\mathfrak{g}}) x_2^+ + \ U_q(\hat{\mathfrak{g}}) x_1^+.$$

Hence, for $v \in V(\mathbf{P})_\mu^+$ we have

$$(x_1^+)^2(x_2^+)^3 x_{1,\pm1}^+.v = 0.$$

For $i = 2$ we consider $x_2^+ x_1^+ (x_2^+)^3 x_{1,\pm1}^+$. As above, we have

$$x_2^+ x_1^+ (x_2^+)^3 x_{1,\pm1}^+ \approx (x_2^+)^4 x_1^+ x_{1,\pm1}^+ + (x_2^+)^3 x_1^+ x_2^+ x_{1,\pm1}^+ + x_1^+ (x_2^+)^4 x_{1,\pm1}^+ + (x_2^+)^2 x_1^+ (x_2^+)^2 x_{1,\pm1}^+.$$

The first three terms on the right-hand side are contained in $U_q(\hat{\mathfrak{g}}) x_2^+ + \ U_q(\hat{\mathfrak{g}}) x_1^+$, so we consider

$$(x_2^+)^2 x_1^+ (x_2^+)^2 x_{1,\pm1}^+ \approx (x_2^+)^2 x_1^+ x_2^+ (x_{1,\pm1}^+ x_2^+ + x_{2,\mp1}^+ x_1^+ + x_1^+ x_{2,\mp1}^+).$$

Now

$$\begin{aligned}(x_2^+)^2 x_1^+ x_2^+ x_1^+ x_{2,\mp1}^+ &\approx (x_2^+)^2 (x_1^+)^2 x_2^+ x_{2,\mp1}^+ + (x_2^+)^3 (x_1^+)^2 x_{2,\mp1}^+ \\ &\in \ U_q(\hat{\mathfrak{g}}) x_2^+ + \ U_q(\hat{\mathfrak{g}}) x_1^+,\end{aligned}$$

hence

$$x_2^+ x_1^+ (x_2^+)^3 x_{1,\pm1}^+ \in \ U_q(\hat{\mathfrak{g}}).x_2^+ + \ U_q(\hat{\mathfrak{g}}).x_1^+.$$

It follows that

$$x_2^+ x_1^+ (x_2^+)^3 x_{1,\pm1}^+.v = 0,$$

and so $x_1^+ (x_2^+)^3 x_{1,\pm1}^+.v \in V(\mathbf{P})_{\mu+\theta}^+$, as required.

Now suppose that $x^+_{1,\pm1}.v \neq 0$ and $x^+_1(x^+_2)^3x^+_{1,\pm1}.v = 0$. Since

$$x^+_2(x^+_2)^3x^+_{1,\pm1} = (x^+_2)^4x^+_{1,\pm1} \in U_q(\hat{\mathfrak{g}})x^+_2,$$

we have

$$x^+_2(x^+_2)^3x^+_{1,\pm1}.v = 0.$$

This implies that $(x^+_2)^3x^+_{1,\pm1}.v \in V(\mathbf{P})^+_{\mu+3\alpha_2+\alpha_1} = 0$ by assumption, hence

$$(x^+_2)^3x^+_{1,\pm1}.v = 0.$$

Now consider

$$\begin{aligned} x^+_1(x^+_2)^2x^+_{1,\pm1} &\approx x^+_1x^+_2x^+_1x^+_{2,\mp1} \\ &\approx (x^+_1)^2x^+_2x^+_{2,\mp1} + x^+_2(x^+_1)^2x^+_{2,\mp1} \\ &\approx (x^+_1)^2x^+_{2,\mp1}x^+_2 + (x^+_2)^2(x^+_2)^2 + (x^+_2x^+_1)^2 \\ &\in U_q(\hat{\mathfrak{g}})x^+_2 + U_q(\hat{\mathfrak{g}})x^+_1. \end{aligned}$$

So $x^+_1(x^+_2)^2x^+_{1,\pm1}.v = 0$ which gives

$$(x^+_2)^2x^+_{1,\pm1}.v \in V(\mathbf{P})^+_{\mu+2\alpha_2+\alpha_1} = 0,$$

and so

$$(x^+_2)^2x^+_{1,\pm1}.v = 0.$$

Next, consider

$$\begin{aligned} x^+_1x^+_2x^+_{1,\pm1} &\approx x^+_1x^+_{1,\pm1}x^+_2 + x^+_{1,\pm1}x^+_1x^+_2 + x^+_2x^+_1x^+_{1,\pm1} + x^+_2x^+_1x^+_{1,\pm1} \\ &\in U_q(\hat{\mathfrak{g}})x^+_2 + U_q(\hat{\mathfrak{g}})x^+_1. \end{aligned}$$

