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Some convergence results using a new iterative algorithm in CAT(0) space

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Abstract

This paper presents a new iterative algorithm for approximating the invariant points of Suzuki's generalized nonexpansive maps. Some strong convergence theorems are developed in the context of CAT(0) space. We also included examples to demonstrate the proposed algorithm's convergence nature. Lastly, the stability of the said iterative algorithm is discussed to validate the results.

Keywords: Strong convergence Stability CAT(0) spaces Iterative algorithm
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1. Introduction

Fixed point theory has an explosive growth for nonexpansive maps over the past five decades. A self map S defined on a nonempty subset of a CAT(0) space is nonexpansive if $d(Sa, Sb) \leq d(a, b)$ for all $a, b \in C$. Once the existence of a invariant point for a mapping has been determined, an algorithm for calculating value of invariant point is essential. Banach contraction principle tells us that the successive iterative method (Picard iterative) can be used to find the invariant point for a contraction map and the sequence $\{a_n\}$ is formed from any arbitrary $a_1 \in C$ using the subsequent algorithm :

$$a_{n+1} = Sa_n, \qquad n \geq 1.$$

However, the Picard iteration in the convergence part has not been successfully employed in approximating the fixed point of some mappings such as a nonexpansive self-mapping on a metric space. Next, we give some example showing the claiming.

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Example 1.1. *Consider a mapping $S : [0, 1] \to [0, 1]$ defined by $Sa = 1 - a$ for all $a \in [0, 1]$. Then S is a nonexpansive mapping with a usual metric and invariant point of S is $\frac{1}{2}$. If one chooses as a starting value $a = a_0$ such that $a_0 \neq \frac{1}{2}$, then Picard iteration of S yield that*

$$\begin{aligned} a_1 &= Sa_0 = 1 - a_0, \\ a_2 &= Sa_1 = a_0, \\ a_3 &= Sa_2 = 1 - a_0, \\ &\vdots \end{aligned}$$

This concludes that Picard iteration does not converge to a fixed point of S. Based on this problem, other approximation techniques are needed to approximate it.

Throughout this paper, $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $(0, 1)$. In the following years, a number of iterative algorithms for approximating the invariant point for nonexpansive maps have been developed by the researchers. To approximate the invariant point for nonexpansive maps, the Mann [13] iterative algorithm has been widely used. The sequence $\{a_n\}$ is formed from any arbitrary $a_1 \in C$ in the subsequent way in this iterative algorithm :

$$a_{n+1} = (1 - \alpha_n)a_n + \alpha_n Sb_n.$$

Next, we give an example Ishikawa [10] proposed another iterative algorithm for approximating the invariant point for a nonexpansive map, in which $\{a_n\}$ is described iteratively from $a_1 \in C$ by

$$\begin{aligned} a_{n+1} &= (1 - \alpha_n)a_n + \alpha_n Sb_n \\ b_n &= (1 - \beta_n)a_n + \beta_n Sa_n \end{aligned}$$

for all $n \geq 1$.
Ullah and Arshad [18] introduced the following M^* iterative algorithm defined as:

$$\begin{aligned} &a_1 \in C,\ a_{n+1} = Sb_n \\ &b_n = S((1 - \alpha_n)a_n + \alpha_n Sc_n) \\ &c_n = (1 - \beta_n)a_n + \beta_n Sa_n \end{aligned}$$

for all $n \geq 1$.
Ullah and Arshad [19] proposed the following M iterative algorithm in 2018: for any arbitrary $a_1 \in C$, the sequence $\{a_n\}$ is defined as

$$\begin{aligned} a_{n+1} =& Sb_n \\ b_n =& Sc_n \\ c_n =& (1 - \beta_n)a_n + \beta_n Sa_n \end{aligned}$$

for all $n \geq 1$.
They established that the maps satisfying condition (C) has weak and strong convergence theorems. They provided a numerical example of a map that satisfied condition (C) and compared the proposed iterative algorithm to the existing algorithms numerically.
The above development in iterations encourage us to introduce a new three step iterative algorithm defined as:

$$\begin{aligned} &a_1 \in C,\ a_{n+1} = Sb_n \\ &b_n = S((1 - \alpha_n)a_n + \alpha_n Sc_n) \\ &c_n = S((1 - \beta_n)a_n + \beta_n Sa_n) \end{aligned}$$

for all $n \geq 1$.
On the other hand in 2008, Suzuki [15] developed the definition of generalized nonexpansive maps, that is the mapping satisfies a condition known as condition (C). A self map S defined on nonempty subset C of a CAT(0) space E, is seen to satisfy condition (C) if

$$\frac{1}{2}d(Sa, Sb) \leq d(a, b) \Longrightarrow d(Sa, Sb) \leq d(a, b), \forall a, b \in C.$$

For such maps, Suzuki [15] found invariant point results and demonstrated that the map which satisfies condition (C) is weaker than nonexpansive and more powerful than quasi-nonexpansive. A number of scholars have recently looked into invariant point theorems for maps (see e.g.[1, 2, 7, 14, 16, 17, 23]).
Motivated by above work, our aim is to establish a new iterative algorithm and show that the map satisfying condition (C) in CAT(0) space has strong convergence theorems. Also, we provide an example of maps that meets condition (C) but is not nonexpansive maps and then prove analytically the stability of our algorithm and compare the convergence of the suggested iterative algorithm against that of existing algorithms numerically.

2. Preliminaries

Let (E, d) be a metric space and $a, b \in E$ with $d(a, b) = l$. A geodesic path from a to b is a isometry $c : [0, l] \to E$ such that $c(0) = a$ and $c(l) = b$. The image of a geodesic path is called a geodesic segment. A metric space E is a (uniquely) geodesic space, if every two points of E are joined by only one geodesic segment. A geodesic triangle $\Delta(a_1, a_2, a_3)$ in a geodesic space E consists of three points a_1, a_2, a_3 of E and three geodesic segments joining each pair of vertices. A comparison triangle of a geodesic triangle $\Delta(a_1, a_2, a_3)$ is the triangle $\overline{\Delta}(a_1, a_2, a_3) := \Delta(\overline{a_1}, \overline{a_2}, \overline{a_3})$ in the Euclidean space $\mathbb{R}^2$ such that

$$d(a_i, a_j) = d_{\mathbb{R}^2}(\overline{a_i}, \overline{a_j}), \qquad \forall\ i,\ j\ =\ 1,2,3.$$

A geodesic space E is a CAT(0) space, if for each geodesic triangle $\Delta(a_1, a_2, a_3)$ in E and its comparison triangle $\overline{\Delta} := \Delta(\overline{a_1}, \overline{a_2}, \overline{a_3})$ in $\mathbb{R}^2$, the CAT(0) inequality $d(a, b) \leq d_{\mathbb{R}^2}(\overline{a}, \overline{b})$ is satisfied for all $a, b \in \Delta$ and $\overline{a}, \overline{b} \in \overline{\Delta}$. A thorough discussion of these spaces and their important role in various branches of mathematics are given [3, 4, 6].
The famous mathematician Kirk [11, 12] developed a more broader outcome to investigate the invariant point outcomes in the context of complete CAT(0) space which is one approach. Since then, a large number of papers have been published on fixed point theory of different maps and iterative algorithms in CAT(0) space. Riemannian manifolds with nonpositive sectional curvature provide a motivating example of CAT(0) space. "We compose $(1 - s)a \bigoplus sb$ for the unique point c in the geodesic segment joining from a to b such that

$$d(c, a) = sd(a, b),\ d(c, b) = (1 - s)d(a, b).$$

We also denote by $[a, b]$ the geodesic segment joining from a to b, i.e., $[a, b] = \{(1 - s)a \bigoplus sb : s \in [0, 1]\}$. For the sake of simplicity, we recall a few definitions, exceptions and conclusions.

Example 2.1. *[5]* When endowed with the induced metric, a convex subset of Euclidean space $\mathbb{E}^n$ is CAT(0) and any real inner product space (not necessarily complete) is a CAT(0) space.

