Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 52 (3) (2023), 560 – 571
DOI : 10.15672/hujms.1096357

Research Article

Convexity and double-sided Taylor's approximations

Y. J. Bagul[1], C. Chesneau[2], M. Kostić[3], T. Lutovac[4], B. Malešević[4*], M. Rašajski[4]

[1] *Department of Mathematics, K.K.M. College Manwath, Dist: Parbhani(M.S.)-43150 India*
[2] *LMNO, University of Caen-Normandie, Caen, France*
[3] *Faculty of Technical Scences, University of Novi Sad, Serbia*
[4] *School of Electrical Engineering, University of Belgrade, Serbia*

Abstract

Using convexity and double-sided Taylor's approximations of functions, we establish new general results in this field which can be used to refine and/or sharp some analytic inequalities in the existing literature.

Mathematics Subject Classification (2020). 33B10, 26D05, 26D07

Keywords. Convexity, double-sided Taylor's approximation, Kober's inequality, Janous inequality

1. Introduction

The inequality

$$1 - \frac{2x}{\pi} \leq \cos x \leq 1 - \frac{x^2}{\pi};\ x \in \left[0, \frac{\pi}{2}\right] \tag{1.1}$$

was first established by H. Kober [3, p. 22] in 1944, and the inequality

$$\sin x \geq \frac{2}{\pi}x;\ x \in \left[0, \frac{\pi}{2}\right]$$

was first established by C. Jordan in 1869 [2]. Another inequality of interest is the Janous inequality

$$\frac{\sin x}{x} \geq \frac{2}{\pi}\left(1 + \frac{\pi^2}{24}\right) - \frac{x^2}{3\pi};\ x \in \left(0, \frac{\pi}{2}\right]. \tag{1.2}$$

The functions $\sin x/x$ and $\cos x$ have been considered many times by researchers, and obtaining sharp boundaries of them has always piqued interest. Concerning the already established results about the polynomial, rational and irrational bounds of functions $\sin x/x$ and $\cos x$, we would like to recommend reading the survey article [10] by F. Qi, D.-W. Niu

*Corresponding Author.
Email addresses: yjbagul@gmail.com (Y. J. Bagul), christophe.chesneau@unicaen.fr (C. Chesneau), marco.s@verat.net (M. Kostić), tatjana.lutovac@etf.bg.ac.rs (T. Lutovac), branko.malesevic@etf.bg.ac.rs (B. Malešević), marija.rasajski@etf.bg.ac.rs (marija.rasajski@etf.bg.ac.rs)
Received: 31.03.2022; Accepted: 28.09.2022

and B.-N. Guo (see also Example 3.4, where we will use some particular results quoted in this article). On the other hand, the double-sided Taylor's approximations play important role in the theory of analytic inequalities; see, e.g., [5, 6, 8, 13] and references cited therein for further information in this direction. As mentioned in the abstract, the main aim of this paper is to establish some general results about the double-sided Taylor's approximations of functions. We essentially use the convexity here and obtain some general results about the double-sided Taylor's approximations of functions which can give better estimates compared to the estimates obtained by applying the well-known result of S. Wu and L. Debnath (see [14, Theorem 2] and Theorem 2.1 and Corollary 2.2 below). The first main result of paper is Theorem 3.2, whose proof is based on a quite simple argument appearing in the proofs of Jordan inequality and Janous inequality. In Theorem 3.5 and Theorem 3.6, our second and third main result, we follow a slightly different method which is probably more efficient from the application point of view. In addition to the above, we propose many illustrative examples, open problems and applications.

The organization of paper can be briefly described as follows. In Section 2, we give some preliminaries necessary for our further work. The main results of paper are presented in Section 3; Section 4 ends the findings with some applications. Section 5 gives some conclusions.

2. Preliminaries

Let $-\infty < a < b < +\infty$. For a real function $f : (a, b) \to \mathbb{R}$, for which there exist finite limits $f^{(k)}(a+) = \lim_{x\to a+} f^{(k)}(x)$, for $k = 0, 1, \ldots, n$, where $n \in \mathbb{N}_0$, we define the *first* TAYLOR*'s approximation in the right neighborhood of* a as follows:

$$T_n^{f,\,a+}(x) := \sum_{k=0}^{n} \frac{f^{(k)}(a+)}{k!}(x-a)^k.$$

Similarly, the *first* TAYLOR*'s approximation in the left neighborhood of* b is defined by

$$T_n^{f,\,b-}(x) := \sum_{k=0}^{n} \frac{f^{(k)}(b-)}{k!}(x-b)^k,$$

where $f^{(k)}(b-) = \lim_{x\to b-} f^{(k)}(x)$, for $k = 0, 1, \ldots, n,\quad n\in\mathbb{N}_0$.

Polynomials

$$\mathbb{T}_n^{f;\,a+,\,b-}(x) := \begin{cases} T_{n-1}^{f,\,a+}(x) + \dfrac{1}{(b-a)^n}\Big(f(b-) - T_{n-1}^{f,\,a+}(b-)\Big)(x-a)^n & : \quad n \geq 1 \\ f(b-) & : \quad n = 0, \end{cases}$$

and

$$\mathbb{T}_n^{f;\,b-,\,a+}(x) := \begin{cases} T_{n-1}^{f,\,b-}(x) + \dfrac{1}{(a-b)^n}\Big(f(a+) - T_{n-1}^{f,\,b-}(a+)\Big)(x-b)^n & : \quad n \geq 1 \\ f(a+) & : \quad n = 0, \end{cases}$$

are called the *second* TAYLOR*'s approximation in the right neighborhood of* a, and the *second* TAYLOR*'s approximation in the left neighborhood of* b, respectively [8].