So $x^+_1x^+_2x^+_{1,\pm1}.v = 0$ which gives

$$x^+_2x^+_{1,\pm1}.v \in V(\mathbf{P})^+_{\mu+\alpha_2+\alpha_1} = 0,$$

and so

$$x^+_2x^+_{1,\pm1}.v = 0.$$

Finally, consider $x^+_1x^+_{1,\pm1}$. Since this $\in U_q(\hat{\mathfrak{g}})x^+_1$, we have

$$x^+_1x^+_{1,\pm1}.v = 0,$$

so that $x^+_{1,\pm1}.v \in V(\mathbf{P})^+_{\mu+\alpha_1} = 0$, which implies that $x^+_{1,\pm1}.v = 0$. This completes the proof of Lemma 4.5.

Proposition 4.6 *Let $V(\mathbf{P})$ be a minimal affinization of $\lambda \in P^+$ with the roots of P_1 forming a q^2-segment with inverse centre a. Then:*

(i) $m_{\lambda-n\theta}(V(\mathbf{P})) = 1$ *for* $0 \le n \le r$*;*

(ii) $m_\mu(V(\mathbf{P})) = 0$ *if μ is of the form $\lambda - n\theta - (\alpha_1+\alpha_2)$ for some $n \in \mathbb{N}$.*

Proof. Both parts are proved by induction on n. The $n = 0$ case is proved in [2]. Assume that (i) and (ii) are proved for all integers $\le n$.

Suppose that $m_{\lambda-(n+1)\theta}(V(\mathbf{P})) > 1$. By Lemma 4.5, the action of $x_1^+(x_2^+)^3x_{1,\pm1}^+$ defines a map

$$V(\mathbf{P})^+_{\lambda-(n+1)\theta} \to V(\mathbf{P})^+_{\lambda-n\theta}.$$

By the induction hypothesis, $m_{\lambda-n\theta}(V(\mathbf{P})) = 1$ so this map has a non-zero kernel. This implies that there exists $0 \ne v \in V(\mathbf{P})^+_{\lambda-(n+1)\theta}$ such that

$$x_1^+(x_2^+)^3x_{1,1}^+.v = 0$$

which, by Lemma 4.5, gives $x_{1,1}^+.v = 0$.

Suppose now that $x_{1,-1}^+.v \ne 0$ and, for $s = 0, 1, \dots, n+1$, define $v_s \in V(\mathbf{P})$ by

$$v_s = (x_1^+(x_2^+)^3x_{1,-1}^+)^s.v.$$

We prove, by induction on s, that v_s has the following properties:

$$\begin{aligned}&(\mathrm{i})_s \quad 0 \ne v_s \in V(\mathbf{P})^+_{\lambda-(n+1-s)\theta} \ \ (0 \le s \le n+1),\\ &(\mathrm{ii})_s \quad x_{i,k}^+.v_s = 0 \quad (i = 1,2,\ \ k \ge 0).\end{aligned}$$

Note that (i)$_0$ holds by assumption, and that (ii)$_0$ holds by the choice of v. Since $x_{1,1}^+.v_s = 0$ and $v_s \ne 0$ for $s = 1, \dots, r$, Lemma 4.4 implies that $x_{1,-1}^+.v_s \ne 0$ and Lemma 4.5 then implies that

$$x_1^+(x_2^+)^3x_{1,-1}^+.v_s \ne 0,$$

so $v_{s+1} \ne 0$ which gives (i)$_{s+1}$.

