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WHEN IS THE SET OF INTERMEDIATE RINGS A FINITE BOOLEAN ALGEBRA

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Abstract. Let $R \subset S$ be an extension of integral domains with identity such that R is not a field and R is integrally closed in S. We determine necessary and sufficient conditions so that the set of intermediate rings $[R, S]$ between R and S is a finite boolean algebra. Several cases are treated, specially when S is the quotient field of R or when R is a Krull domain.

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1. Introduction

Throughout this paper, $R \subset S$ is supposed to be an extension of integral domains with identity such that R is not a field and R is integrally closed in S. We denote by $qf(R)$ the quotient field of R, by $Spec(R)$ the set of all prime ideals of R and by $Max(R) = \{M_i : i \in I\}$ the set of all maximal ideals of R. We also denote by $[R, S]$ the set of all intermediate rings between R and S, and by $Supp(S/R)$ the set of all prime ideals Q of R such that $QS = S$.

If $T_1, T_2, \dots, T_n \in [R, S]$, we denote by $\prod_{i=1}^{n} T_i$ the smallest intermediate ring between R and S containing $\bigcup_{i=1}^{n} T_i$. It is obvious that every element of $\prod_{i=1}^{n} T_i$ can be expressed as a finite sum of the form $\sum t_1 t_2 \cdots t_n$, where $t_i \in T_i$.

Finally, if $\Gamma = \{T_i : i \in I\}$ is a non-empty set of intermediate rings between R and S, and each $T \in [R, S]$ can be written as $\prod_{i \in J} T_i$ for some finite subset J of I, we say that $[R, S]$ is *generated by* Γ. By convention, we may suppose that $R = \prod_{i \in \varnothing} T_i$.

Let us recall some needed definitions:

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A pair of rings (R, S) is said to be a *normal pair* provided that each $T \in [R, S]$ is integrally closed in S. These pairs where first defined and studied by E. D. Davis [3]. He proved that if R is local, then (R, S) is a normal pair if and only if there exists a divided prime ideal P of R (i.e, $PR_P = P$) such that $S = R_P$ and R/P is a valuation ring [3, Theorem 1]. Several other characterizations of such pairs are settled in [2]:

Proposition 1.1. [2, Theorems 2.5, 2.10, Lemma 2.9] *If R is integrally closed in S, then the following conditions are equivalent:*

(i) *(R, S) is a normal pair.*
(ii) *For each $T \in [R, S]$, $Spec(T) = \{PT : PT \subset T, P \in Spec(R)\}$.*
(iii) *For each $T \in [R, S]$, $Spec(T) \to Spec(R)$ is injective.*
(iv) *For each $T \in [R, S]$, and for each $Q \in Spec(T)$; set $P = Q \cap R$, then $R_P = T_Q$.*
(v) *For each $T \in [R, S]$, $T = \bigcap_{P \in Spec(R), PT \subset T} R_P$.*
In particular, if R is local, the above conditions are equivalent to the following:
(vi) *For all $s \in S$, $s \in R$ or $s^{-1} \in R$.*

A *boolean algebra* B is a bounded distributive lattice $(B, \curlywedge, \curlyvee)$ with unary operation $' : B \longrightarrow B$ such that $a \curlywedge a' = 1$ and $a \curlyvee a' = 0$, where 0 is the least element and 1 is the greatest element. Boolean algebras arise in variety of areas of mathematics and computer science.

Our main purpose is to investigate under which conditions $([R, S], ., \cap)$ is a finite boolean algebra. Among other equivalent assertions, we find that $([R, S], ., \cap)$ is a boolean algebra with cardinality 2^n if and only if (R, S) is a normal pair and $Supp(S/R)$ consists of n maximal ideals; or equivalently, there is a maximal chain $R_0 = R \subset R_1 \subset R_2 \subset ... \subset R_n = S$ of length n and every prime ideal of $Supp(S/R)$ is maximal (Theorem 3.2). If S is the quotient field of R, we find that $([R, S], ., \cap)$ is a boolean algebra with cardinality 2^n if and only if R is a 1-dimensional semi-local Prüfer ring with n maximal ideals (Corollary 3.4). If R is a Krull domain and $[R, S]$ is finite, we establish that $([R, S], ., \cap)$ is a boolean algebra of cardinality 2^n, where n is the number of low maximal ideals of R such that $MS = S$ (Corollary 3.6).

