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Determinants and Inverses of Circulant Matrices with Gaussian Pell Numbers

Meral YAŞAR KARTAL[*1]

[1] Niğde Ömer Halisdemir University, Science and Art Faculty, Department of Mathematics, NIGDE

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Abstract: In this paper, by considering the circulant matrix $M_n = circ(GP_1, GP_2, \dots, GP_n)$ whose entries are the Gaussian Pell numbers, we calculate the determinants and inverses of M_n in terms of Gaussian Pell numbers.

Elemanları Gaussian Pell Sayıları Olan Sirkülant Matrislerin Determinantları ve Tersleri

Anahtar Kelimeler
Sirkülant matris,
Determinant,
Matris tersi,
Gauss Pell sayıları

Öz: Bu çalışmada, elemanları Gauss Pell sayıları olan $M_n = circ(GP_1, GP_2, \dots, GP_n)$ sirkülant matrisinin determinantı ve tersi yine Gauss Pell sayıları cinsinden hesaplanmıştır.

*Corresponding Author, email: myasar@ohu.edu.tr

1. Introduction and Preliminaries

Circulant matrices have a wide range of applications, for example in image processing, coding theory, signal processing, numerical computation, self-regress design, etc. For detail one can see [1], [2].
There are many studies in the literature which is about circulant matrices and their properties such as their determinants and inverses involving some famous numbers.
Lind studied on the determinant $D_{n,r}$ of the circulant matrix $circ(F_r, F_{r+1}, \dots, F_{r+n-1})$ in 1970 [3]. In [4], the author gave the bounds for the spectral and Euclidean norms of the circulant matrices involving Fibonacci and Lucas numbers. In [5], [6] the authors defined generalized k-Horadam sequence and investigated some its properties. In addition, a new generalization to compute determinants and inverses of the circulant matrix $C_n(H) = circ(H_{k,1}, H_{k,2}, \dots, H_{k,n})$ where $H_{k,n}$ is the generalized $k-$Horadam numbers was presented. Also, in another study of the same authors, a new upper and lower bounds for the spectral norm of an $r-$circulant matrix H whose entries are generalized $k-$Horadam numbers were presented. Furthermore, they obtained new formulas to calculate the eigenvalues and determinant of the matrix H [7]. Shen et al. obtained the determinants of the circulant matrix with classical Fibonacci and Lucas numbers. In addition, the inverses of these matrices were derived in [8]. In [9], the determinants and inverses of the circulant matrix involving Jacobsthal and Jacobsthal-Lucas numbers were obtained in terms of these numbers. In another study, the same authors studied on the $r-$circulant matrix $\mathbb{W}_n = (W_1, W_2, \dots, W_n)$ associated with the numbers defined by the recurrence relation $W_n = pW_{n-1} + qW_{n-2}$ with initial conditions $W_0 = a$ and $W_1 = b$. They obtained determinants, inverses and some bounds for spectral norms of $r-$circulant matrix $\mathbb{W}_n$ [10]. Jiang et al. studied on some types of circulant matrices. They proved that these matrices with Gaussian Fibonacci numbers were invertible matrices for $n > 2$ and they gave the determinants and inverses of these matrices in [11]. In [12], the authors calculated the determinant of the circulant matrix $\mathbb{F}_n = circ(F_1^*, F_2^*, \dots, F_n^*)$ where F_n^* is the complex Fibonacci numbers. In addition, they showed that this matrix is invertible and inverse matrix can be obtained in terms of complex Fibonacci numbers. In [13], the author used the