Theorem 2 in [14], by S. Wu and L. Debnath, illustrates the importance of the above-mentioned TAYLOR's approximations. We report it below:

Theorem 2.1. *Suppose that* $f(x)$ *is a real function defined on* (a, b)*, and* n *is a positive integer such that* $f^{(k)}(a+)$, $f^{(k)}(b-)$*, for* $k\in\{0, 1, 2, \ldots, n\}$*, exist.*

(i) Supposing that $(-1)^{(n)} f^{(n)}(x)$ *is strictly increasing on* (a, b)*, then for all* $x \in (a, b)$ *the following inequality holds:*

$$\mathbb{T}_n^{f;\,b-,\,a+}(x) < f(x) < T_n^{f,\,b-}(x). \tag{2.1}$$

Furthermore, if $(-1)^n f^{(n)}(x)$ *is strictly decreasing on* (a,b)*, then the reversed inequality of* (2.1) *holds.*

(ii) Supposing that $f^{(n)}(x)$ *is strictly increasing on* (a,b)*, then for all* $x\in(a,b)$ *the following inequality also holds:*

$$\mathbb{T}_n^{f;\,a+,\,b-}(x) > f(x) > T_n^{f,\,a+}(x). \tag{2.2}$$

Furthermore, if $f^{(n)}(x)$ *is strictly decreasing on* (a,b)*, then the reversed inequality of* (2.2) *holds.*

In the papers [4, 7, 9, 11, 12], Theorem 2.1 was specified as Theorem WD. Let us call it now the *Theorem on double-sided* TAYLOR*'s approximations.* Note that the proof of Theorem 2.1 is based on the L'HOSPITAL's rule for monotonicity. The same method has been employed in the proofs of some theorems already published in [13, 15, 16].

For a continuous function $f:[a,b]\to\mathbb{R}$, let the continuous functions $g_{1,2}:[a,b]\to\mathbb{R}$ and $h_{1,2}:[a,b]\to\mathbb{R}$ represent the lower and upper bounds of function f, respectively, such that the following holds:

$$g_1(x)\le f(x)\le h_1(x)$$

and

$$g_2(x)\le f(x)\le h_2(x)$$

for all $x\in[a,b]$. Then, the following double-sided inequality holds:

$$\max\left(g_1(x),g_2(x)\right)\le f(x)\le \min\left(h_1(x),h_2(x)\right)$$

for all $x\in[a,b]$. Let us name the above double-sided inequality the *quad of bounds of the function* f. Based on Theorem 1, the following assertion holds:

Corollary 2.2. *For* $n\in\mathbb{N}$ *and a real function* $f:(a,b)\to\mathbb{R}$*, let there exist* $f^{(k)}(a+)$ *and* $f^{(k)}(b-)$ *for* $k\in\{0,1,2,\dots,n\}$*. Then the following holds:*

(i) *If* $n=2\ell$ *is an even number and* $f^{(n)}(x)$ *is a strictly increasing function on* (a,b)*, then for every* $x\in(a,b)$*:*

$$\max\left(T_n^{f,\,a+}(x),\,\mathbb{T}_n^{f;\,b-,\,a+}(x)\right) < f(x) < \min\left(T_n^{f,\,b-}(x),\,\mathbb{T}_n^{f;\,a+,\,b-}(x)\right).$$

(ii) *If* $n=2\ell+1$ *is an odd number and* $f^{(n)}(x)$ *is a strictly increasing function on* (a,b)*, then for every* $x\in(a,b)$*:*

$$\max\left(T_n^{f,\,a+}(x),T_n^{f,\,b-}(x)\right) < f(x) < \min\left(\mathbb{T}_n^{f;\,a+,\,b-}(x),\,\mathbb{T}_n^{f;\,b-,\,a+}(x)\right).$$

(iii) *If* $n=2\ell$ *is an even number and* $f^{(n)}(x)$ *is a strictly decreasing function on* (a,b)*, then for every* $x\in(a,b)$*:*

$$\max\left(T_n^{f,\,b-}(x),\,\mathbb{T}_n^{f;\,a+,\,b-}(x)\right) < f(x) < \min\left(T_n^{f,\,a+}(x),\,\mathbb{T}_n^{f;\,b-,\,a+}(x)\right).$$

(iv) *If* $n=2\ell+1$ *is an odd number and* $f^{(n)}(x)$ *is a strictly decreasing function on* (a,b)*, then for every* $x\in(a,b)$*:*

$$\max\left(\mathbb{T}_n^{f;\,a+,\,b-}(x),\,\mathbb{T}_n^{f;\,b-,\,a+}(x)\right) < f(x) < \min\left(T_n^{f,\,a+}(x),T_n^{f,\,b-}(x)\right).$$

It is worth noting that Theorem 2.1 and Corollary 2.2 can be reformulated for the increasing (decreasing) functions; we only need to replace any strict inequality with "less than" or "greater than".

3. Main results

1. Let $\phi : [a,b] \to \mathbb{R}$ be a continuous convex function (in connection with this assumption, let us recall that a convex function defined on the open interval (a,b) is always Lipschitz continuous on any subinterval $[c,d] \subseteq (a,b)$). Then

$$(*)_\phi \qquad \phi(x) \le P_1(x) := \frac{\phi(b)-\phi(a)}{b-a}(x-a)+\phi(a)\,,$$

for $x \in (a,b]$. For a continuous function $\varphi : [a,b] \to \mathbb{R}$ and $m \in \mathbb{N}$, let us define the m-th antiderivative of function φ as follows:

$$\varphi(x)^{(-m)} := \int_a^x \int_a^{x_1} \dots \int_a^{x_{m-1}} \varphi(x_m)\, dx_m dx_{m-1} \dots dx_1 .$$