The proofs are mostly based on the notion of Kaplansky ideal transforms. Recall that the *Kaplansky ideal transform* $\Omega_R(I)$ of an ideal I of R is an overring of R defined by

$$\Omega_R(I) = \{x \in qf(R) : \forall y \in I, xy^n \in R \text{ for some integer } n \geq 1\}.$$

We frequently write $\Omega(I)$ instead of $\Omega_R(I)$, when no confusion is possible. Note that $\Omega_R(I)$ can be simply expressed in terms of localizations of R by

$$\Omega_R(I) = \bigcap\{R_P : P \in Spec(R), P \not\supseteq I\}.$$

Further properties of such transform can be found in details in [4].

2. Preliminary results

We say that $R \subset S$ is a *minimal extension* if $[R, S]$ contains only R and S. Because R is not a field and R is assumed to be integrally closed in S, then (R, S) is obviously a normal pair. The following useful characterization due to A. Jaballah precises the relationship between these two concepts. We label it as Lemma 2.1 for the sake of reference.

Lemma 2.1. [5, Lemma 3.2] *The following conditions are equivalent:*

(i) $R \subset S$ *is a minimal extension.*

(ii) (R, S) *is a normal pair and* $Supp(S/R)$ *consists of a maximal ideal of* R.

It is clear that, if $R \subset S$ is a minimal extension, then $[R, S]$ is generated by $\Gamma = \{S\}$. In this section, we will generalize Lemma 2.1 by considering the case where $[R, S]$ is generated by a non-empty set $\Gamma = \{T_i : i \in I\}$ of incomparable intermediate rings. We start by two preparatory Lemmas.

Lemma 2.2. *If* $[R, S]$ *is generated by a non-empty set* $\Gamma = \{T_i : i \in I\}$ *of incomparable intermediate rings, then*

(i) *Each* T_i *is a minimal overring of* R.

(ii) S *is an overring of* R.

(iii) I *is finite.*

Proof. (i) If there is a proper intermediate ring T between R and T_i, then $T = \prod\limits_{j\in J} T_j$ for some non-empty finite subset J of I. Then $T_j \subseteq T_i$ for each $j \in J$, but this is false since by assumption, the rings in Γ are incomparable. Thus $R \subset T_i$ is a minimal extension.

(ii) According to Lemma 2.1, (R, T_i) is a normal pair and $Supp(R/T_i)$ consists of one maximal ideal M_i. By application of Proposition 1.1, T_i can be expressed as

$$T_i = \bigcap_{QT_i \subset T_i} R_Q = \bigcap_{Q \notin Supp(T_i/R)} R_Q = \bigcap_{Q \neq M_i} R_Q = \Omega(M_i)$$

Moreover, $R_{M_i} \subset (T_i)_{M_i}$ is a minimal extension [1, Proposition 2.2]. Since $(R_{M_i}, (T_i)_{M_i})$ is a normal pair, there is a prime ideal P_i of R such that $P_i \subset M_i$ and $(T_i)_{M_i} =$

$(R_{M_i})_{P_i R_{M_i}} = R_{P_i}$ [3, Theorem 1]. Now, we have $S = \prod_{i \in K} T_i$ for some non-empty finite subset K of I, so we can present S as

$$S = \prod_{i \in K} \Omega(M_i) \subseteq \Omega(\prod_{i \in K} M_i) = \bigcap_{Q \neq M_i, i \in K} R_Q$$

In particular, we deduce that S is an overring of R.