algebra methods, the properties of the r-circulant matrix and the geometric circulant matrix to study the upper and lower bound estimate problems for the spectral norms of a geometric circulant matrix involving the generalized k-Horadam numbers and some estimations were obtained.
Horadam and Mahon introduced Pell and Pell–Lucas polynomials. Moreover, some properties related with these sequences were studied in [14]. In [15], sum formulas for squares of terms of complex Pell and Pell-Lucas number sequences were studied and certain products of terms of the Pell and Pell-Lucas sequences were determined. In [16], the gell numbers were defined as the generalization of Pell numbers. Moreover, the authors derived Binet-like formula, generating function and exponential generating function for this sequence. The authors, introduced the quadra Fibona-Pell,Fibona-Jacobsthal and Pell-Jacobsthal and the hexa Fibona-Pell-Jacobsthal sequences. These sequences are the compound sequences of Fibonacci, Pell and Jacobsthal sequences. They derived the Binet-like formulas, the generating functions and the exponential generating functions of these sequences. Also, some binomial identities were obtained for them [17]. In [18], the authors considered the Pell, Pell-Lucas and Modified Pell sequences, and they defined some new 2×2 matrices, then showed that the identities presented before can be produced by using them. In [19], the authors defined the Gaussian Pell and Gaussian Pell-Lucas sequences. They obtained some identities for these numbers.
In this paper, we consider the circulant matrix $M_n = (GP_1, GP_2, \dots, GP_n)$, where GP_n is the Gaussian Pell numbers. Firstly, we obtained the determinant of this matrix in terms of Gaussian Pell numbers. Then we calculate the inverse of the circulant matrix M_n.
We conclude this section with some preliminaries related our study.
The $n \times n$ circulant matrix $C_n = circ(c_0, c_1, \dots, c_{n-1})$, associated with the numbers $c_0, c_1, \dots, c_{n-1}$ is defined as

$$C_n = \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & \cdots & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & \cdots & c_0 \end{bmatrix}.$$

Determinant and inverse of nonsingular circulant matrix C_n are given as in the following

$$detC_n = \prod_{r=0}^{n-1} g(w^r), \quad C_n^{-1} = circ(a_0, a_1, \dots, a_{n-1})$$

where $a_s = \frac{1}{n}\sum_{r=0}^{n-1} g(w^r)^{-1} w^{-rs} \quad (s = 0,1, \dots, n-1),\ \ g(x) = \sum_{i=0}^{n-1} c_i x^i$ and $w = exp\left(\frac{2\pi i}{n}\right)$ [2].

Lemma 1.1. [2] Let $C_n = circ(c_0, c_1, \dots, c_{n-1})$ be a circulant matrix. Then we have the following

i. C_n is invertible if and only if $f(w^k) \neq 0\ (k = 0,1, \dots, n-1)$, where $f(x) = \sum_{j=0}^{n-1} c_j x^j$ and $w = exp\left(\frac{2\pi i}{n}\right)$
ii. If C_n is invertible then its inverse is also a circulant matrix.

2. Main Results

In this section, we consider the circulant matrix M_n with Gaussian Pell numbers. Firstly, we give the determinant of the matrix M_n. Then we prove that M_n is an invertible matrix and we formulate the inverse matrix in terms of Gaussian Pell numbers.

Definition 2.1. The Gaussian Pell numbers are defined as

$$GP_n = 2GP_{n-1} + GP_{n-2}, \qquad n \geq 2$$

with the initial conditions $GP_0 = i, GP_1 = 1$ [19].

Theorem 2.2. Let M_n be a circulant matrix with Gaussian Pell numbers as $M_n = circ(GP_1, GP_2, \dots, GP_n)$. Then we have

$$detM_n = (1 - GP_{n+1})^{n-2}(1 - 2GP_n - iGP_n) + \sum_{k=1}^{n-2}(GP_k - iGP_{k+1})(1 - GP_{n+1})^{k-1}(GP_n - i)^{n-k-1}$$

for $n \geq 3$.

