Fundamentals of Contemporary Mathematical Sciences
doi:10.54974/fcmathsci.1094167
(**2023**) **4**(1) 9 – 15
Research Article

On the Analytical Determination of Geometric Characterizations of Analytic Functions

Alaattin Akyar *
Düzce University, Düzce Vocational School, Düzce, Türkiye

Received: 27 March 2022 **Accepted:** 02 January 2023

Abstract: As it is known, there are many sufficient conditions for the classification complex functions of one variable $f(z)$, which are analytic and univalent in the open unit disc $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$, and are also normalized with $f(0) = 1 - f'(0) = 0$ which are also known as normalization conditions. In this sense, the main goal of present article is to derive some special sufficient conditions for $f(z)$ to be starlike of order 2^{-r} and convex of order 2^{-r} in $\mathcal{U}$, with r is a positive integer.

Keywords: Analytic function, convex function, starlike function, univalent function.

1. Introduction

Let's take $\mathcal{A}$ as the class of functions of the form

$$w = f(z) = z + \sum_{n=2}^{\infty} a_n z^n = z + a_2 z^2 + \dots, \tag{1}$$

which are analytic in the open unit disc $\mathcal{U}$ and additionally satisfy normalization conditions $f(0) = 1 - f'(0) = 0$. If being univalent is imposed as an additional condition on the elements of class $\mathcal{A}$, class $\mathcal{S}$, which is a subclass of class $\mathcal{A}$, is obtained. The fact that a complex-valued function $f(z)$ is univalent in the unit disk $\mathcal{U}$ indicates that $w = f(z)$ for distinct z elements in $\mathcal{U}$ are also distinct [2]. In other words, the equation $f(z) = w$ has at most one root in the unit disk $\mathcal{U}$. Under these conditions, the regions $f(\mathcal{U})$ of the functions $f(z)$ belonging to the class $\mathcal{S}$ exhibits very interesting geometries. Moreover, these functions are classified using common geometries. We denoted by $\mathcal{S}^*$ the subclass of class $\mathcal{S}$ consisting of functions $f(z)$ in class $\mathcal{A}$, which exhibit a starlike geometry with respect to the origin and also denote by $\mathcal{C}$ the subclass of class $\mathcal{S}$ consisting of functions $f(z)$ in class $\mathcal{A}$, which exhibit a convex geometry. These two classes can be given

*Correspondence: alaattinakyar28@gmail.com.tr
2020 *AMS Mathematics Subject Classification*: 30C45, 33A30

analytically as

$$\mathcal{S}^* = \left\{ f(z) \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in \mathcal{U} \right\} \tag{2}$$

and

$$\mathcal{C} = \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, z \in \mathcal{U} \right\}, \tag{3}$$

respectively [2, 3, 6].

Furthermore, we denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{S}$ consisting of functions $f(z)$ in class $\mathcal{A}$, which are satisfies the condition $\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \ (0 \leq \alpha < 1)$, and analytically this subclass is given as

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \alpha \in [0,1), z \in \mathcal{U} \right\}. \tag{4}$$

Similarly, also we denote by $\mathcal{C}(\alpha)$ the subclass of $\mathcal{S}$ consisting of functions $f(z)$ in class $\mathcal{A}$, which are the satisfies the conditions $\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \ (0 \leq \alpha < 1)$. Analytically this subclass is given as

$$\mathcal{C}(\alpha) = \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \alpha \in [0,1), z \in \mathcal{U} \right\}. \tag{5}$$

Functions belonging to subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$, respectively, are called starlike of order α and convex of order α functions in the open unit disc $\mathcal{U}$. For proofs of analytical characterizations given so far, we refer to [1, 9, 12]. Since it is very difficult to classify analytic functions with different domains, it would be good to remember that the domain of definition is taken as the open unit disc $\mathcal{U}$ in the studies carried out in this field in the light of the Riemann mapping theorem. In this sense, the information given is only valid for the analytic $f(z)$ functions defined in the open unit disc $\mathcal{U}$ and satisfying the relevant conditions. Moreover, it is clear that the common geometric characterization of analytic functions belonging to a subclass cannot be generalized to all elements of the classes that cover this class. At this stage, it is clear from the subclass relationship that $\mathcal{C}(\alpha) \subset \mathcal{S}^*(\alpha) \subset \mathcal{S}^* \subset \mathcal{S} \subset \mathcal{A}$ and $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{C}(0) = \mathcal{C}$ for $\alpha = 0$. When interpreting the subclass relationship given above, it will be helpful to remember that any convex region is also a starlike region with respect to every point. Since the functions belonging to the class $\mathcal{A}$ meet the normalization conditions, functions that are starlike according to the origin are mainly used in the studies in this field.