(iii) If $K \neq I$, we can consider an intermediate ring $T_l = \Omega(M_l)$ for some $l \in I - K$. As $S \subseteq R_{M_l}$, it follows that $R_{M_l} \subset (T_l)_{M_l} = R_{P_l} \subseteq S_{M_l} \subseteq R_{M_l}$, a contradiction. Thus $I = K$ is a finite set. □

We will denote $I = \{1, 2, \dots, n\}$. It follows that, if $[R, S]$ is generated by a set $\Gamma = \{T_i : 1 \leq i \leq n\}$ of incomparable intermediate rings, then each T_i is the Kaplansky ideal transform $T_i = \Omega(M_i)$ of a unique maximal ideal M_i of R such that $M_i T_i = T_i$. We will use frequently this fact along this line.

Lemma 2.3. *Let (R, S) be a normal pair and M_1, M_2, …, M_k maximal ideals in Supp(S/R). Set $T_i = \Omega(M_i)$ and $T = \prod_{i=1}^{k} T_i$, then*

(i) *T_i is a minimal overring of S.*

(ii) *$T = \Omega(\prod_{i=1}^{k} M_i)$ and Supp$(T/R) = \{M_i : 1 \leq i \leq k\}$.*

Proof. (i) Let H be an intermediate ring between R and $T_i = \bigcap_{Q \neq M_i} R_Q$. For every prime ideal $Q \neq M_i$ of R, we have $R_Q \subseteq H_Q \subseteq R_Q$, thus $R_Q = H_Q$ and $QH \subset H$. Therefore, either *Supp*$(H/R) = \varnothing$, so $H = R$; or *Supp*$(H/R) = \{M_i\}$, so $H = \bigcap_{QH \subset H} R_Q = \bigcap_{Q \neq M_i} R_Q = T_i$.

(ii) Because of $M_i T_i = T_i$ for each $i \in \{1, 2, \dots, k\}$, then $M_i T = T$. It follows that $\{M_i : 1 \leq i \leq k\} \subseteq$ *Supp*(T/R). To show the reverse containment, notice that

$$T = \prod_{i=1}^{k} \Omega(M_i) \subseteq \Omega(\bigcap_{i=1}^{k} M_i) = \bigcap_{Q \neq M_i, 1 \leq i \leq k} R_Q$$

Therefore, if Q is a prime ideal of R which does not belong to $\{M_i : 1 \leq i \leq k\}$, then $T \subseteq R_Q$. Thus $QT \subset T$ and $Q \notin$ *Supp*(T/R).

Hence *Supp*$(T/R) = \{M_i : 1 \leq i \leq k\}$ and

$$T = \bigcap_{QT \subset T} R_Q = \bigcap_{Q \notin Supp(T/R)} R_Q = \bigcap_{Q \neq M_i, 1 \leq i \leq k} R_Q = \Omega(\prod_{i=1}^{k} M_i).$$

□

We are able to provide the generalization of Lemma 2.1:

Theorem 2.4. *The following conditions are equivalent:*

(i) $[R, S]$ *is generated by a finite non-empty set* $\Gamma = \{T_i : 1 \leq i \leq n\}$ *of incomparable intermediate rings.*

(ii) (R, S) *is a normal pair and* $Supp(S/R)$ *consists of* n *maximal ideals of* R.

Proof. $(i) \Rightarrow (ii)$ Since $[R, S]$ is generated by $\Gamma = \{T_i : 1 \leq i \leq n\}$, then S can be written as $S = \prod_{i=1}^{n} T_i$. In light of [3, Introduction], to prove that (R, S) is a normal pair, it suffices to show that (R_M, S_M) is a normal pair for each maximal ideal M of R. For each i, we have $R_M = (T_i)_M$ or $R_M \subset (T_i)_M$ is a minimal extension. But, according to [1, Theorem 1.2], we know that R_M has at most one minimal overring, then two cases may occur:

- If $R_M = (T_i)_M$ for each $i \in \{1, 2, \ldots, n\}$, then $S_M = \prod_{i=1}^{n} (T_i)_M = R_M$, so (R_M, S_M) is clearly a normal pair.