To prove that (ii)$_{s+1}$ holds it suffices, by the proof of Lemma 4.4, to show that

$$x_{1,1}^+.v_{s+1} = x_{2,1}^+.v_{s+1} = 0.$$

Consider

$$x_{2,1}^+x_1^+(x_2^+)^3x_{1,-1}^+ \approx x_1^+x_{2,1}^+(x_2^+)^3x_{1,-1}^+ + x_{1,1}^+(x_2^+)^4x_{1,-1}^+ + x_2^+x_{1,1}^+(x_2^+)^3x_{1,-1}^+.$$

The first term on the right-hand side is contained in

$$U_q(\hat{\mathfrak{g}})x^+_{2,1} + U_q(\hat{\mathfrak{g}})x^+_2 + U_q(\hat{\mathfrak{g}})x^+_1$$

and the second term is contained in $U_q(\hat{\mathfrak{g}})x^+_2$. Now

$$x^+_2 x^+_{1,1}(x^+_2)^3 x^+_{1,-1} \approx (x^+_2)^4 x^+_{1,1}x^+_{1,-1} + (x^+_2)^3 x^+_{1,1}x^+_2 x^+_{1,-1} + (x^+_2)^2 x^+_{1,1}(x^+_2)^2 x^+_{1,-1}.$$

The first two terms are contained in $U_q(\hat{\mathfrak{g}})x^+_1 + U_q(\hat{\mathfrak{g}})x^+_{1,1}$. Next, consider

$$(x^+_2)^2 x^+_{1,1}(x^+_2)^2 x^+_{1,-1} \approx (x^+_2)^2 x^+_2 x^+_{1,1}x^+_2 x^+_{1,-1} + (x^+_2)^2 x^+_{2,1}x^+_1 x^+_2 x^+_{1,-1} + (x^+_2)^2 x^+_1 x^+_{2,1}x^+_2 x^+_{1,-1}.$$

The first term is contained in $U_q(\hat{\mathfrak{g}})x^+_1 + U_q(\hat{\mathfrak{g}})x^+_{1,1}$ and the second term is contained in $U_q(\hat{\mathfrak{g}})x^+_1$. Finally,

$$(x^+_2)^2 x^+_1 x^+_{2,1}x^+_2 x^+_{1,-1} \approx (x^+_2)^2 x^+_1 x^+_2 (x^+_{1,-1}x^+_{2,1} + x^+_1 x^+_2 + x^+_2 x^+_1),$$

and all three terms are contained in

$$U_q(\hat{\mathfrak{g}})x^+_{2,1} + U_q(\hat{\mathfrak{g}})x^+_2 + U_q(\hat{\mathfrak{g}})x^+_1.$$

So $x^+_{2,1}x^+_1(x^+_2)^3 x^+_{1,-1}$ is contained in

$$U_q(\hat{\mathfrak{g}})x^+_{2,1} + U_q(\hat{\mathfrak{g}})x^+_2 + U_q(\hat{\mathfrak{g}})x^+_1 + U_q(\hat{\mathfrak{g}})x^+_{1,1}.$$

Hence,

$$x^+_{2,1}x^+_1(x^+_2)^3 x^+_{1,-1}.v_s = 0,$$

which gives

$$x^+_{2,1}.v_{s+1} = 0.$$

Now consider

$$\begin{aligned} x^+_{1,1}x^+_1(x^+_2)^3 x^+_{1,-1} &\approx x^+_1 x^+_{1,1}(x^+_2)^3 x^+_{1,-1} \\ &\approx x^+_1 x^+_2 x^+_{1,1}(x^+_2)^2 x^+_{1,-1} + x^+_1 x^+_{2,1}x^+_1(x^+_2)^2 x^+_{1,-1} + (x^+_1)^2 x^+_{2,1}(x^+_2)^2 x^+_{1,-1} \\ &= \mathbf{1} + \mathbf{2} + \mathbf{3}, \end{aligned}$$

say. Now,

$$\mathbf{1} \in U_q(\hat{\mathfrak{g}})x^+_{1,1}x^+_2 x^+_{1,-1} + U_q(\hat{\mathfrak{g}})x^+_{2,1}x^+_2 x^+_{1,-1} + U_q(\hat{\mathfrak{g}})x^+_1 x^+_2 x^+_{1,-1},$$

so

$$\mathbf{1} \in \sum_{i=1}^{2}(U_q(\hat{\mathfrak{g}})x^+_i + U_q(\hat{\mathfrak{g}})x^+_{i,1}).$$