Based on Cauchy's formula [1], we have:

$$\varphi(x)^{(-m)} = \frac{1}{(m-1)!}\int_a^x (x-s)^{m-1}\varphi(s)\,ds \qquad (x \in [a,b])\,.$$

Particularly, for $m=2$, we have:

$$\varphi(x)^{(-2)} = \int_a^x (x-s)\varphi(s)\,ds \qquad (x \in [a,b])\,.$$

Further on, note that the function

$$F_1(x) := P_1(x)^{(-2)} - \phi(x)^{(-2)}$$

is convex on $[a,b]$, because

$$F_1(x)'' = P_1(x) - \phi(x) \ge 0\,,$$

for $x \in [a,b]$. From $(*)_\phi$ for $\phi = F_1$, we have:

$$F_1(x) \le \frac{F_1(b)-F_1(a)}{b-a}(x-a) + F_1(a)\,,$$

i.e.

$$\begin{aligned} P_1(x)^{(-2)} - \phi(x)^{(-2)} \le\ & \frac{\big(P_1(x)^{(-2)} - \phi(x)^{(-2)}\big)|_{x=b} - \big(P_1(x)^{(-2)} - \phi(x)^{(-2)}\big)|_{x=a}}{b-a}(x-a) \\ & + \Big(P_1(x)^{(-2)} - \phi(x)^{(-2)}\Big)|_{x=a}\,. \end{aligned}$$

Remark 3.1. It is worth noting that we obtain the same inequality if we replace the functions $P_1(x)^{(-2)}$ and $\phi(x)^{(-2)}$ with the functions $P_1(x)^{(-2)} + Ax + B$ and $\phi(x)^{(-2)} + Cx + D$, where $A,B,C,D \in \mathbb{R}$. This can be done in any next step of the following procedure but leads to the same results.

Further on, we have

$$\begin{aligned} \phi(x)^{(-2)} \ge P_3(x) \ :=\ & P_1(x)^{(-2)} \\ & - \frac{\Big(P_1(x)^{(-2)} - \phi(x)^{(-2)}\Big)|_{x=b} - \Big(P_1(x)^{(-2)} - \phi(x)^{(-2)}\Big)|_{x=a}}{b-a}(x-a) \\ & - \Big(P_1(x)^{(-2)} - \phi(x)^{(-2)}\Big)|_{x=a} \\ =\ & P_1(x)^{(-2)} + \frac{\Big(\phi(b)^{(-2)} - P_1(b)^{(-2)}\Big) - \Big(\phi(a)^{(-2)} - P_1(a)^{(-2)}\Big)}{b-a}(x-a) \\ & + \phi(a)^{(-2)} - P_1(a)^{(-2)}\,. \end{aligned}$$

Also,

$$F_2(x) := \phi(x)^{(-4)} - P_3(x)^{(-2)}$$

is a convex function on $[a,b]$, because

$$F_2(x)'' = \phi(x)^{(-2)} - P_3(x) \geq 0\,,$$

for $x \in [a,b]$.

Let us define the following sequence of functions

$$F_k(x) := (-1)^k \left(\phi(x)^{(-2k)} - P_{2k-1}(x)^{(-2)}\right),$$

for $x \in [a,b]$ and $k \in \mathbb{N}$. Suppose that the induction hypothesis holds:

$$F_k(x) \text{ is a convex function on } [a,b].$$

From $(*)_\phi$ for $\phi = F_k$, we have:

$$F_k(x) \leq \frac{F_k(b) - F_k(a)}{b-a}\,(x-a) + F_k(a)\,,$$

i.e.

$$\begin{aligned} &(-1)^k \left(\phi(x)^{(-2k)} - P_{2k-1}(x)^{(-2)}\right) \\ &\leq \frac{(-1)^k \left(\phi(x)^{(-2k)} - P_{2k-1}(x)^{(-2)}\right)|_{x=b} - (-1)^k \left(\phi(x)^{(-2k)} - P_{2k-1}(x)^{(-2)}\right)|_{x=a}}{b-a}\,(x-a) \\ &\quad + (-1)^k \left(\phi(x)^{(-2k)} - P_{2k-1}(x)^{(-2)}\right)|_{x=a}\,. \end{aligned}$$

Then

$$\begin{aligned} &(-1)^{k+1}\phi(x)^{(-2k)} \\ &\qquad \geq (-1)^{k+1} P_{2k+1}(x) \\ &\qquad := (-1)^{k+1} P_{2k-1}(x)^{(-2)} \\ &\qquad\quad + (-1)^{k+1} \frac{\left(\phi(b)^{(-2k)} - P_{2k-1}(b)^{(-2)}\right) - \left(\phi(a)^{(-2k)} - P_{2k-1}(a)^{(-2)}\right)}{b-a}\,(x-a) \\ &\qquad\quad + (-1)^{k+1} \left(\phi(a)^{(-2k)} - P_{2k-1}(a)^{(-2)}\right)\,. \end{aligned}$$

Also,

$$F_{k+1}(x) := (-1)^{k+1} \left(\phi(x)^{(-(2k+2))} - P_{2k+1}(x)^{(-2)}\right)$$

is a convex function on $[a,b]$, because

$$F_{k+1}(x)'' = (-1)^{k+1} \left(\phi(x)^{(-2k)} - P_{2k+1}(x)\right) \geq 0\,,$$

for $x \in [a,b]$. Now, by the principle of mathematical induction, $F_k(x)$ is a convex function on $[a,b]$, for all $k \in \mathbb{N}$. Thus, the following theorem is proved.