- If $R_M \subset (T_j)_M$ is a minimal extension for a unique $j \in \{1, 2, \ldots, n\}$, then $S_M = \prod_{i=1}^{n} (T_i)_M = (T_j)_M$, so $R_M \subset S_M$ is a minimal extension. As R_M is integrally closed in S_M, then (R_M, S_M) is a normal pair.

Since each T_i is a minimal overring of R, then $T_i = \Omega(M_i)$ for a maximal ideal M_i of R such that $M_i T_i = T_i$ and $M_i S = S$ for each $i \in \{1, 2, \ldots, n\}$. Thus, according to Lemma 2.3, we have $Supp(S/R) = \{M_1, M_2, \ldots, M_n\}$.

$(ii) \Rightarrow (i)$ Suppose that (R, S) is a normal pair such that $Supp(S/R)$ consists of n maximal ideals M_1, M_2,..., M_n. Set $T_i = \Omega(M_i)$ and $\Gamma = \{T_i : 1 \leq i \leq n\}$. Since each T_i is a minimal overring of R, Lemma 2.3, then the elements of Γ are incomparable. It remains to show that Γ generates $[R, S]$. Let $T \in [R, S]$. Then $Supp(T/R) \subseteq Supp(S/R)$. Therefore, if $Supp(T/R) = \{M_i : i \in J\}$ for some subset J of $\{1, 2, \ldots, n\}$, then

$$T = \bigcap_{QT \subset T} R_Q = \bigcap_{Q \notin Supp(T/R)} R_Q = \bigcap_{Q \neq M_i, i \in J} R_Q = \Omega(\prod_{i \in J} M_i).$$

Again from Lemma 2.3, we get

$$T = \Omega(\prod_{i \in J} M_i) = \prod_{i \in J} \Omega(M_i) = \prod_{i \in J} T_i.$$

□

3. Boolean algebra

Lemma 3.1. *Suppose that* $[R, S]$ *is generated by a finite set* $\Gamma = \{T_i : 1 \leq i \leq n\}$ *of incomparable intermediate rings. Let* φ *be the function from the power set* $P(I)$ *of* $I = \{1, 2, \ldots, n\}$ *to* $[R, S]$ *that maps* $\varnothing$ *to* R *and any non-empty subset* J *of* I

to $\prod_{i\in J} T_i$. *Then* φ *is bijective, and satisfies the following properties for every two subsets* J *and* K *of* I*:*

(i) $J \subseteq K$ *if and only if* $\varphi(J) \subseteq \varphi(K)$.

(ii) $\varphi(J \cup K) = \varphi(J)\varphi(K)$.

(iii) $\varphi(J \cap K) = \varphi(J) \cap \varphi(K)$.

Proof. In view of Theorem 2.4, $Supp(S/R)$ consists of n maximal ideals of R, namely $M_1, M_2, \ldots, M_n$.

(i) Set $H = \varphi(J)$ and $L = \varphi(K)$. It is clear that $J \subseteq K$ implies $H \subseteq L$. Conversely, if $H \subseteq L$, then $Supp(H/R) \subseteq Supp(L/R)$. But, by Lemma 2.3, we have $Supp(H/R) = \{M_i : i \in J\}$ while $Supp(L/R) = \{M_i : i \in K\}$. Hence $J \subseteq K$. In particular, this shows that φ is injective. As φ is also onto by hypothesis on $[R, S]$, then φ is bijective.

