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r-COSTAR MODULES

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Abstract. We give the definition of r-costar modules as a generalization of costar modules under artin algebra situation. We also study characterizations of r-costar modules and give characterizations of r-costar modules with some special properties.

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1. Introduction

The theory of equivalences and dualities between module subcategories, originating in the well-known theory of Morita theorems, had been studied extensively. Both quasi-progenerators and tilting modules over arbitrary rings induce equivalences between certain categories of modules. Menini and Orsatti introduced a generalization of these modules that have come to be called $*$-modules, cf. [5]. Namely, let $\mathcal{C}_A$ be a subcategory of Mod-A closed under submodules and containing A_A, and let $\mathcal{Y}_R$ be a subcategory of Mod-R closed under direct sums and epimorphic images, then any equivalence between $\mathcal{C}_A$ and $\mathcal{Y}_R$ is represented by a bimodule ${}_AP_R$, via the adjoint pair (T_P, H_P). P is called $*$-*module* if we put $A = \mathrm{End}(P_R)$, $T_P = - \otimes_A P$, $H_P = \mathrm{Hom}_R(P, -)$, $K_A = \mathrm{Hom}_R(P, Q)$, where Q is a cogenerator of Mod-A, then the pair of the functors (T_P, H_P) defines an equivalence between $\mathrm{Gen}(P_R)$ and $\mathrm{Cogen}(K_A)$. In [3], Colpi gave that P is a $*$-module if and only if P is selfsmall and, for any exact sequence $0 \to L \to M \to N \to 0$ with $M,\ N \in \mathrm{Gen}(P)$, the induced sequence $0 \to H_P(L) \to H_P(M) \to H_P(N) \to 0$ is exact if and only if $L \in \mathrm{Gen}(P)$.

Wei introduced the notion of $*^s$-modules, where s denotes static, by replacing the subcategory $\mathrm{Gen}(P)$ in the theory of $*$-module with the subcategory $\mathrm{Stat}(P)$ and some results on $*$-module are successively extended to $*^s$-modules, cf. [7].

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On the other side, a dual notion of quasi-progenerators called quasi-duality modules and cotilting modules dual to tilting modules have been a central topic of recent investigation in module theory. Colby and Fuller generalized these modules to costar modules which may be viewed as dual to $*$-modules in a sense, cf. [1].

Inspired by [7], we shall investigate r-costar modules by replacing the subcategory cogen(P) in the theory of costar modules with the subcategory $\mathrm{Ref}(P_\Lambda)$ under artin algebra situation. Some similar results are obtained.

2. Preliminaries

Let R be an associative ring with nonzero identity, Mod-R (R-Mod) denotes the category of all right (left) R-modules, while mod-R and R-mod denote their subcategories of finitely generated modules. We shall let proj_R be the full subcategory of all projective modules in mod-R (or R-mod). Given a module P_R (or ${}_RP$), we denote

$$\mathrm{KerExt}_R^{i\geq e}(-,P)=\{X\in \text{mod-}R(\text{or } R\text{-mod})\mid \mathrm{Ext}_R^i(X,P)=0,\ \text{for all } i\geq e,\ \text{where } e \text{ is a positive integer or } 0\}.$$

If $X\in$ Mod-R (or R-Mod), let $\mathrm{Rej}_P(X)=\bigcap\{\mathrm{Ker}(f)\mid f\in \mathrm{Hom}_R(X,P)\}$.

For every R-module P, Prod$P$ (prod$P$) will denote the class of the R-modules isomorphic to arbitrary (finite) summands of direct products of copies of P. Add(P) (add(P)) will denote the class of the R-modules isomorphic to arbitrary (finite) summands of direct sums of copies of P. Cogen(P) (cogen(P)) consists of the R-modules that embed in arbitrary (finite) direct products of modules isomorphic to P. An R-module $N\in$ Cogen(P) if and only if, for every $0\neq x\in N$, there is a morphism $f\in \mathrm{Hom}_R(N,P)$ such that $f(x)\neq 0$. Gen(P) will denote the class of the R-modules generated by P. For a module M, if there exists an exact sequence $0\to M\to P^X\to P^Y$, where X,Y are sets, we say M is copresented by P_R. A subcategory is resolving if it contains all projective objects and is closed under extensions and kernels of epimorphisms.