Theorem 3.2. *For a continuous convex function $\phi : [a,b] \to \mathbb{R}$ and a sequence of real polynomials $P_1(x), P_3(x), \ldots, P_{2k-1}(x), \ldots,$ where*

$$P_1(x) := \frac{\phi(b) - \phi(a)}{b-a}\,(x-a) + \phi(a)\,, \tag{3.1}$$

and

$$\begin{aligned} P_{2k+1}(x) := P_{2k-1}(x)^{(-2)} + \frac{\left(\phi(t)^{(-2k)} - P_{2k-1}(t)^{(-2)}\right)\Big|_{t=a}^{b}}{b-a}(x-a) \\ + \left(\phi(t)^{(-2k)} - P_{2k-1}(t)^{(-2)}\right)\Big|_{t=a}, \end{aligned} \tag{3.2}$$

the following holds:

$$(-1)^{k+1}\phi(x)^{(-2k)} \geq (-1)^{k+1} P_{2k+1}(x)\,,$$

for all $x \in [a, b]$.

Remark 3.3. It is clear that $dg(P_{2k+1}) = 2k+1$ for all $k \in \mathbb{N}\cup\{0\}$. Further on, assume that ϕ is continuously differentiable on $[a, b]$. Then the mapping ϕ' is monotonically increasing and we can apply Corollary 2.2(ii) in order to see that ($f = \phi^{(-2k)}$)

$$\max\Big(T_{2k+1}^{f;a+}(x), T_{2k+1}^{f;b-}(x)\Big) \le f(x) \le \min\Big(T_{2k+1}^{f;a+,b-}(x), T_{2k+1}^{f;b-,a+}(x)\Big), \tag{3.3}$$

for $x \in [a, b]$. If $k = 0$, then the right inequality in (3.3) is equivalent with the estimate obtained in Theorem 3.2. It is reasonable to ask can the estimate obtained by the use of Theorem 3.2 be better than the corresponding estimate in (3.3)? The answer is affirmative and to explain this, suppose first that $a = 0$ and $\phi(0) = 0$. Then we have

$$P_3(x) \equiv \frac{\phi(b)}{6b}x^3 + \left(\frac{f(b)}{b} - b^2\frac{\phi(b)}{6}\right)x$$

and

$$\max\Big(T_3^{f;0+}(x), T_3^{f;b-}(x)\Big) \equiv \max\left(\frac{\phi'(0)}{6}x^3, f(b)+f'(b)(x-b)+\phi(b)\frac{(x-b)^2}{2}+\phi'(b)\frac{(x-b)^3}{6}\right).$$

The situation in which $P_3(x) > \max(T_3^{f;0+}(x), T_3^{f;b-}(x))$ for $x = b/2$ occurs in the case that $\phi(x) := x^m/m!$, $x \in [0, b]$, where $m > 1$,

$$0 < b < 1/4 \quad \text{and} \quad 1 - \frac{m+2}{2} + \frac{(m+1)(m+2)}{8} - \frac{m(m+1)(m+2)}{48} < 0. \tag{3.4}$$

In actual fact, due to (3.4), we have

$$P_3(b/2) = \frac{b^{m+2}}{(m+2)!}\left[\frac{(m+1)(m+2)}{48} + \frac{1}{2} - b\frac{(m+1)(m+2)}{12}\right] > \frac{1}{2}\frac{b^{m+2}}{(m+2)!}$$

and

$$\max\Big(T_3^{f;0+}(x), T_3^{f;b-}(x)\Big) = \frac{b^{m+2}}{(m+2)!}\max\left(0, 1 - \frac{m+2}{2} + \frac{(m+1)(m+2)}{8} - \frac{m(m+1)(m+2)}{48}\right) = 0.$$

Example 3.4. Define $\phi : (0, \pi/2] \to \mathbb{R}$ by

$$\phi(x) = \frac{2x}{\pi} - \sin x.$$

According to Theorem 3.2, we have $P_1(x) = 0$,

$$P_3(x) = \frac{2}{\pi}\left(\frac{\pi^2}{24} + 1\right)x,$$

and

$$P_5(x) = \frac{2}{5}\left(1 + \frac{\pi^2}{24}\right)\frac{x^2}{6} - \frac{2}{\pi}\left[1 + \frac{2}{\pi}\left(1 + \frac{\pi^2}{24}\right)\frac{\pi^3}{48} - \frac{\pi^4}{1920}\right].$$

Thus $\phi(x)^{(-2)} \ge P_3(x)$ implies the Janous inequality (1.2) and $\phi(x)^{(-4)} \le P_5(x)$ implies the following:

$$\frac{\sin x}{x} \ge \frac{2}{\pi}\left[1 + \frac{2}{\pi}\left(1 + \frac{\pi^2}{24}\right)\frac{\pi^3}{48} - \frac{\pi^4}{1920}\right] - \frac{2}{5}\left(1 + \frac{\pi^2}{24}\right)\frac{x^2}{6} + \frac{x^4}{60\pi}. \tag{3.5}$$

The inequality (3.5) refines the Janous inequality (1.2) on the interval $(0, \xi)$, where $\xi \sim 0.38$.

It is also worth noting that the Abel-Caccia inequality

$$\frac{\sin x}{x} \geq \frac{2}{\pi} + \frac{\pi^2 - 4x^2}{\pi^3}, \quad x \in \left(0, \frac{\pi}{2}\right] \tag{3.6}$$

generalizes Li-Li's inequality [10, (1.6)] on the interval $(0, \xi)$, where $\xi \sim 1.218$, as well as that the Janous inequality, which has not been quoted in [10], generalizes Li-Li's inequality on the interval $(\eta, \pi/2]$, where $\eta \sim 1.469$. On the other hand, the Qi-Guo inequality

$$\frac{\sin x}{x} \geq 1 - \frac{2(\pi - 2)}{\pi^2} x, \quad x \in \left(0, \frac{\pi}{2}\right] \tag{3.7}$$

is not comparable to the Abel-Caccia inequality and provides a better result than the Abel-Caccia inequality on the interval $(0, \zeta)$, where $\zeta \sim 0.222$. In order to partially improve the Abel-Caccia inequality and the Qi-Guo inequality using the methods established in this paper, we will consider the function

$$f(x) := -\sin x - \left(\frac{x^3}{6} - \frac{2(\pi - 2)}{\pi^2} \frac{x^4}{12}\right), \quad x \in \left(0, \frac{\pi}{2}\right].$$