(ii) Since $(T_i)^2 = T_i$ for every $i \in \{1, 2, \ldots, n\}$, we have

$$\varphi(J \cup K) = \prod_{i\in J\cup K} T_i = (\prod_{i\in J} T_i)(\prod_{i\in K} T_i) = \varphi(J).\varphi(K)$$

(iii) This assertion is obvious if there is a containment between J and K. Suppose that $J \nsubseteq K$ and $K \nsubseteq J$. Let $L = J \cap K$ (eventually, we may have $L = \varnothing$). Since the maximal ideals $(M_i)_{1\leq i\leq n}$ are comaximal ideals, then $\prod_{i\in J\setminus K} M_i$ and $\prod_{i\in K\setminus J} M_i$ are also comaximal ideals. It results that

$$\begin{aligned}
\varphi(J) \cap \varphi(K) &= (\prod_{i\in J} T_i) \cap (\prod_{i\in K} T_i) \\
&= \Omega(\prod_{i\in J} M_i) \cap \Omega(\prod_{i\in K} M_i) && \text{by Lemma 2.3} \\
&= \Omega(\prod_{i\in J} M_i + \prod_{i\in K} M_i) && \text{[4, Lemma 3.1]} \\
&= \Omega[\prod_{i\in L} M_i(\prod_{i\in J\setminus K} M_i + \prod_{i\in K\setminus J} M_i)] \\
&= \Omega(\prod_{i\in L} M_i) \\
&= \prod_{i\in L} \Omega(M_i) && \text{by Lemma 2.3} \\
&= \prod_{i\in L} T_i = \varphi(L)
\end{aligned}$$

□

We are ready to provide the main theorem of this paper.

Theorem 3.2. *The following conditions are equivalent for an integer $n \geq 1$:*

(i) $([R, S], ., \cap)$ *is a boolean algebra with cardinality* 2^n.

(ii) $[R, S]$ *is generated by a set* $\Gamma = \{T_i : 1 \leq i \leq n\}$ *of incomparable intermediate rings.*

(iii) (R, S) *is a normal pair and* $Supp(S/R)$ *consists of* n *maximal ideals.*

(iv) $Supp(S/R) \subseteq Max(R)$ *and* $|[R, S]| = 2^n$.

(v) $Supp(S/R) \subseteq Max(R)$, *and there is a maximal chain* $R_0 = R \subset R_1 \subset R_2 \subset ... \subset R_n = S$ *of length* n.

Proof. $(i) \Rightarrow (ii)$ It is known that, if $([R, S], ., \cap)$ is a finite boolean algebra with cardinality 2^n, then it is isomorphic to a boolean algebra of type $(P(I), \cup, \cap)$, where $P(I)$ is the power set of a finite set I with cardinality n. Let $\Psi: P(I) \longrightarrow [R, S]$ be such an isomorphism, and set $T_i = \Psi(\{i\})$ for every $i \in I$. As the sets $(\{i\})_{i \in I}$ are incomparable, then the T_i's, for $i \in I$ are incomparable. Moreover, if $T \in [R, S]$, $T \neq R$, then $T = \Psi(J)$ for some non-empty subset J of I. Thus

$$T = \Psi(\bigcup_{i \in J} \{i\}) = \prod_{i \in J} \Psi(\{i\}) = \prod_{i \in J} T_i.$$

(ii)$\Rightarrow$ (i) By virtue of Lemma 3.1, we deduce that $([R, S], ., \cap)$ is a distributive lattice with least element R and greatest element S. In addition, this lattice is complemented. Indeed, if $T = \prod_{i \in J} T_i \in [R, S]$, where $J \subseteq \{1, 2, \dots, n\}$, then $T' = \prod_{i \notin J} T_i \in [R, S]$ is the complement of T, since

$$T \cap T' = \varphi(J) \cap \varphi(I - J) = \varphi(J \cap (I - J)) = \varphi(\varnothing) = R,$$

and

$$T.T' = \varphi(J).\varphi(I - J) = \varphi(J \cup (I - J)) = \varphi(I) = S.$$

Thus $([R, S], ., \cap)$ is a boolean algebra with cardinality 2^n.

(ii) $\Leftrightarrow$ (iii) results from Theorem 2.4.