Example 2.2. *[3]* Attach together three copies of the ray $[0, \infty) \subset \mathbb{R}$ by gluing at the point 0. The resulting space has nonpositive curvature.

Lemma 2.3. *[5]* Let E be a CAT(0) space. Then
$d((1 - s)a \bigoplus sb, c) \leq (1 - s)d(a, c) + sd(b, c)$ for all $a, b, c \in E$ and $s \in [0, 1]$.

Proposition 2.4. *[15]* Let C be a nonempty subset of a CAT(0) space E and $S : C \to C$ be any mapping. Then :
(i) If S is nonexpansive then S is a Suzuki generalized nonexpansive mapping.
(ii) If S is a Suzuki generalized nonexpansive mapping and has a fixed point, then S is a quasi-nonexpansive

mapping.
(iii) If S is a Suzuki generalized nonexpansive mapping, then
$$d(a, Sb) \leq 3d(a, Sa) + d(a, b) \text{ for all } a, b \in C.$$

Lemma 2.5. *[15]* Let C be a weakly compact convex subset of a CAT(0) space E. Let S be a mapping on C. Assume that S is a Suzuki generalized nonexpansive mapping. Then S has a fixed point.

A mapping $S : C \to C$ is called contraction if there exists $\theta \in (0, 1)$ such that $d(Sa, Sb) \leq \theta d(a, b)$, for all $a, b \in C$. Many other stability outcomes for numerous invariant point iterative algorithms and for different groups of nonlinear maps were established based on the findings of Harder [8], Harder and Hicks [9], who introduced and studied the definition of stable fixed point iterative algorithm.

Definition 2.6. *[9]* Let $\{t_n\}_{n=0}^{\infty}$ be an arbitrary sequence in C. Then, an iterative procedure $a_{n+1} = f(S, a_n)$ converging to fixed point p, is said to be $S-$ stable or stable with respect to S, if for $\epsilon_n = d(t_n, f(S, t_n)), n = 0, 1, 2, ...$, we have
$$\lim_{n\to\infty} \epsilon_n = 0 \text{ if and only if } \lim_{n\to\infty} t_n = p\ .$$

Lemma 2.7. *[20]* Let $\{\psi_n\}_{n=0}^{\infty}$ and $\{\phi_n\}_{n=0}^{\infty}$ be nonnegative real sequences satisfying the following inequality:
$$\psi_{n+1} \leq (1 - \phi_n)\psi_n + \phi_n,$$
where $\phi_n \in (0, 1)$, for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \phi_n = \infty$ and $\dfrac{\phi_n}{\psi_n} \to 0$ as $n \to \infty$ then $\lim_{n\to\infty} \psi_n = 0$.

Lemma 2.8. *[21]* Let $\{p_n\}, \{q_n\}$ and $\{r_n\}$ be sequences of nonnegative numbers satisfying the inequality
$$p_{n+1} \leq (1 + q_n)p_n + r_n \text{ for all } n \geq 1.$$
If $\sum_{n=1}^{\infty} q_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$, then $\lim_{n\to\infty} p_n$ exists.

3. Convergence theorems in CAT(0) space

In this section, we prove convergence theorems in the sense of CAT(0) space via an iterative algorithm (3.1) for the maps which satisfies the condition (C). To approximate the invariant point of maps which meets condition (C), we first transform the new iterative algorithm in the context of CAT(0) space.

$$\begin{cases} c_n = S((1 - \beta_n)a_n \bigoplus \beta_n S a_n) \\ b_n = S((1 - \alpha_n)a_n \bigoplus \alpha_n S c_n) \\ a_{n+1} = S b_n \end{cases} \tag{1}$$

for all $n \geq 1$.
Now, we arrive at the subsequent conclusions :

Theorem 3.1. Consider a self map S defined on a nonempty closed convex subset C of a complete CAT(0) space E with $F(S) \neq \phi$ and also S meets the condition (C). For arbitrary $a_0 \in C$. Let the sequence $\{a_n\}$ be defined as in (1). Then, $\lim_{n\to\infty} d(a_n, p)$ exists for any $p \in F(S)$.