This function is convex since its second derivative is given by

$$f''(x) := \sin x - \left(x - \frac{2(\pi - 2)}{\pi^2} x^2\right), \quad x \in \left(0, \frac{\pi}{2}\right];$$

see (3.7). Arguing as in the proof of Theorem 3.2, we get that

$$\frac{\sin x}{x} \geq \frac{2\,(\pi - 2)}{\pi^2} \frac{x^3}{12} - \frac{x^2}{6} + \frac{2}{\pi}\left(1 + \frac{\pi^3}{48} - \frac{(\pi - 2)\,\pi^2}{96}\right), \quad x \in \left(0, \frac{\pi}{2}\right].$$

The last estimate generalizes (3.7) on the interval $(\eta, \pi/2]$, where $\eta \sim 0.128$, and the estimate (3.6) on the interval $(0, \zeta)$, where $\zeta \sim 0.99$.

2. The computation of coefficients of real polynomials $P_{2k+1}(x)$ defined recursively by (3.1)-(3.2) is rather non-trivial. In this part, we follow a slightly different approach which will enable us to precisely formulate some inequalities for the power series representations of functions.

Let a continuous convex function $\phi : [a, b] \to \mathbb{R}$ and an integer $m \in \mathbb{N}$ be given in advance. Define

$$G_{\phi,m}(x) := -\phi(x)^{(-m)} + \left(\phi(a)\, \frac{x^m}{m!} + \frac{\phi(b) - \phi(a)}{b - a}\left(\frac{x^{m+1}}{(m+1)!} - \frac{x^m}{m!} a\right)\right).$$

Then from the convexity of function ϕ, it holds:

$$\Big(G_{\phi,m}(x)\Big)^{(m)} = -\phi(x) + \left(\phi(a) + \frac{\phi(b) - \phi(a)}{b - a}\,(x - a)\right) \geq 0\,,$$

for $x \in [a, b]$. Thus, we conclude that

$$\Big(G_{\phi,m}(x)\Big)^{(m-1)}$$

is an increasing function on $[a, b]$.

Now, let us focus on the function $(-1)^{(m-1)}(G_{\phi,m}(x))^{(m-1)}$. For an even number $m = 2k$ the function

$$(-1)^{(m-1)}\Big(G_{\phi,m}(x)\Big)^{(m-1)}$$

is a decreasing function on $[a, b]$, while for an odd number $m = 2k + 1$ the function

$$(-1)^{(m-1)}\Big(G_{\phi,m}(x)\Big)^{(m-1)}$$

is an increasing function on $[a,b]$.

Keeping in mind Corollary 2.2 and the above consideration, we obtain the first part of the following results (in the second parts, we replace convexity by monotonicity):

Theorem 3.5.

(i) *For a given even number $m = 2k \in \mathbb{N}$, and a continuous convex function $\phi : [a,b] \to \mathbb{R}$, the following is true:*

$$\min\left(\mathbb{T}_{m-1}^{G_{\phi,m};a+,b-}(x), \mathbb{T}_{m-1}^{G_{\phi,m};b-,a+}(x)\right) \geq G_{\phi,m}(x) \geq \max\left(T_{m-1}^{G_{\phi,m},a+}(x), T_{m-1}^{G_{\phi,m},b-}(x)\right)$$

for $x \in [a,b]$; i.e.

$$\left(\phi(a)\frac{x^m}{m!} + \frac{\phi(b)-\phi(a)}{b-a}\left(\frac{x^{m+1}}{(m+1)!} - \frac{x^m}{m!}a\right)\right) - \max\left(T_{m-1}^{G_{\phi,m},a+}(x), T_{m-1}^{G_{\phi,m},b-}(x)\right)$$

$$\geq (\phi(x))^{(-m)} \geq$$

$$\left(\phi(a)\frac{x^m}{m!} + \frac{\phi(b)-\phi(a)}{b-a}\left(\frac{x^{m+1}}{(m+1)!} - \frac{x^m}{m!}a\right)\right) - \min\left(\mathbb{T}_{m-1}^{G_{\phi,m};a+,b-}(x), \mathbb{T}_{m-1}^{G_{\phi,m};b-,a+}(x)\right)$$

for $x \in [a,b]$.

(ii) (a) *If $m = 2k$ is an even number and $\phi : [a,b] \to \mathbb{R}$ is a continuous increasing function, then for every $x \in [a,b]$:*

$$\max\left(T_m^{\phi^{(-m)},a+}(x), \mathbb{T}_m^{\phi^{(-m)};b-,a+}(x)\right) \leq \phi(x)^{(-m)} \leq \min\left(T_m^{\phi^{(-m)},b-}(x), \mathbb{T}_m^{\phi^{(-m)};a+,b-}(x)\right).$$

(b) *If $m = 2k$ is an even number and $\phi : [a,b] \to \mathbb{R}$ is a continuous decreasing function, then for every $x \in [a,b]$:*

$$\max\left(T_m^{\phi^{(-m)},b-}(x), \mathbb{T}_m^{\phi^{(-m)};a+,b-}(x)\right) \leq \phi(x)^{(-m)} \leq \min\left(T_m^{\phi^{(-m)},a+}(x), \mathbb{T}_m^{\phi^{(-m)};b-,a+}(x)\right).$$

Theorem 3.6.