Next,

$$\begin{aligned}\mathbf{2} &\approx (x_1^+)^2 x_{2,1}^+ (x_2^+)^2 x_{1,-1}^+ + x_{2,1}^+ (x_1^+)^2 (x_2^+)^2 x_{1,-1}^+ \\ &\approx (x_1^+)^2 (x_2^+)^2 x_{2,1}^+ x_{1,-1}^+ + x_{2,1}^+ (x_1^+ x_2^+)^2 x_{1,-1}^+ + x_{2,1}^+ x_2^+ (x_1^+)^2 x_{2,1}^+ x_{1,-1}^+,\end{aligned}$$

so

$$\mathbf{2} \in\ U_q(\hat{\mathfrak{g}}) x_{2,1}^+ +\ U_q(\hat{\mathfrak{g}}) x_2^+ +\ U_q(\hat{\mathfrak{g}}) x_1^+.$$

Finally,

$$\begin{aligned}\mathbf{3} &\approx (x_1^+)^2 (x_2^+)^2 x_{2,1}^+ x_{1,-1}^+ \\ &\approx (x_1^+)^2 (x_2^+)^2 (x_{1,-1}^+ x_{2,1}^+ + x_2^+ x_1^+ + x_1^+ x_2^+),\end{aligned}$$

so

$$\mathbf{3} \in\ U_q(\hat{\mathfrak{g}}) x_{2,1}^+ +\ U_q(\hat{\mathfrak{g}}) x_2^+ +\ U_q(\hat{\mathfrak{g}}) x_1^+.$$

Hence,

$$x_{1,1}^+ x_1^+ (x_2^+)^3 x_{1,-1}^+ \in\ U_q(\hat{\mathfrak{g}}) x_{2,1}^+ +\ U_q(\hat{\mathfrak{g}}) x_2^+ +\ U_q(\hat{\mathfrak{g}}) x_{1,1}^+ +\ U_q(\hat{\mathfrak{g}}) x_1^+,$$

which gives

$$x_{1,1}^+ x_1^+ (x_2^+)^3 x_{1,-1}^+ . v_s = 0$$

and so

$$x_{1,1}^+ . v_{s+1} = 0,$$

which implies that (ii)$_{s+1}$ holds. We have now proved that (i)$_s$ and (ii)$_s$ hold for all $s \geq 0$.

Note that $v_{n+1} = A v_{\mathbf{P}}$ for some $A \in \mathbb{C}^*$. Since $\dim(V(\mathbf{P})_{\lambda-\alpha_1}) = 1$, it follows that

$$(x_2^+)^3 x_{1,-1}^+ . v_n = B x_1^- . v_{\mathbf{P}}$$

for some $B \in \mathbb{C}$. Applying $x_{1,3}^+$ gives

$$x_{1,3}^+ (x_2^+)^3 x_{1,-1}^+ . v_n = B x_{1,3}^+ x_1^- . v_{\mathbf{P}}.$$

But

$$\begin{aligned}x_{1,3}^+ (x_2^+)^3 x_{1,-1}^+ &\approx x_2^+ x_{1,3}^+ (x_2^+)^2 x_{1,-1}^+ + x_{2,1}^+ x_{1,2}^+ (x_2^+)^2 x_{1,-1}^+ + x_{1,2}^+ x_{2,1}^+ (x_2^+)^2 x_{1,-1}^+ \\ &= \mathbf{4} + \mathbf{5} + \mathbf{6},\end{aligned}$$

say. Now,

$$\mathbf{4} \approx (x_2^+)^2 x_{1,3}^+ x_2^+ x_{1,-1}^+ + x_2^+ x_{1,2}^+ x_{2,1}^+ x_2^+ x_{1,-1}^+ + x_2^+ x_{2,1}^+ x_{1,2}^+ x_2^+ x_{1,-1}^+.$$

But, for $k \geq 1$,

$$x^+_{1,k}x^+_2x^+_{1,-1} \approx x^+_{1,k}x^+_{1,-1}x^+_2 + x^+_{1,-1}x^+_2x^+_{1,k} + x^+_{1,-1}x^+_{1,k}x^+_2 + x^+_2x^+_{1,-1}x^+_{1,k} + x^+_2x^+_{1,k}x^+_{1,-1}$$
$$\in\ U_q(\hat{\mathfrak{g}})x^+_2 +\ U_q(\hat{\mathfrak{g}})x^+_k +\ U_q(\hat{\mathfrak{g}})x^+_{k-1} +\ U_q(\hat{\mathfrak{g}})x^+_1$$