For a bimodule ${}_AP_R$, we let $\Delta_{P_R}=\mathrm{Hom}_R(-,P)$ and $\Delta_{{}_AP}=\mathrm{Hom}_A(-,P)$. Both of which will, when convenient, simply be denoted by Δ_P or Δ. We have a pair of contravariant functors

$$\Delta_{P_R}:\text{Mod-}R\rightleftharpoons A\text{-Mod}:\Delta_{{}_AP}.$$

Associated with this adjunction are the evaluation maps

$$\delta_X: X\to \Delta^2X \text{ with } \delta_X(x): f\mapsto f(x)$$

for X in Mod-R or A-Mod, $x \in X$, and $f \in \Delta(X)$. This yields natural transformations

$$\delta : 1_{\text{Mod-}R} \to \Delta^2 \text{ and } \delta : 1_{A\text{-Mod}} \to \Delta^2.$$

A module X is (P-)reflexive if δ_X is an isomorphism, and (P-)torsionless if δ_X is a monomorphism. Note that $\text{Ker}\delta_X = \text{Rej}_P(X)$, so that X is P-torsionless if and only if X is cogenerated by P. We shall let $\Gamma_{P_R} = \text{Ext}^1_R(-, P)$ and $\Gamma_{{}_AP} = \text{Ext}^1_A(-, P)$, both of which will usually be denoted by Γ, cf. [2].

Denoting the class of P-reflexive right-R and left-A modules by Ref(P_R) and Ref(${}_AP$), respectively, it always the case that ${}_AP_R$-induces a duality

$$\Delta : \text{Ref}(P_R) \rightleftharpoons \text{Ref}({}_AP) : \Delta.$$

Let fgd-tl(P_R) denote by the class of torsionless right R-modules whose P-duals are finitely generated over A and fg-tl(${}_AP$) denote the class of finitely generated torsionless left A-modules. we also denote by

$$\text{sf-cp}(P_R) = \{M \in \text{Mod-}R |\ 0 \to M \to P^n \to P^A \text{ with } n \in N\}.$$

Colby and Fuller obtain the following theorem.

Theorem 2.1. ([1]) *Let R be a ring, $P \in$ Mod-R and $A = End(P_R)$. The following are equivalent.*

(a) Δ : fgd-tl(P_R) $\rightleftharpoons$ fg-tl(${}_AP$) : Δ is a duality. That is, P_R is a costar module.

(b) Δ : sf-cp(P_R) $\rightleftharpoons$ fg-tl(${}_AP$) : Δ is a duality and fgd-tl(P_R) = sf-cp(P_R).

(c) δ_M is an epimorphism if $\Delta(M) \in$ A-mod and δ_N is an epimorphism if $N \in$ A-mod.

(d) fgd-tl(P_R) $\subseteq$ sf-cp(P_R) and if $0 \to M \xrightarrow{f} P^n \to L \to 0$ is exact with $L \in$ Cogen(P_R), then $\Delta(f)$ is an epimorphism.

(e) If $0 \to M \xrightarrow{f} P^n \to L \to 0$ is exact then $L \in$ Cogen(P_R) if and only if $\Delta(f)$ is an epimorphism.

A ring Λ is an artin algebra if its center K is an artinian ring and Λ is finitely generated as a K-module. Any finitely generated module over an artin algebra is finitely generated over its endmorphism ring, which is also an artin algebra, cf. [2].

In this paper, we let Λ denote by an artin albebra and we only consider finitely generated module category. If $P \in$ mod-Λ, for $M \in$ mod-Λ(A-mod), then $\Delta(M) \in$ A-mod(mod-Λ). Any exact sequence $0 \to L \to P^n \to M \to 0$ in mod-Λ(or A-mod), then $L,\ M \in$ mod-Λ(or A-mod) and $\Delta(L),\ \Delta(M) \in$ A-mod(or Λ-mod), since Λ is an artin algebra. If Λ is an artin algebra and ${}_AP_\Lambda$ is a bimodule, then Cogen(P_Λ) $\cap$ mod-Λ=cogen(P_Λ) and Cogen(${}_AP$) $\cap$ A-mod=cogen(${}_AP$), cf. [2]($P115$). So that from Theorem 2.1 (e), P is a costar module provided that any exact sequence $0 \to M \xrightarrow{f} P^n \to L \to 0$ remains exact after applying the functor Δ_P if and only if $L \in \text{cogen}(P_\Lambda)$.

