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TELESCOPIC NUMERICAL SEMIGROUPS WITH MULTIPLICITY TEN AND EMBEDDING DIMENSION THREE

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Abstract. In this work, we give parametrizations of telescopic numerical semigroups with multiplicity ten and embedding dimension three. We also express some of its invariants in terms of generators of these semigroups such as the Frobenius number, genus and Sylvester number.

1. Introduction

Let $\mathbb{Z}$ be the set of integer numbers and $\mathbb{N} = \{x : x \geq 0, x \in \mathbb{Z}\}$. If $M \subseteq \mathbb{N}$ contains the zero element, is closed under usual addition, and such that $\mathbb{N} \setminus M$ is finite, then M is called a numerical semigroup. Numerical semigroups have an important place in semigroup theory of algebra with applications in ring theory, algebraic geometry and coding theory (see, for instance, [1, 6, 7, 11, 12]).

The starting point of numerical semigroups is known as the Frobenius problem, given by Sylvester in 1882. The Frobenius problem revolves around the set of positive integers that cannot be represented by the linear form $h_1x_1, \ldots, h_nx_n$. For brevity, let us chose $H = \{h_1, \ldots, h_n\}$. For this solution to be finite set, it is necessary and sufficient that $gcd(H) = 1$. The Frobenius problem operates under this assumption. Sylvester solved this problem for $n = 2$ in 1884 [17].

On the other hand, for a numerical semigroup M, the Frobenius number is defined as the largest integer in the set of integers but not in the set M, and this number is denoted by $f(M)$. Specifically, if $M = \mathbb{N}$, then $f(M) = -1$.

Modern studies on the Frobenius problem started with Brauer's article in 1942 and continued until today (see, [2, 3, 5]). This problem has applications in number theory, automata theory, sorting algorithms, and in many fields of mathematics [13].

Let M be a numerical semigroup and $H = \{h_1, \ldots, h_n\} \subset \mathbb{N}$ such that $h_1 < \ldots < h_n$. If there are $n > 0$ and $h_1 < \ldots < h_n$ satisfying

$$M = \langle H \rangle = \{a_1h_1 + \cdots + a_nh_n : a_i \in \mathbb{N}, h_i \in H\}$$

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we say that M is generated by $h_1, \ldots, h_n$. In addition, we say that M is minimally generated by $h_1, \ldots, h_n$, if any proper subset of $\{h_1, \ldots, h_n\}$ does not generate M, we denote by $M = \langle h_1, \ldots, h_n \rangle$. Since M is a numerical semigroup, $\mathbb{N} \setminus M$ is finite, $gcd(h_1, \ldots, h_n) = 1$.

The number $n(M)$, expressed as $n(M) = |\{0, \ldots, f(M)\} \cap M|$, is sometimes called the Sylvester number of M (where $|A|$ denotes the number of elements of the set A). However, the number of elements of the minimal generator set of the numerical semigroup M is also called the embedding dimension of M, denoted by $e(M)$. The smallest nonzero element of the numerical semigroup M is called the multiplicity of this semigroup, denoted by $\mu(M)$. It is also known that $e(M) \leq \mu(M)$ for M. If $M = \langle h_1, \ldots, h_n \rangle$, then $e(M) = n$ and $\mu(M) = h_1$. Also, we denote M by $\{x_0 = 0, x_1 \ldots, x_r, \longrightarrow\} = \{x_0 = 0, x_1 \ldots, x_r,\} \cup \{x \in \mathbb{N} : x \geq x_r + 1\}$ and $n(M) = r$ (where $x_i < x_{i+1}$ for any $i = 0, 1 \ldots, r = n(M)$) [5, 13].

We say $x \in \mathbb{N}$ is a gap of M if $x \notin M$. The largest gap of M is the Frobenius number of M. The set of all gaps of M is denoted by $G(M)$. The number expressed as $g(M) = |G(M)|$ is called the genus of M. Furthermore, in a numerical semigroup M, it is known that the equality $f(M) + 1 = g(M) + n(M)$ (see [13]).