Proof. Suppose $p \in F(S)$ and $c \in C$. As S satisfies condition (C),
$$\frac{1}{2} d(p, Sp) \leq 0 \leq d(p, c) \Rightarrow d(Sp, Sc) \leq d(p, c).$$
Using Proposition 2.4, we get
$$\begin{aligned} d(c_n, p) &= d(S((1 - \beta_n)c_n \bigoplus \beta_n S a_n), p) \\ &\leq d((1 - \beta_n)a_n \bigoplus \beta_n S a_n, p) \\ &\leq (1 - \beta_n)d(a_n, p) + \beta_n d(S a_n, p) \\ &\leq (1 - \beta_n)d(a_n, p) + \beta_n d(a_n, p) \end{aligned}$$

$$
\begin{aligned}
&= d(a_n, p).\\
d(b_n, p) &= d(S((1-\alpha_n)a_n \bigoplus \alpha_n Sc_n), p)\\
&\le d((1-\alpha_n)a_n \bigoplus \alpha_n Sc_n, p)\\
&\le (1-\alpha_n)d(a_n, p) + \alpha_n d(Sc_n, p)\\
&\le (1-\alpha_n)d(a_n, p) + \alpha_n d(c_n, p)\\
&\le (1-\alpha_n)d(a_n, p) + \alpha_n d(a_n, p)\\
&= d(a_n, p).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
d(a_{n+1}, p) &= d(Sb_n, p)\\
&= d(Sb_n, Sp)\\
&\le d(b_n, p)\\
&\le d(a_n, p).
\end{aligned}
$$

Hence, $\{d(a_n, p)\}$ is a non-increasing sequence of real numbers that is bounded below by zero and so convergent. Therefore, $\lim_{n\to\infty} d(a_n, p)$ exists for all $p \in F(S)$. □

Theorem 3.2. Assume C, E, S and $\{a_n\}$ are the same as in Theorem 3.1. If $\{a_n\}$ is a sequence defined as in (1), then $lim_{n\to\infty} d(Sa_n, a_n) = 0$.

Proof. By Theorem 3.1, it follows $lim_{n\to\infty} d(a_n, p)$ exists, say $lim_{n\to\infty} d(a_n, p) = x$.
$\Rightarrow$ $\qquad lim_{n\to\infty} sup d(b_n, p) \le x$ and $lim_{n\to\infty} sup d(c_n, p) \le x$.
Since S meets condition(C), we get
$d(Sa_n, p) \le d(a_n, p), d(Sb_n, p) \le d(b_n, p)$ and $d(Sc_n, p) \le d(c_n, p)$. This suggests that

$$
\begin{aligned}
&\limsup_{n\to\infty} d(Sa_n, p) \le x\\
&\limsup_{n\to\infty} d(Sb_n, p) \le x\\
&\limsup_{n\to\infty} d(Sc_n, p) \le x.
\end{aligned}
$$

Now, $x = \lim_{n\to\infty} d(a_n, p) = \lim_{n\to\infty} d(a_{n+1}, p) = \lim_{n\to\infty} d(Sb_n, p)$,

$$x = \lim_{n\to\infty} d(Sb_n, p) \le \lim_{n\to\infty} d(b_n, p)$$

implying that $\lim_{n\to\infty} d(b_n, p) = x$.

$$
\begin{aligned}
d(Sb_n, p) &\le d(Sb_n, Sc_n) + d(Sc_n, p)\\
&\le d(Sb_n, Sc_n) + d(c_n, p).
\end{aligned}
$$

Taking limit as $n \to \infty$ on both sides, we have

$$\lim_{n\to\infty} d(Sb_n, Sc_n) = 0.$$

$$
\begin{aligned}
d(b_n, p) &\le (1-\alpha_n)d(a_n, p) + \alpha_n d(Sc_n, p)\\
&= (1-\alpha_n)d(a_n, p) + \alpha_n d(Sc_n, Sa_n) + \alpha_n d(Sa_n, p)\\
&= d(a_n, p) + \alpha_n d(Sc_n, Sa_n)
\end{aligned}
$$

which gives $\lim_{n\to\infty} inf d(Sa_n, Sc_n) = 0$

$$
\begin{aligned}
\text{Now, } d(Sa_n, a_n) &= d(Sa_n, Sb_n)\\
&\le d(Sa_n, Sc_n) + d(Sc_n, Sb_n).
\end{aligned}
$$