3. Basic properties on r-costar modules

Firstly, we give the definition of r-costar modules.

Definition 3.1. Let $P \in \text{mod-}\Lambda$. A Λ-module P is said to be an *r-costar module* provided that any exact sequence $0 \to L \to M \to N \to 0$ with $L,\ M \in \text{Ref}(P_\Lambda)$ remains exact after applying the functor Δ_{P_Λ} if and only if $N \in \text{Ref}(P_\Lambda)$.

If P is an r-costar module and $\text{cogen}(P_\Lambda) = \text{Ref}(P_\Lambda)$, then P is a costar module.

For an r-costar module P_Λ, the subcategory $\text{Ref}(P_\Lambda)$ has the following properties.

Proposition 3.2. *Let $P_\Lambda \in$ mod-Λ be an r-costar module. Then the following hold.*

(1) *The functor Δ_{P_Λ} preserves short exact sequences in Ref(P_Λ).*

(2) *For any $M \in$ Ref(P_Λ), there is an infinite exact sequence $0 \to M \to P_1 \to \cdots \to P_n \to \cdots$ which remains exact after applying the functor Δ_{P_Λ} where $P_i \in$ prod$P$ for each i.*

(3) For any exact sequence $0 \to L \to M \to N \to 0$ which is also exact after applying the functor Δ_{P_Λ}, if two of its terms are in Ref(P_Λ), then so is the third one.

Proof. (1) Follows from the definition of r-costar modules.

(2) If X is finitely generated, then $\Delta(X)$ is finitely generated as A-module. So, for any $M \in \text{Ref}(P_\Lambda)$, let $\Delta(M)$ is generated by $f_1, \cdots, f_n$ as A-module. Then we have an exact sequence $0 \to M \xrightarrow{f} P^n \to M_1 \to 0$ where $f = (f_1, \cdots, f_n)$ and $P^n \in \text{prod}P$. Since the sequence is clearly exact after applying the functor Δ_{P_Λ} and $P^n \in \text{Ref}(P_\Lambda)$, we obtain that $M_1 \in \text{Ref}(P_\Lambda)$. By repeating the process to M_1, and so on, we finally obtain the desired exact sequence.

(3) If $L, M \in \text{Ref}(P_\Lambda)$, then $N \in \text{Ref}(P_\Lambda)$ by the definition of r-costar modules.

Now let $N \in \text{Ref}(P_\Lambda)$. By applying the functor Δ^2 to the sequence in (3), we have the following commutative diagram by assumptions.

$$\begin{array}{ccccccccc} 0 & \to & L & \to & M & \to & N & \to & 0 \\ & & \downarrow{\scriptstyle \delta_L} & & \downarrow{\scriptstyle \delta_M} & & \downarrow{\scriptstyle \delta_N} & & \\ 0 & \to & \Delta^2(L) & \to & \Delta^2(M) & \to & \Delta^2(N) & & \end{array}$$

since $N \in \mathrm{Ref}(P_\Lambda)$, we have that δ_N is an isomorphism. It follows easily that δ_L is an isomorphism if and only if δ_M is an isomorphism. So that, Ł $\in \mathrm{Ref}(P_\Lambda)$ if and only if $M \in \mathrm{Ref}(P_\Lambda)$. □

The following proposition gives some properties of the subcategory $\mathrm{Ref}({}_AP)$ for P an r-costar module.

Proposition 3.3. *Let $P \in$ mod-Λ be an r-cotar module with $A = End(P_\Lambda)$. Then the following hold.*

(1) *$proj_A \subseteq Ref({}_AP) \subseteq KerExt_A^{i\geq 1}(-, P)$. In particular, the functor $\Delta_{{}_AP}$ preserves short exact sequences in $Ref({}_AP)$.*

(2) *$Ref({}_AP)$ is a resolving subcategory.*

Proof. Since $A \in \mathrm{Ref}({}_AP)$, we have $\mathrm{proj}_A \subseteq \mathrm{Ref}({}_AP)$. Now for any $N \in \mathrm{Ref}({}_AP)$, then $\Delta(N) \in \mathrm{Ref}(P_\Lambda)$. By proposition 3.2(2), there is an infinite exact sequence $0 \to \Delta(N) \to P_1 \to \cdots \to P_n \to \cdots$ which remains exact after applying the functor Δ_{P_Λ} to the sequence. We obtain an exact sequence $\cdots \to \Delta(P_n)(\cong A_n) \to \cdots \to \Delta(P_1)(\cong A_1) \to \Delta^2(N) \to 0$ with $A_i \in \mathrm{proj}_A$. It remains exact after applying the functor $\Delta_{{}_AP}$. We obtain that $\mathrm{Ref}({}_AP) \subseteq \mathrm{KerExt}_A^{i\geq 1}(-, P)$ by dimension shifting.