Theorem 1.1 *The classes $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ satisfies Alexander duality relation*

$$zf'(z) \in \mathcal{S}^*(\alpha) \Leftrightarrow f(z) \in \mathcal{C}(\alpha), \ 0 \leq \alpha < 1. \tag{6}$$

The basic argument provided by this theorem, also known as the Alexander's theorem that $f(z)$ is univalent and convex if and only if $zf'(z)$ can be univalent and convex [2]. In this case, it can be said immediately that f is convex, according to the argument of Alexander's theorem. This brilliant theorem, which is not difficult to prove, is used as a very useful mathematical tool in obtaining many results set forth in univalent function theory.

Definition 1.2 *Let's take $f(z)$ and $g(z)$ be analytic in the open unit disc $\mathcal{U}$. If there is an analytic function $w(z)$ in $\mathcal{U}$ that satisfies the conditions $w(0)=0$, $|w(z)|<1$ and $f(z)=g(w(z))$, then the $f(z)$ function is said to be subordinate to the $g(z)$ function and denoted as $g(z)\prec f(z)$ [2].*

Lemma 1.3 *(Jack's Lemma) Let the (none constant) function $w(z)$ be analytic in the open unit disc $\mathcal{U}$ with $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at a point $z_0\in\mathcal{U}$, then $c=\dfrac{z_0w'(z_0)}{w(z_0)}$, where c is a real number and $c\geq 1$.*

It is well known that Jack's lemma is a very useful mathematical tool used in many applications in the theory of geometric functions [2, 5]. In this sense, we start by reminding that it is also used as a basic tool in the proof of our results.

2. Main Results

Theorem 2.1 *Let $f(z)$ be a function in class $\mathcal{A}$. If $f(z)$ satisfies*

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)<\frac{3+2^{-r}}{2\left(1+2^{-r}\right)},\ |z|<1 \tag{7}$$

for some $2+2^{-r}(r\in\mathbb{N})$, then

$$\frac{zf'(z)}{f(z)}\prec\frac{\left(2+2^{-r}\right)(1-z)}{\left(2+2^{-r}\right)-z},\ |z|<1 \tag{8}$$

and

$$\left|\frac{zf'(z)}{f(z)}-\frac{2+2^{-r}}{3+2^{-r}}\right|<\frac{2+2^{-r}}{3+2^{-r}},\ |z|<1. \tag{9}$$

This implies that $f(z)\in\mathcal{S}^*\left(2^{-r}\right)$.

Proof As in many studies in this field, let's define the function $w(z)$ by

$$\frac{zf'(z)}{f(z)}=\frac{\left(2+2^{-r}\right)(1-w(z))}{\left(2+2^{-r}\right)-w(z)},\ w(z)\neq 2+2^{-r} \tag{10}$$

to prove the result of the theorem under the given conditions. It is clear that the function $w(z)$ is analytic in the open unit disc $\mathcal{U}$ and also $w(0)=0$. Thus, we need to prove that $|w(z)|<1$ in $\mathcal{U}$ according to the Jack's Lemma Lemma 1.3. Since

$$1+\frac{zf''(z)}{f'(z)}=\frac{(2+2^{-r})(1-w(z))}{(2+2^{-r})-w(z)}-\frac{zw'(z)}{1-w(z)}+\frac{zw'(z)}{(2+2^{-r})-w(z)}, \tag{11}$$

we have that

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)=\Re\left(\frac{(2+2^{-r})(1-w(z))}{(2+2^{-r})-w(z)}-\frac{zw'(z)}{1-w(z)}+\frac{zw'(z)}{(2+2^{-r})-w(z)}\right),$$

so $\Re\left(1+\frac{zf''(z)}{f'(z)}\right)=\dfrac{3+2^{-r}}{2(1+2^{-r})}, |z|<1$ for some $2+2^{-r}(r\in\mathbb{N})$. At this stage, using the exponential form of the complex number provides ease of operation. Now, if there is a point z_0 in $\mathcal{U}$ such that $\max\limits_{|z|\leq|z_0|}=|w(z_0)|=1$, then $w(z_0)=e^{i\theta}$ and $c=\dfrac{z_0w'(z_0)}{w(z_0)}$, $c\geq 1$ by Jack's Lemma 1.3. So, we have

$$\begin{aligned}1+\frac{z_0f''(z_0)}{f'(z_0)}&=\frac{(2+2^{-r})(1-w(z_0))}{(2+2^{-r})-w(z_0)}-\frac{zw'(z_0)}{1-w(z_0)}+\frac{zw'(z_0)}{(2+2^{-r})-w(z_0)}\\&=(2+2^{-r})+(2+2^{-r})(1-(2+2^{-r})+c)\frac{1}{(2+2^{-r})-e^{i\theta}}-\frac{c}{1-e^{i\theta}}.\end{aligned}$$