(since $x^+_2x^+_{1,k}x^+_{1,-1} \approx x^+_2x^+_{1,-1}x^+_{1,k} + x^+_2x^+_1x^+_{1,k-1} + x^+_2x^+_{1,k}x^+_1$) and

$$x^+_2x^+_{1,2}x^+_{2,1}x^+_2x^+_{1,-1} \approx x^+_2x^+_{1,2}x^+_2(x^+_{1,-1}x^+_{2,1} + x^+_2x^+_1 + x^+_1x^+_2),$$

so

$$\mathbf{4} \in\ U_q(\hat{\mathfrak{g}})x^+_{2,1} +\ U_q(\hat{\mathfrak{g}})x^+_2 +\ U_q(\hat{\mathfrak{g}})x^+_{1,3} +\ U_q(\hat{\mathfrak{g}})x^+_{1,2} +\ U_q(\hat{\mathfrak{g}})x^+_{1,1} +\ U_q(\hat{\mathfrak{g}})x^+_1.$$

Next,

$$\mathbf{5} \approx x^+_{2,1}x^+_2x^+_{1,2}x^+_2x^+_{1,-1} + (x^+_{2,1})^2x^+_{1,1}x^+_2x^+_{1,-1} + x^+_{2,1}x^+_{1,1}x^+_{2,1}x^+_2x^+_{1,-1},$$

so

$$\mathbf{5} \in\ U_q(\hat{\mathfrak{g}})x^+_{2,1} +\ U_q(\hat{\mathfrak{g}})x^+_2 +\ U_q(\hat{\mathfrak{g}})x^+_{1,2} +\ U_q(\hat{\mathfrak{g}})x^+_1,$$

and

$$\mathbf{6} \in\ U_q(\hat{\mathfrak{g}})x^+_{2,1} +\ U_q(\hat{\mathfrak{g}})x^+_2 +\ U_q(\hat{\mathfrak{g}})x^+_1.$$

Finally,

$$x^+_{1,3}(x^+_2)^3x^+_{1,-1} \in\ U_q(\hat{\mathfrak{g}})x^+_{2,1} + U_q(\hat{\mathfrak{g}})x^+_2 + U_q(\hat{\mathfrak{g}})x^+_{1,3} + U_q(\hat{\mathfrak{g}})x^+_{1,2} + U_q(\hat{\mathfrak{g}})x^+_{1,1} + U_q(\hat{\mathfrak{g}})x^+_1,$$

which gives

$$x^+_{1,3}(x^+_2)^3x^+_{1,-1}.v_n = 0$$

and hence

$$\Phi^+_{1,3}.v_{\mathbf{p}} = 0.$$

But,

$$P_1(u) = \prod_{j=1}^{r}(1 - uaq_1^{-r+1+2j}).$$

By Proposition 1.4,

$$\sum_{k=0}^{\infty}\Phi^+_{1,k}u^kv_{\mathbf{p}} = q_1^{\deg P_1}\frac{P_1(uq_1^{-2})}{P_2(u)}v_{\mathbf{p}}$$
$$= q_1^r(1 - uaq_1^{-r-1})(1 - uaq_1^{r-1})v_{\mathbf{p}},$$

so by comparing coefficients of u^3 we get

$$\Phi^+_{1,3}.v_{\mathbf{p}} = a^3q_1^{3(r-1)}(q_1^r - q_1^{-r})v_{\mathbf{p}}.$$

This is non-zero since $r > 0$, so this gives a contradiction which completes the proof of part (i) of the proposition.

To prove part (ii), assume that the multiplicity $m_{\lambda-n\theta-(\alpha_1+\alpha_2)}(V(\mathbf{P})) = 0$ and let $v \in V(\mathbf{P})^+_{\lambda-(n+1)\theta-(\alpha_1+\alpha_2)}$. Since, by Proposition 4.3,

$$m_{\lambda-(n+1)\theta-\alpha_i}(V(\mathbf{P})) = 0$$

and

$$\lambda \neq \lambda - n\theta - (\alpha_1 + \alpha_2)$$

for any $n \in \mathbb{N}$, to prove part (ii) it suffices to show, by Lemma 4.4, that

$$x^+_{2,\pm1}.v = 0.$$

By Proposition 4.3, to do this it is enough to show that

$$x^+_{2,\pm1}.v \in V(\mathbf{P})^+_{\lambda-(n+1)\theta-\alpha_1}.$$

By (7),

$$x^+_2 x^+_{2,\pm1}.v = 0.$$

To show that

$$x^+_1 x^+_{2,\pm1}.v = 0$$

it is enough to show that

$$x^+_1 x^+_{2,\pm1}.v \in V(\mathbf{P})^+_{\lambda-(n+1)\theta} \tag{13}$$
$$x^+_{1,s} x^+_1 x^+_{2,\pm1}.v = 0 \quad \text{for } s = \pm1.$$