Proof. For $n \geq 3$, set

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -2 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -1 & 0 & 0 & 0 & \cdots & 1 & -2 \\ 0 & 0 & 0 & 0 & \cdots & -2 & -1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 1 & -2 & \cdots & 0 & 0 \\ 0 & 1 & -2 & -1 & \cdots & 0 & 0 \end{bmatrix}$$

and

$$L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \left(\frac{GP_n - i}{1-GP_{n+1}}\right)^{n-2} & 0 & 0 & \cdots & 0 & 1 \\ 0 & \left(\frac{GP_n - i}{1-GP_{n+1}}\right)^{n-3} & 0 & 0 & \cdots & 1 & 0 \\ 0 & \left(\frac{GP_n - i}{1-GP_{n+1}}\right)^{n-4} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \left(\frac{GP_n - i}{1-GP_{n+1}}\right) & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Then we have the following matrix

$$KM_nL_1 = \begin{bmatrix} 1 & f_n & GP_{n-1} & GP_{n-2} & GP_{n-3} & \cdots & GP_3 & GP_2 \\ i & g_n & GP_{n-2} & GP_{n-3} & GP_{n-4} & \cdots & GP_2 & GP_1 \\ 0 & 0 & GP_1 - GP_{n+1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & GP_0 - GP_n & GP_1 - GP_{n+1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & GP_0 - GP_n & GP_1 - GP_{n+1} & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & GP_0 - GP_n & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & GP_0 - GP_n & GP_1 - GP_{n+1} \end{bmatrix}$$

where

$$f_n = \sum_{k=1}^{n-1} GP_{k+1} \left(\frac{GP_n - i}{1 - GP_{n+1}}\right)^{n-k-1}$$

and

$$g_n = 1 - 2GP_n + \sum_{k=1}^{n-2} GP_k \left(\frac{GP_n - i}{1 - GP_{n+1}}\right)^{n-k-1}.$$

So, we obtain

$$detKdetM_ndetL_1 = \left[1 - 2GP_n + \sum_{k=1}^{n-2} GP_k \left(\frac{GP_n - i}{1 - GP_{n+1}}\right)^{n-k-1} - i\sum_{k=1}^{n-1} GP_{k+1} \left(\frac{GP_n - i}{1 - GP_{n+1}}\right)^{n-k-1}\right](1 - GP_{n+1})^{n-2}$$

while

$$detK = detL_1 = \begin{cases} 1, & n \equiv 1,2(mod4) \\ -1, & n \equiv 0,3(mod4) \end{cases}.$$

Hence, we get

$$detM_n = (1-GP_{n+1})^{n-2}(1-2GP_n-iGP_n) + \sum_{k=1}^{n-2}(GP_k - iGP_{k+1})(1-GP_{n+1})^{k-1}(GP_n - i)^{n-k-1}.$$

Thus, the proof is completed.

Theorem 2.3. Let $M_n = circ(GP_1, GP_2, \ldots, GP_n)$ be a circulant matrix. For $n \geq 3, M_n$ is invertible.

Proof. Let $n \geq 3$. The Binet formula for Gaussian Pell number is

$GP_n = c\alpha^n + d\beta^n$

where $c = \frac{1+(\sqrt{2}-1)i}{2\sqrt{2}}$ and $d = \frac{-1+(\sqrt{2}+1)i}{2\sqrt{2}}$. It is clear that $\alpha+\beta = 2$ and $\alpha\beta = -1$. Hence we have

$$\begin{aligned} f(w^k) &= \sum_{j=1}^{n} GP_j (w^k)^{j-1} \\ &= c\alpha\left(\frac{1-\alpha^n}{1-\alpha w^k}\right) + d\beta\left(\frac{1-\beta^n}{1-\beta w^k}\right) \\ &= \frac{c\alpha(1-\alpha^n)(1-\beta w^k) + d\beta(1-\beta^n)(1-\alpha w^k)}{1-2w^k-w^{2k}} \\ &= \frac{1-GP_{n+1}+w^k(i-GP_n)}{1-2w^k-w^{2k}} \end{aligned}$$