Thus, if follows that

$$\Re\left(\frac{1}{(2+2^{-r})-e^{i\theta}}\right)=\frac{1}{2(2+2^{-r})}+\frac{(2+2^{-r})^2-1}{2(2+2^{-r})\left(1+(2+2^{-r})^2-2cos\theta\right)}$$

and so $\Re\left(\dfrac{1}{1-w(z_0)}\right)=\dfrac{1}{2}$. This implies that, for $2+2^{-r}$ $(r\in\mathbb{N})$,

$$\begin{aligned}\Re\left(1+\frac{z_0f''(z_0)}{f'(z_0)}\right)&\geq\frac{3+2^{-r}}{2}+\frac{(3+2^{-r})(1-(2+2^{-r})+c)}{2(1+2^{-r})}\\&\geq\frac{3+2^{-r}}{2}+\frac{(3+2^{-r})(-2^{-r})}{2(1+2^{-r})}\\&=\frac{3+2^{-r}}{2(1+2^{-r})}.\end{aligned}$$

This contradicts the hypothesis of our theorem. Therefore, there is no $z_0\in\mathcal{U}$ such that $|w(z_0)|=1$ for all $z\in\mathcal{U}$, that is

$$\frac{zf'(z)}{f(z)}\prec\frac{(2+2^{-r})(1-z)}{(2+2^{-r})-z},\ |z|<1.$$

Furthermore, since

$$w(z) = \frac{(2+2^{-r})\left(\frac{zf'(z)}{f(z)} - 1\right)}{\frac{zf'(z)}{f(z)} - (2+2^{-r})}$$

and $|w(z)| < 1$ $(|z| < 1)$, we conclude that

$$\left| \frac{zf'(z)}{f(z)} - \frac{2+2^{-r}}{3+2^{-r}} \right| < \frac{2+2^{-r}}{3+2^{-r}}, \ |z| < 1,$$

which implies that $f(z) \in \mathcal{S}^*$. □

While $r \to \infty$ in the theorem, the following corollary due to Singh R. and Singh S. is obtained with a different calculation [11]. The proof is obtained directly from the proof of Theorem 2.1.

Corollary 2.2 *Let $f(z)$ be a function in class $\mathcal{A}$. If $f(z)$ satisfies*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{3}{2}, \ |z| < 1,$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{2(1-z)}{2-z}, \ |z| < 1,$$

and

$$\left| \frac{zf'(z)}{f(z)} - \frac{2}{3} \right| < \frac{3}{2}, \ |z| < 1.$$

Theorem 2.3 *Let $f(z)$ be a function in class $\mathcal{A}$. If $f(z)$ satisfies*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{7+3.2^{-r}}{2\,(2+2^{-r})\,(3+2^{-r})}, \ |z| < 1 \tag{12}$$

for some $1+2^{-r}(r \in \mathbb{N})$, then $f(z) \in \mathcal{C}\left(\frac{3+2^{-r}}{2(2+2^{-r})}\right)$.

Proof We define the function in $\mathcal{U}$ by

$$\frac{f(z)}{zf'(z)} = \frac{(2+2^{-r})\,(1-w(z))}{(2+2^{-r}) - w(z)}, \ (w(z) \neq 2+2^{-r}, r \in \mathbb{N}), \tag{13}$$

so that $w(z)$ is analytic in $\mathcal{U}$ and $|w(z)| < 1$. In this case, if the logarithmically derivative of both sides of equation (13) is taken and necessary simplifying are made,

$$1 + \frac{zf''(z)}{f'(z)} = \frac{(2+2^{-r}) - w(z)}{(2+2^{-r})\,(1-w(z))} + \frac{zw'(z)}{1-w(z)} + \frac{zw'(z)}{(2+2^{-r}) - w(z)}$$

and hence

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)=\Re\left(\frac{(2+2^{-r})-w(z)}{(2+2^{-r})(1-w(z))}+\frac{zw'(z)}{1-w(z)}+\frac{zw'(z)}{(2+2^{-r})-w(z)}\right)$$
$$>\frac{7+3.2^{-r}}{2(2+2^{-r})(3+2^{-r})}$$