To show (13), consider

$$\begin{aligned}(x^+_1)^2(x^+_2)^2 x^+_1 x^+_{2,\pm1} &\approx x^+_1 x^+_2 x^+_1 x^+_2 x^+_1 x^+_{2,\pm1} + x^+_2 (x^+_1)^2 x^+_2 x^+_1 x^+_{2,\pm1} \\ &\approx x^+_1 x^+_2 (x^+_1)^2 x^+_2 x^+_{2,\pm1} + x^+_1 (x^+_2)^2 (x^+_1)^2 x^+_{2,\pm1} + x^+_2 (x^+_1)^3 x^+_2 x^+_{2,\pm1} \\ &\qquad + x^+_2 x^+_1 x^+_2 (x^+_1)^2 x^+_{2,\pm1} \\ &\in\ U_q(\hat{\mathfrak{g}}) x^+_2 +\ U_q(\hat{\mathfrak{g}}) x^+_1.\end{aligned}$$

So

$$(x^+_1)^2(x^+_2)^2 x^+_1 x^+_{2,\pm1}.v = 0$$

and

$$x^+_2 x^+_1 (x^+_2)^2 x^+_1 x^+_{2,\pm1} \approx x^+_2 x^+_1 (x^+_2)^2 x^+_{2,\pm1} x^+_1 + x^+_2 x^+_1 (x^+_2)^2 x^+_{1,\mp1} x^+_2 + x^+_2 x^+_1 (x^+_2)^3 x^+_{1,\mp1}.$$

The first two terms on the right-hand side are in $U_q(\hat{\mathfrak{g}})x_2^+ + U_q(\hat{\mathfrak{g}})x_1^+$ and

$$\begin{aligned} x_2^+x_1^+(x_2^+)^3x_{1,\mp1}^+ &\approx x_1^+(x_2^+)^4x_{1,\mp1}^+ + (x_2^+)^4x_1^+x_{1,\mp1}^+ + (x_2^+)^3x_1^+x_2^+x_{1,\mp1}^+ \\ &\qquad + (x_2^+)^2x_1^+(x_2^+)^2x_{1,\mp1}^+ \\ &\in U_q(\hat{\mathfrak{g}})x_2^+ + U_q(\hat{\mathfrak{g}})x_1^+, \end{aligned}$$

since

$$(x_2^+)^2x_1^+(x_2^+)^2x_{1,\mp1}^+ \approx (x_2^+)^2x_1^+x_2^+(x_{1,\mp1}^+x_2^+ + x_{2,\mp1}^+x_1^+ + x_1^+x_{2,\mp1}^+).$$

This gives

$$x_2^+x_1^+(x_2^+)^2x_1^+x_{2,\pm1}^+.v = 0.$$

So

$$x_i^+x_1^+(x_2^+)^2x_1^+x_{2,\pm1}^+.v = 0 \quad (i = 1, 2),$$

which implies that

$$x_1^+(x_2^+)^2x_1^+x_{2,\pm1}^+.v \in V(\mathbf{P})^+_{\lambda-(n+1)\theta-(\alpha_1+\alpha_2)}.$$

So

$$x_1^+(x_2^+)^2x_1^+x_{2,\pm1}^+.v = 0$$

by the induction hypothesis. On the other hand, by using relations (7) and (8) we see that

$$(x_2^+)^3x_1^+x_{2,\pm1}^+ \in U_q(\hat{\mathfrak{g}})x_2^+ + U_q(\hat{\mathfrak{g}})x_1^+.$$

Hence,

$$(x_3^+)^2x_1^+x_{2,\pm1}^+.v = 0$$

and so

$$(x_2^+)^2x_1^+x_{2,\pm1}^+.v \in V(\mathbf{P})^+_{\lambda-n\theta-(2\alpha_1+\alpha_2)}.$$

By Proposition 4.3,

$$(x_2^+)^2x_1^+x_{2,\pm1}^+.v = 0.$$

Now consider

$$x_1^+x_2^+x_1^+x_{2,\pm1}^+ \approx (x_1^+)^2x_2^+x_{2,\pm1}^+ + x_2^+(x_1^+)^2x_{2,\pm1}^+ \in U_q(\hat{\mathfrak{g}})x_2^+ + U_q(\hat{\mathfrak{g}})x_1^+.$$