Therefore, $\lim_{n\to\infty} d(Sa_n, a_n) = 0$. □

Theorem 3.3. Assume C, E, S and $\{a_n\}$ are the same as in Theorem 3.1. Also C is compact. Then $\{a_n\}$ strongly converges to a invariant point of S.

Proof. By Theorem 3.2, we have $\lim_{n\to\infty} d(Sa_n, a_n) = 0$. Since C is compact, so there is a subsequence $\{a_{n_q}\}$ of $\{a_n\}$ such that $\{a_{n_q}\}$ converges strongly to p for some $p \in C$. By Proposition 2.4, we have

$$d(a_{n_q}, Sp) \le 3d(a_{n_q}, a_{n_q}) + d(a_{n_q}, p), \text{ for all } n \ge 1.$$

Letting $k \to \infty$ we get $p \in F(S)$. Since, by Theorem 3.1, $\lim_{n\to\infty} d(a_n, p)$ exists for any $p \in F(S)$, so $\{a_n\}$ strongly converge to p. □

4. New three step iterative algorithm and its convergence analysis

We demonstrate that our iterative algorithm (1) is stable and has a fast convergence rate when compared to other iterative algorithms in this section.

Theorem 4.1. Let S be a self contraction map defined on a nonempty closed convex subset of a complete CAT(0) space E. Let $\{a_n\}$ be an iterative sequence formed by (1) with real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in [0,1] pleasing $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$. Then the iterative algorithm (1) converges to the unique invariant point p of S.

Proof. Following Xue [22], S has a unique a invariant point. Thus, we will prove that $a_n \to p$ for $n \to \infty$. Using (1) we get

$$\begin{aligned} d(c_n, p) &= d(S((1-\beta_n)a_n \bigoplus \beta_n S a_n), p) \\ &\leq \theta d((1-\beta_n)a_n \bigoplus \beta_n S a_n, p) \\ &\leq \theta(1-\beta_n)d(a_n, p) + \beta_n d(S a_n, p) \\ &\leq \theta(1-\beta_n)d(a_n, p) + \theta\beta_n d(a_n, p) \\ &= \theta[1-\beta_n(1-\theta)]d(a_n, p). \end{aligned}$$

Similarly,

$$\begin{aligned} d(b_n, p) &= d(S((1-\alpha_n)a_n \bigoplus \alpha_n S c_n), p) \\ &\leq \theta d((1-\alpha_n)a_n \bigoplus \alpha_n S c_n, p) \\ &\leq \theta(1-\alpha_n)d(a_n, p) + \alpha_n d(S c_n, p) \\ &\leq \theta(1-\alpha_n)d(a_n, p) + \alpha_n \theta d(c_n, p) \\ &\leq \theta^2 \left[(1-\alpha_n)d(a_n, p) + \alpha_n d(a_n, p)\theta[1-\beta_n(1-\theta)]d(a_n, p)\right] \\ &\leq \theta^2 \left[(1-\alpha_n) + \alpha_n\theta(1-\beta_n(1-\theta))\right] d(a_n, p) \\ &\leq \theta^2 \left[(1-\alpha_n\beta_n(1-\theta)\right] d(a_n, p). \end{aligned}$$

Hence

$$\begin{aligned} d(a_{n+1}, p) = d(S b_n, p) &\leq \theta d(b_n, p) \\ &\leq \theta^3 \left[1-\alpha_n\beta_n(1-\theta)\right] d(a_n, p). \end{aligned}$$