One of the important class of numerical semigroup theory is telescopic numerical semigroups. Let $M = \langle h_1, \ldots, h_n \rangle$ and $d = gcd(h_1, \ldots, h_{n-1})$. If $h_n \in \langle \frac{h_1}{d}, \ldots, \frac{h_{n-1}}{d} \rangle$, then M is a telescopic numerical semigroup. Many researchers have studied Lipman semigroups and other properties of these (see [4, 8, 9, 15, 16]). In particular, the authors classified telescopic numerical semigroups with multiplicity less than ten and embeding dimension three and gave some results in these classes [15, 16]. Let's also note that the multiplicity of a numerical semigroup with embedding dimension three cannot be a prime number. Because, the numerical semigroups with prime multiplicity are generated two elements. In this study, we will classify all telescopic numerical semigroups with multiplicity ten and embedding dimension three and give formulas for the Frobenius number, genus, and Sylvester number in these classes.

2. Telescopic Numerical Semigroups

In this section, we will characterization of all telescopic numerical semigroups with multiplicity ten and embedding dimension three. We will also express some invariants of these semigroups in terms of generators.

Proposition 1. [17] Let m_1 and m_2 be two positive coprime integers.

(1) $f(\langle m_1, m_2 \rangle) = m_1 \cdot m_2 - m_1 - m_2$,
(2) $g(\langle m_1, m_2 \rangle) = \frac{m_1 \cdot m_2 - m_1 - m_2 + 1}{2}$.

Theorem 2.1. *Let M be a numerical semigroup with multiplicity ten and embedding dimension three. Then M is a telescopic numerical semigroup if and only if M is a member of any of the following families:*

$$(1) \quad \Pi = \{\langle 10, 10k+2, B \rangle : B = 10k + 2 + (2i+1), \quad i \neq 0, 5, \ldots, 5(\lceil \frac{k}{2} \rceil - 1) \quad \textit{and} \quad i \neq 3, 5+3, \ldots, 5(k-1)+3 \quad \textit{for} \quad k \in \mathbb{Z}^+, i \in \mathbb{N}\}$$

(where $\lceil a \rceil$ denotes the smallest integer not less than a)

$$(2)\quad \Sigma = \{\langle 10, 10k{+}4, B\rangle : B = 10k{+}4{+}(2i{+}1), \quad (k \neq 1) i \neq 3, 5{+}3, \ldots, 5(\lfloor \frac{k}{2} \rfloor{-}1){+}3 \quad and \quad i \neq 4, 5+4, \ldots, 5(k-1)+4 \quad for \quad k \in \mathbb{Z}^+, i \in \mathbb{N}\}$$

(where $\lfloor a \rfloor$ denotes the largest integer not greater than a)

$$(3)\quad \Upsilon = \{\langle 10, 10k+5, B\rangle : B > 10k+5 \quad and \quad 5 \nmid B \quad for \quad k \in \mathbb{Z}^+\}$$

$$(4)\quad \Phi = \{\langle 10, 10k{+}6, B\rangle : B = 10k{+}6{+}(2i{+}1), \quad i \neq 1, 5{+}1, \ldots, 5(\lceil \frac{k}{2} \rceil{-}1){+}1 \quad and \quad i \neq 0, 5, \ldots, 5k \quad for \quad k \in \mathbb{Z}^+, i \in \mathbb{N}\}$$

$$(5)\quad \Psi = \{\langle 10, 10k{+}8, B\rangle : B = 10k{+}8{+}(2i{+}1), \quad (k \neq 1) i \neq 4, 5{+}4, \ldots, 5(\lfloor \frac{k}{2} \rfloor{-}1){+}4 \quad and \quad i \neq 1, 5+1, \ldots, 5k+1 \quad for \quad k \in \mathbb{Z}^+, i \in \mathbb{N}\}$$

Proof. ($\Rightarrow$) Let $M = \langle 10, A, B\rangle$ be the telescopic numerical semigroup with multiplicity ten and embedding dimension three. Let $\Pi, \Sigma, \Upsilon, \Phi, \Psi$ families be given as in the Theorem. Due to the definition of the telescopic numerical semigroup, $d = gcd(10, A)$ and $B \in \langle \frac{10}{d}, \frac{A}{d}\rangle$. In this case, we have $d \in \{1, 2, 5, 10\}$. Let's examine the conditions of the numerical semigroup M according to the values that d can take.