(2) From (1), we know that $\mathrm{Ref}({}_AP)$ contains all projective A-modules. For any exact sequence $0 \to K \to M \to N \to 0$ with $N \in \mathrm{Ref}({}_AP)$. As $\mathrm{Ref}({}_AP) \subseteq \mathrm{KerExt}_A^{i\geq 1}(-, P)$ by (1), we obtain the following exact commutative diagram by applying the functor Δ^2.

$$\begin{array}{ccccccccc} 0 & \to & K & \to & M & \to & N & \to & 0 \\ & & \downarrow^{\delta_K} & & \downarrow^{\delta_M} & & \downarrow^{\delta_N} & & \\ 0 & \to & \Delta^2(K) & \to & \Delta^2(M) & \to & \Delta^2(N) & & . \end{array}$$

Since $N \in \mathrm{Ref}({}_AP)$, then δ_N is an isomorphism. So that, δ_K is an isomorphism if and only if δ_M is an isomorphism. That is, $K \in \mathrm{Ref}({}_AP)$ if and only if $M \in \mathrm{Ref}({}_AP)$. It follows that $\mathrm{Ref}({}_AP)$ is closed under kernels of epimorphisms and under extensions. Thus, $\mathrm{Ref}({}_AP)$ is a resolving subcategory. □

4. Characterizations of r-costar modules

In this section, we shall give an equivalent condition for r-costar modules.

Proposition 4.1. *Let P be an right Λ-module with $A = End(P_\Lambda)$. Assume that $Ref({}_AP) \subseteq KerExt_A^{i\geq 1}(-, P)$ and $KerExt_A^{i\geq 0}(-, P) = 0$. Then P is an r-costar module.*

Proof. Let $0 \to L \to M \to N \to 0$ be an exact sequence with $L,\ M \in \mathrm{Ref}(P_\Lambda)$. Assume the sequence is exact after applying the functor Δ, then we have the induced exact sequence $0 \to \Delta(N) \to \Delta(M) \to \Delta(L) \to 0$. Since $\Delta(L) \in \mathrm{Ref}({}_AP) \subseteq \mathrm{KerExt}_A^{i\geq 1}(-,P)$, after applying the functor Δ, we obtain an exact sequence $0 \to \Delta^2(L) \to \Delta^2(M) \to \Delta^2(N) \to 0$. Note that $L,\ M \in \mathrm{Ref}(P_\Lambda)$, so $\delta_L,\ \delta_M$ are isomorphisms. It follows that δ_N is an isomorphism and $N \in \mathrm{Ref}(P_\Lambda)$.

Now, suppose that $N \in \mathrm{Ref}(P_\Lambda)$. By applying the functor Δ, we obtain an induced exact sequences $0 \to \Delta(N) \to \Delta(M) \to X \to 0$ and $0 \to X \xrightarrow{f} \Delta(L) \to Y \to 0$ for some $X,\ Y \in A$-mod. After applying the functor Δ to the first sequence, we then have the following exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & L & \to & M & \to & N & \to & 0 \\
 & & \downarrow^{h} & & \downarrow^{\delta_M} & & \downarrow^{\delta_N} & & \downarrow \\
0 & \to & \Delta(X) & \to & \Delta^2(M) & \to & \Delta^2(N) & \to & \mathrm{Ext}_A^1(X,P) & \to 0.
\end{array}$$