for $1+2^{-r}(r\in\mathbb{N})$. Now, if there is a point z_0 in $\mathcal{U}$ such that $\max_{|z|\le|z_0|}=|w(z_0)|=1$, then $w(z_0)=e^{i\theta}$ and $c=\dfrac{z_0w'(z_0)}{w(z_0}$, $c\ge 1$ by Jack's Lemma 1.3. So, we have

$$\Re\left(1+\frac{z_0f''(z_0)}{f'(z_0)}\right)=\frac{1}{2}+\frac{1}{2(2+2^{-r})}-\frac{c\left((2+2^{-r})^2+1\right)}{2\left(1+(2+2^{-r})^2-2(2+2^{-r})\,cos\theta\right)}$$

and, for $1+2^{-r}>0$ $(r\in\mathbb{N})$,

$$\Re\left(1+\frac{z_0f''(z_0)}{f'(z_0)}\right)\le\frac{1}{2}+\frac{1}{2(2+2^{-r})}-\frac{1+2^{-r}}{2(3+2^{-r})}$$
$$=\frac{7+3.2^{-r}}{2(2+2^{-r})(3+2^{-r})}.$$

This contradicts the hypothesis of our theorem. Therefore, there is no $z_0\in\mathcal{U}$ such that $|w(z_0)|=1$ for all $z\in\mathcal{U}$, that is

$$\frac{f(z)}{zf'(z)}\prec\frac{(2+2^{-r})(1-z)}{(2+2^{-r})-z},\ |z|<1.$$

Furthermore, since

$$w(z)=\frac{(2+2^{-r})\left(1-\frac{zf'(z)}{f(z)}\right)}{1-(2+2^{-r})\frac{zf'(z)}{f(z)}}$$

and $|w(z)|<1$ $(|z|<1)$, we conclude that $f(z)\in\mathcal{C}\left(\frac{3+2^{-r}}{2(2+2^{-r})}\right)$. □

Corollary 2.4 *Letting $r\to\infty$ in Theorem 2.3, then $f(z)\in\mathcal{C}(3/4)$.*

3. Conclusion

As it is known, geometric functions basically aim to classify complex functions that are analytic on the open unit disk, provided that they meet some additional conditions such as being univalent and satisfying the normalization conditions. While doing this, a relationship is established between the analytical properties of the functions in question and the geometric properties of their images, as an interaction of analysis and geometry. If the function can be easily graphed, it will be fairly easy to classify the image set according to its geometric characterization. Unfortunately, that may not always be the case. In this case, there are many analytical methods that can be used in the literature. In this sense, we tried to give a different perspective to the conditions existing in the literature.

Declaration of Ethical Standards
The author declares that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Conflict of Interest
The author declares no conflicts of interest.

References

[1] Akyar A., *A new subclass of certain analytic univalent functions associated with hypergeometric functions*, Turkish Journal of Mathematics, 46(1), 145-156, 2022.

[2] Duren P.L., Univalent Functions, Springer Science, Business Media, Vol. 259, 2001.

[3] Goodman A.W., Univalent Functions, Polygonal Publishing House, Vol. I Vol. II, 1983.

[4] Guo L., Wang Y., Liu J., *On the starlikeness for certain analytic functions*, International Journal of Mathematical Research, 1(2), 21-25, 2012.

[5] Jack I.S., *Functions starlike and convex of order* α, Journal of the London Mathematical Society, 2(3), 469-474, 1971.

[6] Noor K.I., On Close-to-Convex and Related Functions, Ph.D. Thesis, University of Wales, Swansea, UK, 1972.

[7] Noor, K.I., *On quasi-convex functions and related topics*, International Journal of Mathematics and Mathematical Sciences, 10(2), 241-258, 1978.

[8] Nunokawa M., Aydoğan M., Kuroki K., Yildiz I., Owa S., *Some properties concerning close-to-convexity of certain analytic functions*, Journal of Inequalities and Applications, 245, 2012.

[9] Owa S., Saitoh H., Srivastava H.M., Yamakawa R., *Geometric properties of solutions of a class of differential equations*, Computers and Mathematics with Applications, 47(10-11), 1689-1696, 2004.

[10] Pommerenke C., Univalent Functions, Vandenhoeck and Ruprecht, 1975.

[11] Singh R., Singh S., *Some sufficient conditions for univalence and starlikeness*, Colloquium Mathematicum, 47(2), 309-314, 1982.

[12] Yildiz I., Akyar A., *An analytical investigatıon on starlikeness and convexity properties for hypergeometric functions*, Turkish Journal of Mathematics, 44(4), 1453-1465, 2022.