The following inequalities result from repeating the above algorithms

$$\begin{aligned} &d(a_{n+1}, p) \leq \theta^3(1-\alpha_n\beta_n(1-\theta))d(a_n, p) \\ &d(a_n, p) \leq \theta^3(1-\alpha_{n-1}\beta_{n-1}(1-\theta))d(a_{n-1}, p) \\ &d(a_{n-1}, p) \leq \theta^3(1-\alpha_{n-2}\beta_{n-2}(1-\theta))d(a_{n-2}, p) \\ &\vdots \\ &d(a_1, p) \leq \theta^3(1-\alpha_0\beta_0(1-\theta))d(a_0, p). \end{aligned}$$

We can quickly deduce

$$d(a_n, p) \leq d(a_0, p)\theta^{3(n+1)} \prod_{k=0}^{n} (1-\alpha_k\beta_k(1-\theta))$$

where $1-\alpha_n\beta_n(1-\theta) < 1$ because $\theta \in (0,1)$ and $\alpha_n\beta_n \in [0,1]$ for all $n \in N$. As we know $1-a \leq e^{-a}$ for all $a \in [0,1]$. Therefore, we get

$$d(a_{n+1}, p) \leq \frac{d(a_0, p)\theta^{3(n+1)}}{e^{(1-\theta)} \sum_{k=0}^{n} \alpha_n\beta_n}.$$

Taking the limit on both sides of above inequality produces $\lim_{n\to\infty} d(a_n, p) = 0$, i.e. $a_n \to p$ for $n \to \infty$, as necessary. □

Theorem 4.2. Assume C, E, S and $\{a_n\}$ be same as in Theorem 4.1. Then the iterative algorithm (1) is S-stable.

Proof. Let $\{t_n\} \subset E$ be any arbitrary sequence in C. Assume the sequence formed by (1) is $a_{n+1} = f(S, a_n)$ converging to invariant point p and $\epsilon_n = d(t_{n+1}, f(S, c_n))$. We will demonstrate that $\lim_{n\to\infty} \epsilon_n = 0 \iff \lim_{n\to\infty} t_n = p$.
Consider $\lim_{n\to\infty} \epsilon_n = 0$. Using Theorem 4.1, we have

$$\begin{aligned} d(t_{n+1}, p) &\leq d(t_{n+1}, f(S, t_n)) + d(f(S, t_n), p) \\ &= \epsilon_n + d(t_{n+1}, p) \\ &\leq \epsilon_n + \theta^3(1 - \alpha_n\beta_n(1-\theta))d(t_n, p). \end{aligned}$$

Define $\psi = d(t_n, p), \phi = \alpha_n\beta_n(1-\theta) \in (0,1)$ and $\varphi_n = \epsilon_n$, since $\theta \in (0,1), \alpha_n, \beta_n \in [0,1]$, for all $n \in N$ and $\lim_{n\to\infty} \epsilon_n = 0$ implies that Lemma 2.7 conditions have been met. Hence $\lim_{n\to\infty} d(t_n, p) = 0 \implies \lim_{n\to\infty} t_n = p$.
For the converse part, let us consider that $\lim_{n\to\infty} t_n = p$, we get

$$\begin{aligned} \epsilon_n &= d(t_{n+1}, f(S, t_n)) \\ &\leq d(t_{n+1} + p) + d(f(S, t_n), p) \\ &\leq d(t_{n+1} + p) + \theta^3(1 - \alpha_n\beta_n(1-\theta))d(t_n,). \end{aligned}$$

It gives, $\lim_{n\to\infty} \epsilon_n = 0$. So, the iterative algorithm (1) is S-stable. □

Next, we first gave some examples of a Suzuki generalized nonexpansive mapping which is not nonexpansive.

Example 4.3. *[16] Let $E = \mathbb{R}$ be a CAT(0) space and $C = [0, 1]$. We can see that C is a compact convex subset of E. Define a mapping $S : C \to C$ by*

$$Sa = \begin{cases} 1 - a : a \in [0, 0.2), \\ \frac{a+4}{5} : a \in [0.2, 1]. \end{cases}$$

Also, Thakur et al. [16] showed that S satisfies condition (C) but is not a nonexpansive mapping.
Here, using Example 4.4, we illustrate the efficiency of our new iterative algorithm (1).