(i) If $d = 1$, then $B \in \langle 10, A\rangle$. But this case contradicts that M has embedding dimension three.
(ii) If $d = 10$, then this contradicts that M has embedding dimension three.
(iii) If $d = 5$, then $A = 10k+5$ for $k \in \mathbb{Z}^+$. By definition of minimal generators, $10 < 10k+5 < B$ and $1 = gcd(10, 10k+5, B)$. So $gcd(10, 10k+5, B) = gcd(5, B) = 1$, namely, $5 \nmid B$. In this case, it is obtained as $M \in \Upsilon$.
(iv) If $d = 2$, then $A = 10k+2$, $A = 10k+4$, $A = 10k+6$ or $A = 10k+8$ for $k \in \mathbb{Z}^+$. Now let's examine M according to the values of A.
(a) Let $A = 10k{+}2$ for $k \in \mathbb{Z}^+$. By definition of minimal generators $10 < 10k+2 < B$ and $1 = gcd(10, 10k+2, B)$. So $gcd(10, 10k+2, B) = gcd(2, B) = 1$, namely, $2 \nmid B$. In this case, it is obtained as $B = 10k+2+(2i+1)$ for $i \in \mathbb{N}$. Consider the values of B for $i \equiv r \pmod 5$.

- If $i \equiv 0 \pmod 5$, then $i = 5u$ for $u \in \mathbb{N}$.

$$B = 10k+2+(2i+1) = 10k+10u+3 = 2(5k+1)+2(5u)+1.$$

Since M is telescopic, we can assume $B = 2(5k+1)+2(5u)+1 = 5x_1 + (5k+1)x_2$ for $u = 0, 1, \ldots, (\lceil \frac{k}{2} \rceil - 1)$. And so, $x_2 = 2$ and $5x_1 = 2(5u)+1$. In this case we get the contradiction $5(x_1 - 2u) = 1$. On the other hand, $B = 10k+2+(2i+1) \in \langle 5, 5k+1\rangle$ for $u \geq \lceil \frac{k}{2} \rceil$. Indeed, if $u \geq \lceil \frac{k}{2} \rceil$, then $u = \lceil \frac{k}{2} \rceil + t$, $t \in \mathbb{N}$.

$$B = 10k{+}2{+}(2i{+}1) = 10k{+}10u{+}3 = 10k{+}10(\lceil \frac{k}{2} \rceil{+}t)\rceil{+}3 = 10k{+}(2\lceil \frac{k}{2} \rceil)5{+}10t{+}3$$

In this case,

- If k is a positive odd integer, then $2\lceil \frac{k}{2} \rceil = k+1$ and

$$B = 10k+2+(2i+1) = 10k+(k+1)5+10t+3 = 3(5k+1)+(2t+1)5 \in \langle 5, 5k+1\rangle$$

- If k is a positive even integer, then $2\lceil \frac{k}{2} \rceil = k$ and

$$B = 10k + 2 + (2i + 1) = 10k + 5k + 10t + 3 = 3(5k + 1) + (2t)5 \in \langle 5, 5k + 1\rangle$$

Thus, $M \in \Pi$ for $i \neq 0, 5, \dots, 5(\lceil \frac{k}{2} \rceil - 1)$ and $(k \neq 1)k \in \mathbb{Z}^+, i \in \mathbb{N}$.

- If $i \equiv 1 \pmod 5$, then $i = 5u + 1$ for $u \in \mathbb{N}$.

$$B = 10k + 2 + (2i + 1) = 10k + 10u + 5 = 5(2k + 2u + 1) \in \langle 5, 5k + 1\rangle$$

Hence, $M \in \Pi$.

- If $i \equiv 2 \pmod 5$, then $i = 5u + 2$ for $u \in \mathbb{N}$.

$$B = 10k + 2 + (2i + 1) = 10k + 10u + 7 = 2(5k + 1) + (2u + 1)5 \in \langle 5, 5k + 1\rangle$$

Therefore, $M \in \Pi$.