Since $GP_n = P_n + iP_{n-1}, n > 1, w^k = cos\theta + isin\theta$ where $\theta = \frac{2k\pi}{n}$ and $0 < \theta < 2\pi$. Then

$$\begin{aligned} x &= 1-GP_{n+1}+w^k(i-GP_n) \\ &= [1-P_{n+1}-P_n cos\theta + (P_{n-1}-1)sin\theta] + i[-P_n + (1-P_{n-1})cos\theta - P_n sin\theta]. \end{aligned}$$

We assume that

$Re(x) = 1 - P_{n+1} - P_n cos\theta + (P_{n-1}-1)sin\theta$

and

$Im(x) = -P_n + (1-P_{n-1})cos\theta - P_n sin\theta.$

We prove that $Re(x) \neq 0$ or $Im(x) \neq 0$ for $1-2w^k-w^{2k} \neq 0$. For the reason that Pell sequence is an increasing sequence, we have the followings.

If $sin\theta > 0$ and $cos\theta > 0, Re(x) < 0$.
If $sin\theta < 0$ and $cos\theta < 0, Im(x) > 0$.
If $sin\theta > 0$ and $cos\theta < 0, Re(x) < 0$.
If $sin\theta < 0$ and $cos\theta > 0, Im(x) < 0$.

It is verified that when $sin\theta = 0$ or $cos\theta = 0, x \neq 0$.
Hence, $1-GP_{n+1}+w^k(i-GP_n) \neq 0$ for any w^k $(k = 1,2,\ldots,n-1)$, that is $f(w^k) \neq 0$. By Lemma 1.1 the proof is completed.

Lemma 2.4. Let $B = (b_{ij})$ be an $(n-2)\times(n-2)$ matrix of the form

$$b_{ij} = \begin{cases} GP_1 - GP_{n+1}, & i = j \\ GP_0 - GP_n, & i = j+1 \\ 0, & otherwise \end{cases}$$

then the inverse $B^{-1} = (b'_{ij})$ is given by

$$b'_{ij} = \begin{cases} \frac{(GP_n - GP_0)^{i-j}}{(GP_1 - GP_{n+1})^{i-j+1}}, & i \geq j \\ 0, & i < j \end{cases}.$$

Proof. Let $c_{ij} = \sum_{k=1}^{n-2} b_{ik} b'_{kj}$. It is clear that $c_{ij} = 0$ for $i < j$. For $i = j$, we get

$$c_{ii} = b_{ii} b'_{ii} = (GP_1 - GP_{n+1}) \frac{1}{(GP_1 - GP_{n+1})} = 1.$$

For $i \geq j + 1$, we obtain

$$\begin{aligned} c_{ij} &= \sum_{k=1}^{n-2} b_{ik} b'_{kj} = b_{i,i-1} b'_{i-1,j} + b_{ii} b'_{ij} \\ &= (GP_0 - GP_n) \frac{(GP_n - GP_0)^{i-j-1}}{(GP_1 - GP_{n+1})^{i-j}} + (GP_1 + GP_{n+1}) \frac{(GP_n - GP_0)^{i-j}}{(GP_1 - GP_{n+1})^{i-j+1}} \\ &= 0. \end{aligned}$$

So, we see that $BB^{-1} = I_{n-2}$, where I_{n-2} is $(n-2) \times (n-2)$ identity matrix. Similarly, it can be shown that $B^{-1}B = I_{n-2}$. Hence, the proof is completed.

Theorem 2.5. Let $n \geq 3$, then the inverse of M_n is

$$M_n^{-1} = circ(m_1, m_2, \dots, m_n)$$

where

$$m_1 = \frac{i}{f_n} - \frac{2(1-i)}{GP_0 f_n - GP_1 g_n} \sum_{k=1}^{n-2} (-1)^{k+1} \frac{(GP_0 - GP_n)^{k-1}}{(GP_1 - GP_{n+1})^k} P_{n-k}$$

$$m_2 = \frac{1-2i}{f_n} + \frac{2(1-i)}{GP_0 f_n - GP_1 g_n} \sum_{k=1}^{n-2} (-1)^k \frac{(GP_0 - GP_n)^{k-1}}{(GP_1 - GP_{n+1})^k} P_{n-k-1}$$