Since $M,\ N \in \mathrm{Ref}(P_\Lambda)$, δ_M, δ_N are isomorphisms. It follows from the diagram that $\mathrm{Ext}_A^1(X,P) = 0$ and $h = \Delta(f)\delta_L$ is an isomorphism. Then $\Delta(f)$ is an isomorphism since δ_L is an isomorphism. Note that $\Delta(M),\ \Delta(N) \in \mathrm{Ref}({}_AP) \subseteq \mathrm{KerExt}_\Lambda^{i\geq 1}(-,P)$, so we have $\mathrm{Ext}_A^i(X,P) = 0$ for all $i \geq 2$ by dimension shifting. Hence $X \in \mathrm{KerExt}_A^{i\geq 1}(-,P)$. On the other hand, by applying the functor Δ to the sequence $0 \to X \to \Delta(L) \to Y \to 0$, we have an induced exact sequence

$$0 \longrightarrow \Delta(Y) \longrightarrow \Delta^2(L) \xrightarrow{\Delta(f)} \Delta(X) \longrightarrow \mathrm{Ext}_A^1(Y,P) \longrightarrow 0$$

and that $Y \in \mathrm{KerExt}_A^{i\geq 2}(-,P)$, since $\Delta(L) \in \mathrm{Ref}({}_AP) \subseteq \mathrm{KerExt}_A^{i\geq 1}(-,P)$. Hence we obtain that $\mathrm{Ext}_A^1(Y,P) = 0 = \Delta(Y)$. It follows that $Y \in \mathrm{KerExt}_A^{i\geq 0}(-,P)$. So that $Y = 0$ by assumptions and hence $X \cong \Delta(L)$ canonically. Therefore, we deduce that the functor Δ preserves the exactness of the exact sequence $0 \to L \to M \to N \to 0$ in $\mathrm{Ref}(P_\Lambda)$.

Finally, we conclude that P is an r-costar module. □

Lemma 4.2. *Let $P \in$ mod-Λ and $A = End(P_\Lambda)$. Then $\Omega^2(KerExt_A^{i\geq 1}(-,P)) \subseteq Ref({}_AP)$ where Ω^2 denotes the second syzygy module.*

Proof. For any $N \in \Omega^2(\mathrm{KerExt}_A^{i\geq 1}(-,P))$, we have an exact sequence $0 \to N \to A_2 \to A_1 \to M \to 0$ with $A_1,\ A_2 \in \mathrm{proj}_A$ and $M \in \mathrm{KerExt}_A^{i\geq 1}(-,P)$. We have an induced exact sequence

$$0 \to \Delta(M) \to \Delta(A_1) \to \Delta(A_2) \to \Delta(N) \to 0$$

by applying the functor Δ. Now after applying the functor Δ to the last sequence, we obtain an induced exact sequence

$$0 \to \Delta^2(N) \to \Delta^2(A_2) \to \Delta^2(A_1).$$

Since δ_{A_1}, δ_{A_2} are isomorphisms, we deduce that δ_N is also an isomorphism. Hence $N \in \mathrm{Ref}({}_AP)$. $\square$

Proposition 4.3. *Let P_Λ be an r-costar module. Then $KerExt_A^{i\geq 0}(-, P) = 0$.*

Proof. For any $M \in \mathrm{KerExt}_A^{i\geq 0}(-, P)$, we have an exact sequence $0 \to N \to A_2 \to A_1 \to M \to 0$ with $A_1, A_2 \in \mathrm{proj}_A$ and $N \in \Omega^2(M)$. Then, by applying the functor Δ to the sequence, we obtain an induced exact sequence $0 \to \Delta(A_1) \to \Delta(A_2) \to \Delta(N) \to 0$. By Lemma 4.2, $N \in \mathrm{Ref}({}_AP)$ and then $\Delta(N) \in \mathrm{Ref}(P_\Lambda)$. So, after applying the functor Δ, we obtain that the induced sequence

$$0 \to \Delta^2(N) \to \Delta^2(A_2) \to \Delta^2(A_1) \to 0$$

is exact by Proposition 3.2. It follows that $M = \mathrm{Coker}(A_2 \to A_1) \cong \mathrm{Coker}(\Delta^2(A_2) \to \Delta^2(A_1)) = 0$. $\square$

From Propositions 3.3, 4.1 and 4.3, we have the following characterization of r-costar modules.

Theorem 4.4. *Let $P \in mod\text{-}\Lambda$ and $A = End(P_\Lambda)$. Then P is an r-costar module if and only if $proj_A \subseteq Ref({}_AP) \subseteq KerExt_A^{i\geq 1}(-, P)$ and $KerExt_A^{i\geq 0}(-, P) = 0$.*

From Theorem 4.4, we obtain the following result.