Example 4.4. *Consider a map $S : [0, 1] \to [0, 1]$ defined as*

$$Sa = \begin{cases} 1 - a : a \in [0, \frac{1}{3}), \\ \frac{a+2}{3} : a \in [\frac{1}{3}, 1]. \end{cases} \tag{2}$$

We must demonstrate that S satisfies the condition(C) but not nonexpansive. If $a = \frac{33}{100}, b = \frac{1}{7}$, we can easily derive

$$d(Sa, Sb) = |Sa - Sb| = |1 - \tfrac{33}{100} - \tfrac{7}{9}| = \tfrac{97}{900} > \tfrac{1}{300} = d(a, b).$$

Therefore, S is not a nonexpansive map. Consider the following cases to see that S is a Suzuki generalized nonexpansive map:
Case a: *Consider $a \in [0, \frac{1}{3})$, so $\frac{1}{2}d(a, Sb) = \frac{1-2a}{2} \in (\frac{1}{6}, \frac{1}{2}]$. For $\frac{1}{2}d(a, Sa) \leq d(a, b)$, we come to the fact that $\frac{1-2a}{2} \leq b - a$, i.e., $b \geq \frac{1}{2}$, so $b \in [\frac{1}{2}, 1]$. We get*

$$d(Sa, Sb) = |\tfrac{b+2}{3} - (1-a)| = |\tfrac{b+2-3+3a}{3}| = |\tfrac{b+3a-1}{3}| < \tfrac{1}{3}.$$

Also $\quad d(a, b) = |a - b| > |\frac{1}{3} - \frac{1}{2}| = \frac{1}{6}$
Therefore, $\quad \frac{1}{2}d(a, Sb) \leq d(a, b) \implies d(Sa, Sb) \leq d(a, b)$.
Case b: *Consider $a \in [\frac{1}{3}, 1)$, so $\frac{1}{2}d(a, Sb) = \frac{1}{2}|\frac{a+2}{3} - a| = \frac{1}{6}(2 - 2a)$. So, for $\frac{1}{2}d(a, Sa) \leq d(a, b)$, we get $\frac{2-2a}{6} \leq |a - b|$, then there are two options:*
(A) *: $b > a$, then $\frac{2-2a}{6} \leq b - a \implies b \geq \frac{2+4a}{6} \implies b \in [\frac{5}{9}, 1] \subset [\frac{1}{3}, 1]$. So, $d(Sa, Sb) = |\frac{a+2}{3} - \frac{b+2}{3}| = \frac{1}{3}d(a, b) \leq d(a, b)$.*
Therefore, $\quad \frac{1}{2}d(a, Sb) \leq d(a, b) \implies d(Sa, Sb) \leq d(a, b)$.
(B) : *Consider $a > b$, then $\frac{2-2a}{6} \leq a - b \implies b \leq \frac{8a-2}{6} \implies b \in [\frac{1}{9}, 1]$. So, here is the situation: $a \in [\frac{1}{3}, 1]$ and $b \in [\frac{1}{9}, 1]$. We can conclude $a \in [\frac{1}{3}, 1]$ and $b \in [\frac{1}{3}, 1]$ is already in (A). Hence consider $a \in [\frac{1}{3}, 1]$ and $b \in [\frac{1}{9}, \frac{1}{3}] \subset [0, \frac{1}{3})$, then $d(Sa, Sb) = |\frac{a+2}{3} - (1 - b)| = |\frac{a+3b-1}{3}|$. Then $d(Sa, Sb) \leq \frac{1}{3}$ and $d(a, b) > \frac{1}{3}$. Hence $d(Sa, Sb) \leq d(a, b)$.*