$$m_3 = \frac{2(1-i)}{GP_0 f_n - GP_1 g_n} \frac{1}{(GP_1 - GP_{n+1})}$$

$$m_4 = \frac{-2(1-i)}{GP_0 f_n - GP_1 g_n} \frac{GP_0 - GP_n}{(GP_1 - GP_{n+1})^2}$$

and

$$m_j = \frac{(-1)^{j-1} 2(1-i)}{GP_0 f_n - GP_1 g_n} \frac{(GP_0 - GP_n)^{j-3}}{(GP_1 - GP_{n+1})^{j-2}}, \qquad j = 5,6, \dots, n.$$

Proof. Let

$$L_2 = \begin{bmatrix} 1 & -\frac{f_n}{GP_1} & \frac{f_n GP_{n-2} - g_n GP_{n-1}}{GP_1 g_n - GP_0 f_n} & \frac{f_n GP_{n-3} - g_n GP_{n-2}}{GP_1 g_n - GP_0 f_n} & \cdots & \frac{f_n GP_1 - g_n GP_2}{GP_1 g_n - GP_0 f_n} \\ 0 & 1 & \frac{GP_{n-1} GP_0 - GP_{n-2} GP_1}{GP_1 g_n - GP_0 f_n} & \frac{GP_{n-2} GP_0 - GP_{n-3} GP_1}{GP_1 g_n - GP_0 f_n} & \cdots & \frac{GP_2 GP_0 - GP_1 GP_1}{GP_1 g_n - GP_0 f_n} \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

where

$$f_n = \sum_{k=1}^{n-1} GP_{k+1}\left(\frac{GP_n - i}{1 - GP_{n+1}}\right)^{n-k-1}$$

and

$$g_n = 1 - 2GP_n + \sum_{k=1}^{n-2} GP_k\left(\frac{GP_n - i}{1 - GP_{n+1}}\right)^{n-k-1}.$$

Then we can write

$$KM_nL_1L_2 = U \oplus B,$$

where $U = \begin{bmatrix} GP_1 & 0 \\ GP_0 & -\frac{f_n GP_0}{GP_1} \end{bmatrix}$ is 2×2 matrix, B is as in Lemma 2.4 and $U \oplus B$ is the direct sum of the matrices U and B. Let $L = L_1L_2$, then we obtain

$$M_n^{-1} = L(U^{-1} \oplus B^{-1})K.$$

By Lemma 1.1., the inverse of the matrix M_n is circulant. Let

$$M_n^{-1} = circ(m_1, m_2, \dots, m_n).$$

Since the last row of the matrix L are

$$0,1, -\frac{(GP_0GP_{n-1} - GP_1GP_{n-2})}{GP_0f_n - GP_1g_n}, -\frac{(GP_0GP_{n-2} - GP_1GP_{n-3})}{GP_0f_n - GP_1g_n}, \cdots, -\frac{(GP_0GP_2 - GP_1GP_1)}{GP_0f_n - GP_1g_n}$$

and Y_{nj} be the nj-th entry of the product $L(U^{-1} \oplus)B^{-1}$ for $1 \le j \le n$, we have

$$Y_{n1} = \frac{1}{f_n}, Y_{n2} = \frac{i}{f_n}$$

and for $3 \le j \le n$

$$Y_{nj} = \frac{2(1-i)}{GP_0f_n - GP_1g_n}\sum_{k=1}^{n-j+1}(-1)^{k-1}\frac{(GP_0 - GP_n)^{k-1}}{(GP_1 - GP_{n+1})^k}P_{n-k-j+2}.$$