Theorem 4.5. *Let $P \in mod\text{-}\Lambda$ and $A = End(P_\Lambda)$. If $Ref({}_AP) = KerExt_A^{i\geq 1}(-, P)$, then P is an r-costar module.*

Proof. Under assumptions we clearly have that $\mathrm{proj}_A \subseteq \mathrm{Ref}({}_AP)$. For any $M \in \mathrm{KerExt}_A^{i\geq 0}(-, P)$. Then $M \in \mathrm{Ref}({}_AP) = \mathrm{KerExt}_A^{i\geq 1}(-, P)$. Hence $M \cong \Delta^2(M) = 0$. By Theorem 4.4, we obtain that P is an r-costar module. $\square$

P_L in condition (2) in the following proposition can be replaced by P^n. So that, from the following proposition, we know that r-costar modules are obtained by replacing the subcategory $\mathrm{cogen}(P_\Lambda)$ with the subcategory $\mathrm{Ref}(P_\Lambda)$ in costar modules theory under artin algebra situation.

Proposition 4.6. *Let $P \in mod\text{-}\Lambda$. The following are equivalent.*

(1) *P is an r-costar module.*

(2) *For any exact sequence $0 \to L \to P_L \to M \to 0$ with $P_L \in prodP$ and $L \in$*

Ref(P_Λ), *then* $M \in$ *Ref*(P_Λ) *if and only if the functor* Δ *preserves the exactness of the sequence.*

Proof. (1) $\Rightarrow$ (2) is followed from the definition of r-costar module.

(2) $\Rightarrow$ (1) As the proof in Proposition 3.2(2), for any $L \in \mathrm{Ref}(P_\Lambda)$, we have an infinite exact sequence $0 \to L \to P_1 \to \cdots \to P_n \to \cdots$ which remains exact after applying the functor Δ, where $P_i \in \mathrm{prod}P$ for each i. Consequently, we have $\mathrm{proj}_A \subseteq \mathrm{Ref}({}_AP) \subseteq \mathrm{KerExt}_A^{i\geq 1}(-,P)$ as in the proof of Proposition 3.3(1).

At last, the same proof as in Proposition 4.3 yields that $\mathrm{KerExt}_A^{i\geq 0}(-,P) = 0$. Thus P is an r-costar module by Theorem 4.4. $\square$

5. r-Costar modules with special properties

In this section, we shall study r-costar modules P such that the subcategory $\mathrm{Ref}(P_\Lambda)$ has some special properties.

Proposition 5.1. *Let* $P \in$ *mod-*Λ *with* $A =$ *End*(P_Λ). *If* ${}_AP$ *is an injective cogenerator in* A*-mod, then* P_Λ *is an* r*-costar module.*

Proof. Let $0 \to L \to M \to N \to 0$ be an exact sequence with $L,\ M \in \mathrm{Ref}(P_\Lambda)$. Assume first that the sequence is exact after applying the functor Δ, then we have an induced exact sequence $0 \to \Delta(N) \to \Delta(M) \to \Delta(L) \to 0$. Since ${}_AP$ is injective, after applying the functor Δ, we obtain that the sequence $0 \to \Delta^2(L) \to \Delta^2(M) \to \Delta^2(N) \to 0$ is exact. Note that $L,\ M \in \mathrm{Ref}(P_\Lambda)$, so, δ_L and δ_M are isomorphisms. It follows that δ_N is an isomorphism and $N \in \mathrm{Ref}(P_\Lambda)$.

Now, suppose that $N \in \mathrm{Ref}(P_\Lambda)$. After applying the functor Δ, we obtain an induced exact sequence $0 \to \Delta(N) \to \Delta(M) \to \Delta(L) \to D \to 0$ for some D. Since ${}_AP$ is injective, we have an exact sequence $0 \to \Delta(D) \to \Delta^2(L) \to \Delta^2(M) \to \Delta^2(N) \to 0$ by applying the functor Δ. Because $\delta_L,\ \delta_M,\ \delta_N$ are isomorphisms, then $\Delta(D) = 0$. As ${}_AP$ is a cogenerator, we then obtain that $D = 0$. Hence $0 \to \Delta(N) \to \Delta(M) \to \Delta(L) \to 0$ is exact. Therefore, P is an r-costar module. $\square$

For an r-costar module P, we know that $\mathrm{Ref}({}_AP)$ is a resolving subcategory from Proposition 3.3(2). Now we consider when $\mathrm{Ref}(P_\Lambda)$ is a resolving subcategory. The following theorem gives some characterizations of this case.