Hence,

$$x_1^+x_2^+x_1^+x_{2,\pm1}^+.v = 0,$$

and so

$$x_2^+x_1^+x_{2,\pm1}^+.v \in V(\mathbf{P})^+_{\lambda-n\theta-(2\alpha_1+2\alpha_2)},$$

which implies that

$$x_2^+ x_1^+ x_{2,\pm 1}^+ .v = 0.$$

But clearly,

$$(x_1^+)^2 x_{2,\pm 1}^+ .v = 0,$$

so

$$x_1^+ x_{2,\pm 1}^+ .v \in V(\mathbf{P})^+_{\lambda-(n+1)\theta}.$$

This proves (13). To prove (14), we have from [2, p. 910],

$$(x_2^+)^2 x_{1,s}^+ x_1^+ x_{2,\pm 1}^+ \in\ U_q(\hat{\mathfrak{g}}) x_2^+ x_1^+ x_{2,\pm 1}^+ +\ U_q(\hat{\mathfrak{g}}) x_2^+ x_{2,\pm 1}^+$$

for $s = \pm 1$. So, in particular,

$$x_1^+ (x_2^+)^3 x_{1,s}^+ x_1^+ x_{2,\pm 1}^+ .v = 0.$$

Let $w = x_1^+ x_{2,\pm 1}^+ .v$. Then,

$$x_1^+ (x_3^+)^2 x_{1,s}^+ .w = 0.$$

Lemma 4.5 now implies that

$$x_{1,s}^+ .w = 0.$$

This completes the proof of Proposition 4.6.

The following theorem is an immediate consequence of Proposition 4.6.

Theorem 4.7 *Let $\mathfrak{g}$ be of type G_2, $r \geq 0$, $\lambda = r\lambda_1 + \lambda_2 \in P^+$ and $V(\mathbf{P})$ a minimal affinization of λ. Then, as a $U_q(\mathfrak{g})$-module,*

$$V(\mathbf{P}) \cong \bigoplus_{j=0}^{r} V_q(\lambda - j\lambda_1).$$

References

[1] J. Beck, *Braid group action and quantum affine algebras*, Comm. Math. Phys., 165 (1994), 555-568.

[2] V. Chari, *Minimal affinizations of representations of quantum groups. The rank 2 case*, Publ. Math. RIMS (Kyoto Univ.), 31 (1995), 873-911.

[3] V. Chari, *On the fermionic formula and the Kirillov-Reshetikhin conjecture*, Internat. Math. Res. Notices, 12 (2001), 629-54.

[4] V. Chari and A. N. Pressley, *Fundamental representations of Yangians and singularities of R-matrices*, J. reine und angew. Math., 417 (1991), 87-128.

[5] V. Chari and A. N. Pressley, *Small representations of quantum affine algebras*, Lett. Math. Phys., 30 (1994), 131-145.

[6] V. Chari and A. N. Pressley, Quantum affine algebras and their representations, *Representations of Groups, B. N. Allison G. H. Cliff (eds.)*, CMS Conference Proceedings, Vol. 16, Canadian Mathematical Society, 1995, 59-78.

[7] V. Chari and A. N. Pressley, A Guide to Quantum Groups, Cambridge University Press, Cambridge, 1994.

[8] V. Chari and A. N. Pressley, *Minimal affinizations of representations of quantum groups: the simply-laced case*, J. Algebra, 184 (1996), 1-30.

[9] V. G. Drinfeld, *A new realization of Yangians and quantum affine algebras*, Soviet Math. Dokl., 36 (1988), 212-216.

[10] E. Frenkel and E. Mukhin, *Combinatorics of q-characters of finite dimensional representations of quantum affine algebras*, Comm. Math. Phys., 216 (2000), 23-57.

[11] E. Frenkel and N. Yu. Reshetikhin, *The q-characters of representations of quantum affine algebras and deformations of W-algebras*, Contemp. Math., 248 (2000), 163-205.

M. G. Moakes and A. N. Pressley
Department of Mathematics, King's College London
e-mails: m_moakes@tiscali.co.uk (M.G. Moakes),
andrew.pressley@kcl.ac.uk (A.N. Pressley)