(i) $\Rightarrow$ (iv) and (v) Since (ii) and (iii) hold, we can say that $Supp(S/R)$ consists of n maximal ideals $M_1, M_2, \dots, M_n$, and $[R, S]$ is generated by $\Gamma = \{T_i = \Omega(M_i) : 1 \leq i \leq n\}$. Now, if $R_j = \prod_{1 \leq i \leq j} T_i$, then

$$R_0 = R \subset R_1 \subset R_2 \subset ... \subset R_n = S$$

is a maximal chain of length n. Indeed, if $T = \prod_{i \in J} T_i$ is an intermediate ring between R_j and R_{j+1} and different from R_j and R_{j+1}, where $J \subseteq \{1, 2, \dots, n\}$, then $\{1, 2, \dots, j\} \subset J \subset \{1, 2, \dots, j, j + 1\}$ by Lemma 3.1, a contradiction.

(v) $\Rightarrow$ (iii) Assume that $Supp(S/R) \subseteq Max(R)$, and there is a maximal chain $R_0 = R \subset R_1 \subset R_2 \subset ... \subset R_n = S$ of length n.

First, we will prove that (R, S) is a normal pair. According to [3, Introduction], it suffices to show that (R_M, S_M) is a normal pair for every maximal ideal M of R. Let M be a maximal ideal of R. Then

$$R_M = (R_o)_M \subseteq (R_1)_M \subseteq \cdots \subseteq (R_n)_M = S_M$$

is a chain between R_M and S_M such that either $(R_i)_M = (R_{i+1})_M$ or $(R_i)_M \subset (R_{i+1})_M$ is a minimal extension. By refining this last chain, we obtain a finite maximal chain between R_M and S_M. Without loss of generality, we may suppose that R is local with maximal ideal M. It is clear that (R, R_1) is a normal pair, since by assumption R is supposed to be integrally closed in S (so in R_1) and $R \subset R_1$ is a minimal extension. Therefore, there is a prime ideal P of R such that $P \subset M$ and $R_1 = R_P$ [3, Theorem 1]. Thus R_1 is also local. In the other way, $R_1 = R_P$ is integrally closed in S_P (so in R_2) and $R_1 \subset R_2$ is a minimal extension. It results that (R_1, R_2) is a normal pair and R_2 is local. Likewise, we can establish that (R_i, R_{i+1}) is a normal pair and R_{i+1} is local for each $0 \leq i \leq n-1$. Consequently, if $z \in S = R_n$, then $z \in R_{n-1}$ or $z^{-1} \in R_{n-1}$ (Proposition 1.1 (vi)). Progressively, we find that $z \in R_i$ or $z^{-1} \in R_i$ for each $0 \leq i \leq n$, and again Proposition 1(vi) ensures that (R, S) is a normal pair.

Now, we will prove that $Supp(S/R)$ consists of n maximal ideals. Since (R_i, R_{i+1}) is a minimal extension, then $Supp(R_{i+1}/R_i)$ consists of a unique prime ideal Q_i of R_i (Lemma 2.1). By virtue of Proposition 1.1 (ii), we have $Q_i = H_iR_i$ for some prime ideal H_i of R. We claim that

$$Supp(S/R) = \{H_0, H_1, \ldots, H_{n-1}\}.$$

Indeed, if $Q \in Supp(S/R)$, then $QR_0 = Q$ and $QR_n = R_n$. Let i be the first index $i \geq 1$ such that $QR_i = R_i$. We necessarily have $QR_{i-1} \subset R_{i-1}$ and $QR_{i-1} \in Supp(R_i/R_{i-1})$. Thus $QR_{i-1} = Q_{i-1} = H_{i-1}R_{i-1}$. By contraction on R, we obtain $Q = H_{i-1}$ (Proposition 1.1 (iii)). So $Supp(S/R) \subseteq \{H_0, H_1, \ldots, H_{n-1}\}$. To see the reverse inclusion, it suffices to note that $Q_iR_{i+1} = R_{i+1}$, so $H_iS = (H_iR_i)S = Q_iS = (Q_iR_{i+1})S = R_{i+1}S = S$ for each $i \in \{0, 1, \ldots, n-1\}$.

Furthermore, the H_i's are distinct. If $H_i = H_j$ for $0 \leq i < j \leq n-1$, then $Q_iR_j = Q_j$, and this leads to the contradiction $Q_j = Q_jR_j = Q_iR_j = (Q_iR_{i+1})R_j = R_{i+1}R_j = R_j$.