Theorem 5.2. *Let* $P \in$ *mod-*Λ. *The following are equivalent.*

(1) *P is an r-costar module such that Ref(P_Λ) is a resolving subcategory.*

(2) *Ref*$(P_\Lambda) =$ *KerExt*$_\Lambda^{i\geq 1}(-,P)$.

(3) *Ref*$(P_\Lambda) \subseteq$ *KerExt*$^{i\geq 1}_{\Lambda}(-,P) \subseteq$ *cogen*(P_Λ).
(4) *proj*$_\Lambda \subseteq$ *Ref*$(P_\Lambda) \subseteq$ *KerExt*$^{i\geq 1}_{\Lambda}(-,P)$ *and proj*$_A \subseteq$ *Ref*$({}_AP) \subseteq$ *KerExt*$^{i\geq 1}_{A}(-,P)$.

Proof. (1) $\Rightarrow$ (2) Let $M \in \mathrm{Ref}(P_\Lambda)$ and $0 \to M_1 \to \Lambda_1 \to M \to 0$ be an exact sequence with $\Lambda_1 \in \mathrm{proj}_\Lambda$. Since $\mathrm{Ref}(P_\Lambda)$ is a resolving subcategory, $\mathrm{proj}_\Lambda \subseteq \mathrm{Ref}(P_\Lambda)$ and $\mathrm{Ref}(P_\Lambda)$ is closed under kernels of epimorphisms. Therefore $M_1 \in \mathrm{Ref}(P_\Lambda)$. It follows from Proposition 3.2 that the sequence remains exact after applying the functor Δ. Then we have that $\mathrm{Ext}^1_\Lambda(M,P) = 0$ for any $M \in \mathrm{Ref}(P_\Lambda)$. Hence, $\mathrm{Ref}(P_\Lambda) \subseteq \mathrm{KerExt}^{i\geq 1}_{\Lambda}(-,P)$, since $\mathrm{Ref}(P_\Lambda)$ is a resolving subcategory.

On the other hand, we may consider the exact sequence $0 \to X \to \Lambda_2 \to \Lambda_1 \to M \to 0$ with $\Lambda_1,\ \Lambda_2 \in \mathrm{proj}_\Lambda$ for any $M \in \mathrm{KerExt}^{i\geq 1}_{\Lambda}(-,P)$. It clearly remains exact after applying the functor Δ. Hence we have an induced exact sequence $0 \to \Delta(M) \to \Delta(\Lambda_1) \to \Delta(\Lambda_2) \to \Delta(X) \to 0$. Since $\mathrm{Ref}(P_\Lambda)$ is a resolving subcategory, then $\mathrm{proj}_\Lambda \subseteq \mathrm{Ref}(P_\Lambda)$. By applying the functor Δ to the sequence, we obtain that $X \cong \Delta^2(X)$. That is, $X \in \mathrm{Ref}(P_\Lambda)$. Since P is an r-costar modules, then we have that $\mathrm{Im}(\Lambda_2 \to \Lambda_1) \in \mathrm{Ref}(P_\Lambda)$ and similarly $M \in \mathrm{Ref}(P_\Lambda)$. Hence we have $\mathrm{KerExt}^{i\geq 1}_{\Lambda}(-,P) \subseteq \mathrm{Ref}(P_\Lambda)$.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (4) Since $\mathrm{prod}P \subseteq \mathrm{Ref}(P_\Lambda)$, we have that $\mathrm{prod}P \subseteq \mathrm{KerExt}^{i\geq 1}_{\Lambda}(-,P)$. For any $M \in \mathrm{KerExt}^{i\geq 1}_{\Lambda}(-,P)$, let $\Delta(M)$ is generated by $f_1,\cdots,f_n$ as A-module. Then we have an exact sequence $0 \to M \xrightarrow{f} P^n \to M_1 \to 0$ where $f = (f_1,\cdots,f_n)$. Note that the sequence clearly stays exact after applying the functor Δ and since $M,\ P^n \in \mathrm{KerExt}^{i\geq 1}_{\Lambda}(-,P)$, we obtain that $M_1 \in \mathrm{KerExt}^{i\geq 1}_{\Lambda}(-,P)$ too. By repeating the process to M_1, and so on, we finally obtain an infinite exact sequence $0 \to M \to P_1 \to \cdots \to P_n \to \cdots$ with $P_i \in \mathrm{prod}P$ for each i, such that the sequence remains exact after applying the functor Δ. It follows that $M \in \mathrm{Ref}(P_\Lambda)$. Hence we deduce that $\mathrm{Ref}(P_\Lambda) = \mathrm{KerExt}^{i\geq 1}_{\Lambda}(-,P)$, therefore $\mathrm{proj}_\Lambda \subseteq \mathrm{Ref}(P_\Lambda)$. Moreover, by an argument similar to the proof of Proposition 3.3(1), we obtain that $\mathrm{proj}_A \subseteq \mathrm{Ref}({}_AP) \subseteq \mathrm{KerExt}^{i\geq 1}_{A}(-,P)$.