If the row matrix $(Y_{n1}, Y_{n2}, \dots, Y_{nn})$ and the matrix K is multiplied, the last row of M_n^{-1} is obtained. Namely,

$$\begin{aligned} m_1 &= Y_{n2} - 2Y_{n3} - Y_{n4} \\ &= \frac{i}{f_n} - \frac{4(1-i)}{GP_0f_n - GP_1g_n}\sum_{k=1}^{n-2}(-1)^{k-1}\frac{(GP_0 - GP_n)^{k-1}}{(GP_1 - GP_{n+1})^k}P_{n-k-1} - \frac{2(1-i)}{GP_0f_n - GP_1g_n}\sum_{k=1}^{n-3}(-1)^{k-1}\frac{(GP_0 - GP_n)^{k-1}}{(GP_1 - GP_{n+1})^k}P_{n-k-2} \\ &= \frac{i}{f_n} - \frac{2(1-i)}{GP_0f_n - GP_1g_n}\sum_{k=1}^{n-2}(-1)^{k+1}\frac{(GP_0 - GP_n)^{k-1}}{(GP_1 - GP_{n+1})^k}P_{n-k} \end{aligned}$$

$$\begin{aligned} m_2 &= Y_{n1} - 2Y_{n2} - Y_{n3} \\ &= \frac{1}{f_n} - \frac{2i}{f_n} - \frac{2(1-i)}{GP_0f_n - GP_1g_n}\sum_{k=1}^{n-2}(-1)^{k-1}\frac{(GP_0 - GP_n)^{k-1}}{(GP_1 - GP_{n+1})^k}P_{n-k-1} \\ &= \frac{1-2i}{f_n} - \frac{2(1-i)}{GP_0f_n - GP_1g_n}\sum_{k=1}^{n-2}(-1)^{k-1}\frac{(GP_0 - GP_n)^{k-1}}{(GP_1 - GP_{n+1})^k}P_{n-k-1} \end{aligned}$$

$$m_3 = Y_{nn} = \frac{2(1-i)}{GP_0 f_n - GP_1 g_n} \frac{1}{GP_1 - GP_{n+1}}$$

$$\begin{aligned} m_4 &= Y_{n,n-1} - 2Y_{nn} \\ &= \frac{2(1-i)}{GP_0 f_n - GP_1 g_n} \sum_{k=1}^{2} (-1)^{k-1} \frac{(GP_0 - GP_n)^{k-1}}{(GP_1 - GP_{n+1})^k} P_{3-k} - \frac{4(1-i)}{GP_0 f_n - GP_1 g_n} \frac{1}{GP_1 - GP_{n+1}} P_{2-k} \\ &= -\frac{2(1-i)}{GP_0 f_n - GP_1 g_n} \frac{GP_0 - GP_n}{(GP_1 - GP_{n+1})^2} \end{aligned}$$

and for $5 \leq j \leq n$

$$\begin{aligned} m_j &= Y_{n,n-j+3} - 2Y_{n,n-j+4} - Y_{n,n-j+5} \\ &= \frac{2(1-i)}{GP_0 f_n - GP_1 g_n} \sum_{k=1}^{j-2} (-1)^{k-1} \frac{(GP_0 - GP_n)^{k-1}}{(GP_1 - GP_{n+1})^k} P_{j-k-1} - \frac{4(1-i)}{GP_0 f_n - GP_1 g_n} \sum_{k=1}^{j-3} (-1)^{k-1} \frac{(GP_0 - GP_n)^{k-1}}{(GP_1 - GP_{n+1})^k} P_{j-k-2} \\ &\quad - \frac{2(1-i)}{GP_0 f_n - GP_1 g_n} \sum_{k=1}^{j-4} (-1)^{k-1} \frac{(GP_0 - GP_n)^{k-1}}{(GP_1 - GP_{n+1})^k} P_{j-k-3} \\ &= \frac{(-1)^{j-1} 2(1-i)}{GP_0 f_n - GP_1 g_n} \frac{(GP_0 - GP_n)^{j-3}}{(GP_1 - GP_{n+1})^{j-2}}. \end{aligned}$$

3. Conclusion

In conclusion, we obtain formulas for the determinant and inverse of circulant matrices whose entries are Gaussian Pell numbers.

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