(i) *For a given odd number $m = 2k+1 \in \mathbb{N}$, and a continuous convex function $\phi : [a,b] \to \mathbb{R}$, the following is true:*

$$\min\left(\mathbb{T}_{m-1}^{G_{\phi,m};a+,b-}(x), T_{m-1}^{G_{\phi,m},b-}(x)\right) \geq G_{\phi,m}(x) \geq \max\left(T_{m-1}^{G_{\phi,m},a+}(x), \mathbb{T}_{m-1}^{G_{\phi,m};b-,a+}(x)\right)$$

for $x \in [a,b]$; i.e.

$$\left(\phi(a)\frac{x^m}{m!} + \frac{\phi(b)-\phi(a)}{b-a}\left(\frac{x^{m+1}}{(m+1)!} - \frac{x^m}{m!}a\right)\right) - \max\left(T_{m-1}^{G_{\phi,m},a+}(x), \mathbb{T}_{m-1}^{G_{\phi,m};b-,a+}(x)\right)$$

$$\geq (\phi(x))^{(-m)} \geq$$

$$\left(\phi(a)\frac{x^m}{m!} + \frac{\phi(b)-\phi(a)}{b-a}\left(\frac{x^{m+1}}{(m+1)!} - \frac{x^m}{m!}a\right)\right) - \min\left(\mathbb{T}_{m-1}^{G_{\phi,m};a+,b-}(x), T_{m-1}^{G_{\phi,m},b-}(x)\right)$$

for $x \in [a,b]$.

(ii) (a) *If $m = 2k+1 \in \mathbb{N}$ and $\phi : [a,b] \to \mathbb{R}$ is a continuous increasing function, then for every $x \in [a,b]$:*

$$\max\left(T_m^{\phi^{(-m)},a+}(x), T_m^{\phi^{(-m)},b-}(x)\right) \leq \phi(x)^{(-m)} \leq \min\left(\mathbb{T}_m^{\phi^{(-m)};a+,b-}(x), \mathbb{T}_m^{\phi^{(-m)};b-,a+}(x)\right).$$

(b) *If $m = 2k+1 \in \mathbb{N}$ and $\phi : [a,b] \to \mathbb{R}$ is a continuous decreasing function, then for every $x \in [a,b]$:*

$$\max\left(\mathbb{T}_m^{\phi^{(-m)};a+,b-}(x), \mathbb{T}_m^{\phi^{(-m)};b-,a+}(x)\right) \leq \phi(x)^{(-m)} \leq \min\left(T_m^{\phi^{(-m)},a+}(x), T_m^{\phi^{(-m)},b-}(x)\right).$$

Regarding Theorem 3.5 and Theorem 3.6, we would like to raise some new issues:

Remark 3.7. It is worth noting that Theorem 3.5 and Theorem 3.6 continue to hold if we replace the function $\phi(x)^{(-m)}$ in the definition of function $G_{\phi,m}$ and their formulations by any function whose m-th derivative is a function ϕ, i.e., by a function of the form

$$\phi(x)^{(-m)} + \sum_{j=0}^{m-1} a_j x^j,$$

where $a_j \in \mathbb{R}$ for $0 \leq j \leq m-1$. As a very simple argumentation shows, we obtain the same result by applying Theorem 3.5 (Theorem 3.6) separately to the function $\phi(x)^{(-m)}$ and to the function $\phi(x)^{(-m)} + \sum_{j=0}^{m-1} a_j x^j$. Further on, it is clear that the second part of Theorem 3.5 (Theorem 3.6) is a simple reformulation of Corollary 2.2, only. It seems very plausible that, in some concrete situations, Theorem 3.5(i) (Theorem 3.6(i)) can give better estimates than Theorem 3.5(ii) (Theorem 3.6(ii)).

We continue by providing the following illustrative example in support of Corollary 2.2:

Example 3.8. Applying Theorem 3.6(i) with $\phi(x) := e^x$, $x \in [0,1]$, $m = 3$ and $k = 1$, we obtain the approximation of function $y = e^x - 1 - x$ by the polynomial of fourth order $P_4(x)$. If we apply Corollary 2.2 here, and approximate the function $y = e^x - 1 - x$ by the polynomial $Q_4(x)$ of fourth order, with the meaning clear, then we obtain an extremely similar but a slightly better result. It is also worth noting that we can extend the validity of our result to the interval $[-1,0]$; on the other hand, the reverse inequality $Q_4(x) \geq e^x - 1 - x$, $x \in [-1,0]$ holds true. See https://www.desmos.com/calculator/msygc6b8c6 for more details, and [14, *Proposition* 1].

As a certain drawback of Corollary 2.2, we would like to note that, if n is an odd natural number and $f^{(n)}(x)$ is a decreasing function on (a,b), then the majorization of $f(x)$ is possible only if we use the usually considered Taylor's polynomials of function f at the points $a+$ and $b-$; see the right-hand side of the inequality in Corollary 2.2/(iv). But, applying Theorem 3.5(i) [Theorem 3.6(i)] we can majorize the function f by the polynomial whose order is odd [even] and whose coefficients really depend on the both parameters, a and b; cf. also Remark 3.3.

The classical situation is $f(x) = \ln x$, $x \in [a,b]$, where $[a,b] \subseteq (0,\infty)$. Then $f^{(m)}(x) = (-1)^{m-1}(m-1)!x^{-m}$ for all $x \in [a,b]$ and $m \in \mathbb{N}$, so that the mapping $f^{(m)}$ is strictly increasing (decreasing) if m is an even (odd) integer; see also [14, Proposition 2] which has been stated a little bit incorrect since we must write $\ln a$ in the sums appearing in the equation [14, (13)], if $k = 0$.