As by assumption $Supp(S/R) \subseteq Max(R)$, then $Supp(S/R)$ consists of n maximal ideals.

(iv)⇒ (v) Suppose that $Supp(S/R) \subseteq Max(R)$ *and* $|[R,S]| = 2^n$. Since $[R,S]$ is finite, we can consider a finite maximal chain

$$R_0 = R \subset R_1 \subset R_2 \subset ... \subset R_m = S$$

of length m from R to S. Since the conditions (i) and (v) are actually equivalent for the integer m, we obtain $|[R,S]| = 2^m$. Henceforth, $m = n$. □

As consequences of Theorem 3.2, we recover the following corollaries. Our first application concerns the case where R is a Prüfer ring and S is an overring of R. In this case, it is known that (R,S) is a normal pair.

Corollary 3.3. *If R is a Prüfer ring and S is an overring of R, then the following conditions are equivalent for an integer $n \geq 1$:*

(i) $([R,S], ., \cap)$ *is a boolean algebra with cardinality* 2^n.
(ii) $Supp(S/R)$ *consists of n maximal ideals.*

Now, if R is an integrally closed domain with quotient field K, then

$$Supp(K/R) = Spec(R) - \{0\}.$$

We can derive the following nice result:

Corollary 3.4. *If R is integrally closed with quotient field K, then the following conditions are equivalent for an integer $n \geq 1$:*

(i) $([R,K], ., \cap)$ *is a boolean algebra with cardinality* 2^n.
(ii) $[R,K]$ *is generated by a set* $\{T_i : 1 \leq i \leq n\}$ *of incomparable proper overrings of R.*
(iii) *R is a $1-$dimensional semi-local Prüfer ring with n maximal ideals.*
(iv) $dimR = 1$ *and* $|[R,K]| = 2^n$.
(v) $dimR = 1$, *and there is a maximal chain* $R_0 = R \subset R_1 \subset \ldots \subset R_n = K$ *of length n.*

The following result provides a method for building more examples of extensions $R \subset S$ such that $[R,S]$ is a finite boolean algebra.

Corollary 3.5. *Let S be an integral domain, M a maximal ideal of S, D a subring of the residue field $L = S/M$ and $R = \varphi^{-1}(D)$ the inverse image of D by the canonical epimorphism $\varphi : S \to L$. If D is integrally closed in L, then $([R,S], ., \cap)$ is a boolean algebra with cardinality 2^n if and only if D is a 1-dimensional semi-local Prüfer ring with n maximal ideals and quotient field L.*

Proof. R is the pullback illustrated by the following square:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ S & \longrightarrow & L = S/M \end{array}$$

Note that R is integrally closed in S. Therefore, this result is a direct consequence of Corollary 3.4 and the fact that $[R,S]$ is generated by a set $\{T_i : 1 \leq i \leq n\}$ of intermediate rings between R and S if and only if $[D,L]$ is generated by the set $\{\varphi(T_i) : 1 \leq i \leq n\}$ of intermediate rings between D and L. $\square$

Our last application is a significant result concerning Krull rings.

Corollary 3.6. *If R is a Krull domain and $[R,S]$ is finite, then $([R,S],.,\cap)$ is a boolean algebra of cardinality 2^n, where n is the number of height-one maximal ideals of R such that $MS = S$.*

Proof. Since $[R,S]$ is finite, we can consider a finite maximal chain between R and S. To apply Theorem 3.2(v), it remains to show that every prime ideal of $Supp(S/R)$ is maximal. Let $Q \in Supp(S/R)$. Then $Q \neq (0)$ and Q is contained in a maximal ideal $M \in Supp(S/R)$. In view of Lemma 2.3, $\Omega(M)$ is a minimal overring of R. Finally, according to [1, Theorem 5.7], we necessarily have $ht_R(M) = 1$ and $Q = M$. $\square$

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