(4) $\Rightarrow$ (1) Since $\mathrm{proj}_\Lambda \subseteq \mathrm{Ref}(P_\Lambda) \subseteq \mathrm{KerExt}^{i\geq 1}_{\Lambda}(-,P)$, we easily check that $\mathrm{Ref}(P_\Lambda)$ is a resolving subcategory as the proof in Proposition 3.3(2). Now it remains to show that P is an r-costar module. By Theorem 4.4, we need only to prove that $\mathrm{KerExt}^{i\geq 0}_{A}(-,P) = 0$. Let $M \in \mathrm{KerExt}^{i\geq 0}_{A}(-,P)$ and take an exact sequence $0 \to N \to A_2 \to A_1 \to M \to 0$ with $A_2,\ A_1 \in \mathrm{proj}_A$. Then we have an induced exact sequence $0 \to \Delta(A_1) \to \Delta(A_2) \to \Delta(N) \to 0$ by applying the functor Δ. Note that $N \in \mathrm{Ref}({}_AP)$ by Lemma 4.2, so that $\Delta(N) \in \mathrm{Ref}(P_\Lambda) \subseteq \mathrm{KerExt}^{i\geq 1}_{\Lambda}(-,P)$. It follows that there is an induced exact sequence $0 \to \Delta^2(N) \to$

$\Delta^2(A_2) \to \Delta^2(A_1) \to 0$ by applying the functor Δ. Hence we obtain that $M = \mathrm{Coker}(A_2 \to A_1) \cong \mathrm{Coker}(\Delta^2(A_2) \to \Delta^2(A_1)) = 0$. □

Proposition 5.3. *Let P be an r-costar module. Then Ref(P_Λ) is closed under extensions if and only if $\mathrm{Ext}^1_\Lambda(M,P) = 0$ for any $M \in$ Ref(P_Λ).*

Proof. $\Rightarrow$. Every exact sequence of the form $0 \to P \xrightarrow{f} N \to M \to 0$ where $M \in \mathrm{Ref}(P_\Lambda)$. By hypothesis, $N \in \mathrm{Ref}(P_\Lambda)$. Then Δ preserves the exactness of the sequence by the definition of r-costar modules. So that, we have $g : N \to P$ such that $gf = 1_P$. Hence the sequence splits and $\mathrm{Ext}^1_\Lambda(M,P) = 0$.

$\Leftarrow$. Let any exact sequence $0 \to L \to M \to N \to 0$ with $L,\ N \in \mathrm{Ref}(P_\Lambda)$. Then Δ preserves the exactness of the sequence by assumption. We have the following commutative diagram

$$\begin{array}{ccccccccccc} 0 & \to & L & \to & M & \to & N & \to & 0 & & \\ & & \downarrow^{\delta_L} & & \downarrow^{\delta_M} & & \downarrow^{\delta_N} & & \downarrow & & \\ 0 & \to & \Delta^2(L) & \to & \Delta^2(M) & \to & \Delta^2(N) & \to & X & \to & 0. \end{array}$$

Note that $\delta_L,\ \delta_N$ are isomorphisms, since $L,\ N \in \mathrm{Ref}(P_\Lambda)$. Therefore, we have that δ_M is an isomorphism. That is, $M \in \mathrm{Ref}(P_\Lambda)$. □

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