We close this section by applying our results to Kober's inequality:

Example 3.9. Define $\phi : [0,\pi/2] \to \mathbb{R}$ by $\phi(x) := 1 - \cos x$, $x \in [0,\pi/2]$. Since $\phi''(x) = \cos x \geq 0$ for all $x \in [0,\pi/2]$, we can apply Theorem 3.2 to get several inequalities. In particular, for $k = 0$, we have $\phi(x) \leq P_1(x)$ where $P_1(x) = (2/\pi)x$, i.e., $1-(2/\pi)x \leq \cos x$, $x \in [0,\pi/2]$, which is the left side of Kober's inequality (1.1); for $k = 1$, we get a refined inequality

$$1 + \frac{2}{\pi}\left(\frac{\pi^2}{12} - 1\right)x - \frac{x^2}{2} + \frac{x^3}{3\pi} \leq \cos x, \quad x \in \left(0, \frac{\pi}{2}\right]. \tag{3.8}$$

The inequality (3.8) can be also obtained using Theorem 3.5. Putting $a = 0$, $b = \pi/2$ and $k = 1$, i.e., $m = 2$ in Theorem 3.5, we get

$$\frac{x^3}{3\pi} - \min\left(\mathbb{T}_{m-1}^{G_{\phi,m};\,a+,\,b-}(x), \mathbb{T}_{m-1}^{G_{\phi,m};\,b-,\,a+}(x)\right) = -1 - \frac{2}{\pi}\left(\frac{\pi^2}{12} - 1\right)x \leq \frac{x^2}{2} + \cos x,$$

for $x \in (0, \frac{\pi}{2}]$ and

$$\cos x \leq -\frac{x^2}{2} + \frac{x^3}{3\pi} - \max\left(T_{m-1}^{G_{\phi,m},a-}(x), T_{m-1}^{G_{\phi,m},b+}(x)\right) = 1 - \frac{x^2}{2} + \frac{x^3}{3\pi}, \tag{3.9}$$

for $x \in \left(0, \frac{\pi}{2}\right]$. The inequality (3.9) refines the right inequality of (1.1). Concerning the obtained results, we feel it is our duty to note that the use of Corollary 2.2 provides here better results compared with (3.8)-(3.9); see also [10, (1.12)-(1.13)].

4. Applications in the theory of analytical inequalities

The functions e^x, $\ln x$, $\sin x$, $\cos x$, $\sinh x$ and $\cosh x$ have been considered in [14, Proposition 1 - Proposition 5], respectively. We first observe that these results can be further generalized by using Corollary 2.2.

We finish by providing some application of our main results, Theorem 3.2-Theorem 3.6.

Suppose that $\phi : [a, b] \to \mathbb{R}$ is a real analytic function and can be represented by a power series $\phi(x) = \sum_{n=0}^{\infty} a_n x^n/n!$, $x \in [a, b] \subseteq \mathbb{R}$. Put, formally, $x^j/j! \equiv 0$ if $j \in -\mathbb{N}$. Let

$$G_{\phi,m}(x) := -\sum_{n=0}^{\infty} a_n \frac{x^{n+m}}{(n+m)!} + \left(\sum_{n=0}^{\infty} a_n \frac{a^n}{n!} + \frac{\sum_{n=0}^{\infty} a_n \frac{(b^n - a^n)}{n!}}{b-a}\left(\frac{x^{m+1}}{(m+1)!} - \frac{x^m}{m!}a\right)\right),$$

for $x \in [a, b]$. Then, for every $j \in \mathbb{N}$ and $x \in [a, b]$, we have

$$G_{\phi,m}^{(j)}(x) = -\sum_{n=0}^{\infty} a_n \frac{x^{n+m-j}}{(n+m-j)!} + \left(\frac{\sum_{n=0}^{\infty} a_n \frac{(b^n - a^n)}{n!}}{b-a}\left(\frac{x^{m+1-j}}{(m+1-j)!} - \frac{x^{m-j}}{(m-j)!}a\right)\right).$$

If ϕ is convex, then we can apply Theorem 3.5(i) to get that, for every even natural number m and for every real number $x \in [a, b]$, we have:

$$\begin{aligned}
&\min\Bigg(\sum_{j=0}^{m-2}\left(-\sum_{n=0}^{\infty} a_n \frac{a^{n+m-1-j}}{(n+m-1-j)!} - (m-1)\left(\frac{\sum_{n=0}^{\infty} a_n \frac{(b^n-a^n)}{n!}}{b-a}\frac{a^{m-j}}{(m-j)!}\right)\right)\frac{(x-a)^j}{j!} \\
&+ \frac{1}{(b-a)^{m-1}}\Bigg(G_{\phi,m}(b) + \sum_{j=0}^{m-2}\Bigg(\sum_{n=0}^{\infty} a_n \frac{a^{n+m-1-j}}{(n+m-1-j)!} \\
&+ (m-1)\left(\frac{\sum_{n=0}^{\infty} a_n \frac{(b^n-a^n)}{n!}}{b-a}\frac{a^{m-j}}{(m-j)!}\right)\Bigg)\frac{(b-a)^j}{j!}\Bigg)(x-a)^{m-1}, \\
&\sum_{j=0}^{m-2}\left(-\sum_{n=0}^{\infty} a_n \frac{b^{n+m-1-j}}{(n+m-1-j)!} + \left(\frac{\sum_{n=0}^{\infty} a_n \frac{(b^n-a^n)}{n!}}{b-a}\left(\frac{b^{m-j}}{(m-j)!} - \frac{b^{m-1-j}}{(m-1-j)!}a\right)\right)\right)\frac{(x-b)^j}{j!} \\
&+ \frac{1}{(a-b)^{m-1}}\Bigg(G_{\phi,m}(a) + \sum_{j=0}^{m-2}\Bigg(\sum_{n=0}^{\infty} a_n \frac{b^{n+m-1-j}}{(n+m-1-j)!} \\
&+ \left(\frac{\sum_{n=0}^{\infty} a_n \frac{(b^n-a^n)}{n!}}{b-a}\left(\frac{b^{m-j}}{(m-j)!} - \frac{b^{m-1-j}}{(m-1-j)!}a\right)\right)\Bigg)\frac{(a-b)^j}{j!}\Bigg)(x-b)^{m-1}\Bigg) \\
&\geq G_{\phi,m}(x) \\
&\geq \max\Bigg(\sum_{j=0}^{m-1}\left(-\sum_{n=0}^{\infty} a_n \frac{a^{n+m-j}}{(n+m-j)!} - m\left(\frac{\sum_{n=0}^{\infty} a_n \frac{(b^n-a^n)}{n!}}{b-a}\frac{a^{m+1-j}}{(m+1-j)!}\right)\right)\frac{(x-a)^j}{j!}, \\
&\sum_{j=0}^{m-1}\left(-\sum_{n=0}^{\infty} a_n \frac{b^{n+m-j}}{(n+m-j)!} + \left(\frac{\sum_{n=0}^{\infty} a_n \frac{(b^n-a^n)}{n!}}{b-a}\left(\frac{b^{m+1-j}}{(m+1-j)!} - \frac{b^{m-j}}{(m-j)!}a\right)\right)\right)\frac{(x-b)^j}{j!}\Bigg).
\end{aligned}$$