Table 1: **Iterative values of** (1), **M*** **and M algorithms**

n	new iteration (1)	M* iteration	M iteration
1	0.9	0.9	0.9
2	0.993574074086111	0.992944444480556	0.9922222223888890
3	0.999587074760719	0.999502191360573	0.999395061741358
4	0.999973465729994	0.999964876834885	0.999952949246550
5	0.999998294927465	0.999997521865572	0.999996340496954
6	0.999999890433302	0.999999825153849	0.999999715371985
7	0.999999992959325	0.999999987663633	0.999999977862266
8	0.999999999547572	0.999999999129601	0.999999998278176
9	0.999999999970927	0.999999999938589	0.999999999866080
10	0.999999999998132	0.999999999995667	0.999999999989584
11	0.999999999999880	0.999999999999694	0.999999999999190
12	0.999999999999992	0.999999999999978	0.999999999999937
13	1	0.999999999999999	0.999999999999995
14	1	1	1

Table 1 shows the iterative values generated by (1), M* and M algorithms. Graphic description is given in the Figure 1, where sequence of each iterative algorithm is depicted by a_n. We can see right away that our new iterative algorithm (1) is the first to converge than the other schemes.

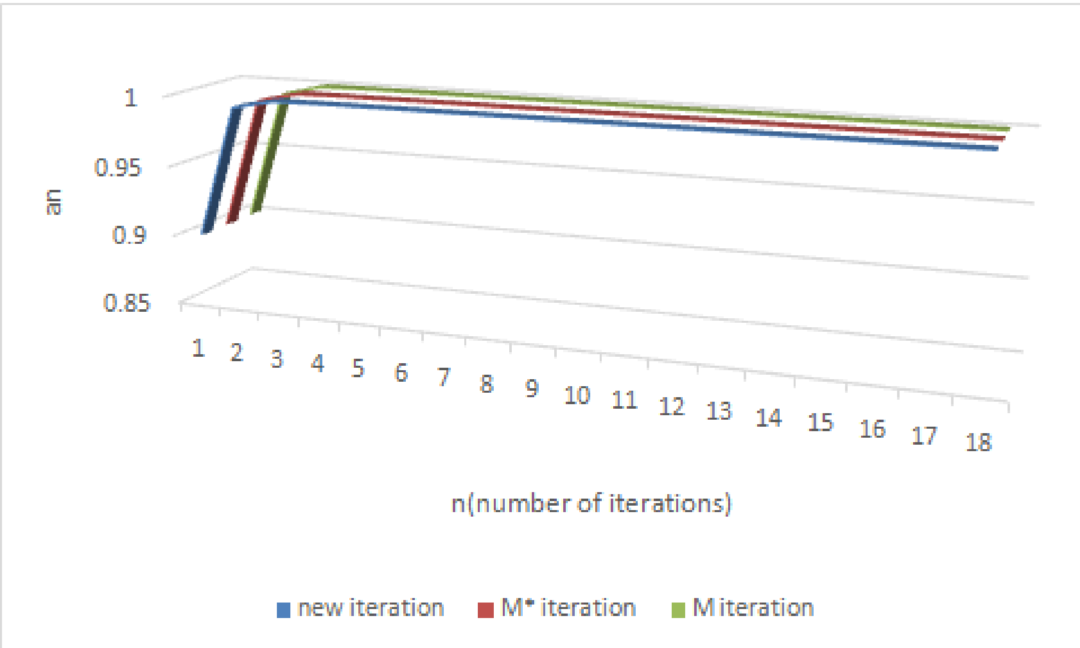

Figure 1: Convergence of (1), M* and M iterative algorithms to the invariant point 1 of map S.

5. Conclusion

In this paper, new iterative algorithm to approximate invariant points of generalized nonexpansive maps is introduced. This scheme has its own importance when the linear version of invariant point outcomes is extended to nonlinear domains. We broaden a linear version of convergence outcomes at the invariant point for a map that meets the condition (C) for the newly implemented iterative algorithm to nonlinear CAT(0) spaces in this article. Our new iterative algorithm is now accessible to computer scientists, engineers, physicists and mathematicians to solve various problems more efficiently.

6. Some open problems

1. It will be fascinating to achieve a generalization of the convergence and stability theorems to commutative, amenable semigroups as in the case of general metric space.
2. It will be fascinating to achieve a generalization of convergence and stability results in the framework of CAT(k) space($k > 0$).

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