We can similarly apply Theorem 3.5(ii) or Theorem 3.6 here; it is clear that the obtained inequality is very complicated and almost unusable in general case.

5. Conclusion

In this paper, double-sided Taylor's approximations and convexity of some functions were discussed. Through a series of assertions and theorems, examples of quad of bounds of appropriate functions were shown. Also, some new results in the theory of analytical inequalities were obtained.

Declaration. The authors declare that they have no known conflict of interests. The authors declare no funding. All the authors contributed equally to the findings.

Acknowledgment. The third author is partially supported by grant 451-03-68/2020/14/ 200156 of Ministry of Education, Science and Technological Development, Republic of Serbia. Research of the fourth, fifth and sixth authors was partially supported by the Serbian Ministry of Education, Science and Technological Development, grant 451-03-68/2022-14/200103.

References

[1] A.L. Cauchy, *Trente-Cinquième Leçon*, Résumé des leçons données à l'Ecole royale polytechnique sur le calcul infinitésimal, Imprimerie Royale, Paris, 133–140, 1823. Reprint: Œuvres complètes II(4), Gauthier-Villars, Paris, 5-261.

[2] J. Dieudonne (ed.), *Oeuvres de Camille Jordan I-IV*, Gauthier-Villars, Paris, 1961-1964.

[3] H. Kober, *Approximation by integral functions in the complex domain,* Trans. Amer. Math. Soc. **56** (1), 7-31, 1944.

[4] T. Lutovac, B. Malešević, and M. Rašajski, *A new method for proving some inequalities related to several special functions*, Results Math. **73**:100, 15 pp, 2018.

[5] B. Malešević, T. Lutovac M. Rašajski, B. Banjac, *Double-Sided Taylor's Approximations and Their Applications in Theory of Trigonometric Inequalities*, in: eds M.Th. Rassias, A. Raigorodskii, Trigonometric Sums and Their Applications, 159–167, Springer, 2020.

[6] B. Malešević, T. Lutovac, M. Rašajski, B. Banjac, *Error-Functions in Double-Sided Taylor's Approximations*, Appl. Anal. Discrete Math. **14** (3), 599–613, 2020.

[7] B. Malešević, T. Lutovac, M. Rašajski and C. Mortici, *Extensions of the natural approach to refinements and generalizations of some trigonometric inequalities*, Adv. Difference Equ. **2018**:90, 15 pp, 2018.

[8] B. Malešević, M. Rašajski, and T. Lutovac, *Double-sided Taylor's approximations and their applications in Theory of analytic inequalities*, in: eds. Th. Rassias and D. Andrica, Differential and Integral Inequalities, Optimization and Its Applications **151**, 569–582, Springer, 2019.

[9] M. Nenezić and L. Zhu, *Some improvements of Jordan-Steckin and Becker-Stark inequalities*, Appl. Anal. Discrete Math. **12**, 244–256, 2018.

[10] F. Qi, D.-W. Niu, and B.-N. Guo, *Refinements, Generalizations and Applications of Jordan's inequality and related problems*, J. Inequal. Appl. **2009**, Article ID: 271923, 52 pp., 2009. Doi: 10.1155/2009/271923

[11] M. Rašajski, T. Lutovac, and B. Malešević, *Sharpening and generalizations of Shafer-Fink and Wilker type inequalities*: *a new approach*, J. Nonlinear Sci. Appl. **11** (7), 885–893, 2018.

[12] M. Rašajski, T. Lutovac, and B. Malešević, *About some exponential inequalities related to the sinc function*, J. Inequal. Appl. **2018**:150, 10 pp, 2018.

[13] S.-H. Wu and L. Debnath, *Jordan-type inequalities for differentiable functions and their applications*, Appl. Math. Lett. **21** (8), 803–809, 2008.

[14] S.-H. Wu and L. Debnath, *A generalization of L'Hospital-type rules for monotonicity and its application*, Appl. Math. Lett. **22** (2), 284–290, 2009.

[15] S.-H. Wu and H.M. Srivastva, *A further refinement of a Jordan type inequality and its applications*, Appl. Math. Comput. **197**, 914–923, 2008.

[16] S.-H. Wu and H.M. Srivastava, *A further refinement of Wilker's inequality*, Integral Transforms Spec. Funct. **19** (10), 757–765, 2008.