- If $i \equiv 3 \pmod 5$, then $i = 5u + 3$ for $u \in \mathbb{N}$.

$$B = 10k + 2 + (2i + 1) = 10k + 10u + 9 = (2u + 1)5 + 2(5k + 1) + 2$$

Since M is telescopic, $B = 10k + 2 + (2i + 1) = (2u + 1)5 + 2(5k + 1) + 2 \in \langle 5, 5k + 1\rangle$. For $u = 1, 2, \dots, k - 1$,

$$B = 10k + 2 + (2i + 1) = (2u + 1)5 + 2(5k + 1) + 2 = 5x_1 + (5k + 1)x_2.$$

And so, $x_2 = 2$ and $5x_1 = 5(2u + 1) + 2$. In this case we get the contradiction $5(x_1 - 2u - 1) = 2$. On the other hand, $B = 10k + 2 + (2i + 1) \in \langle 5, 5k + 1\rangle$ for $u \geq k$. Indeed, if $u \geq k$, then $u = k + t$, $t \in \mathbb{N}$ and

$$B = 10k + 2 + (2i + 1) = 20k + 10t + 9 = (2t + 1)5 + 4(5k + 1) \in \langle 5, 5k + 1\rangle.$$

Hence, $M \in \Pi$ for $i \neq 3, 5 + 3, \dots, 5(k - 1) + 3$ and $k \in \mathbb{Z}^+, i \in \mathbb{N}$.

- If $i \equiv 4 \pmod 5$, then $i = 5u + 4$ for $u \in \mathbb{N}$.

$$B = 10k + 2 + (2i + 1) = 10k + 10u + 11 = 5(k + 2u + 2) + (5k + 1) \in \langle 5, 5k + 1\rangle$$

So, $M \in \Pi$.

(b) Let $A = 10k + 4$ for $k \in \mathbb{Z}^+$. By definition of minimal generators $10 < 10k + 4 < B$ and $1 = gcd(10, 10k + 4, B)$. So, $gcd(10, 10k + 4, B) = gcd(2, B) = 1$, namely, $2 \nmid B$. In this case, it is obtained as $B = 10k + 4 + (2i + 1)$ for $i \in \mathbb{N}$. Consider the values of B for $i \equiv r \pmod 5$.

- If $i \equiv 0 \pmod 5$, then $i = 5u$ for $u \in \mathbb{N}$.

$$B = 10k + 4 + (2i + 1) = 10k + 10u + 5 = 5(2k + 2u + 1) \in \langle 5, 5k + 2\rangle$$

So, $M \in \Sigma$.

- If $i \equiv 1 \pmod 5$, then $i = 5u + 1$ for $u \in \mathbb{N}$.

$$B = 10k + 4 + (2i + 1) = 10k + 10u + 7 = 5(k + 2u + 1) + (5k + 2) \in \langle 5, 5k + 2\rangle$$

So, $M \in \Sigma$.

- If $i \equiv 2 \pmod 5$, then $i = 5u + 2$ for $u \in \mathbb{N}$.

$$B = 10k + 4 + (2i + 1) = 10k + 10u + 9 = 5(2u + 1) + (5k + 2)2 \in \langle 5, 5k + 2\rangle$$

So, $M \in \Sigma$.

- If $i \equiv 3 \pmod 5$, then $i = 5u + 3$ for $u \in \mathbb{N}$ and $B = 10k + 4 + (2i + 1) = 10k + 10u + 11 = 5(2u + 1) + 2 + (5k + 2)2$. Since M is telescopic, we may suppose $B = 5(2u + 1) + 2 + (5k + 2)2 = 5x_1 + (5k + 2)x_2$ for $u = 0, 1, \dots, (\lfloor \frac{k}{2} \rfloor - 1)$. And so, $x_2 = 2$ and $5x_1 = 5(2u + 1) + 2$. In this case we get the contradiction $5(x_1 - 2u - 1) = 2$. On the other hand,

$B = 10k + 4 + (2i+1) \in \langle 5, 5k+2 \rangle$ for $u \geq \lfloor \frac{k}{2} \rfloor$. Indeed, if $u \geq \lfloor \frac{k}{2} \rfloor$, then $u = \lfloor \frac{k}{2} \rfloor + t$, $t \in \mathbb{N}$ and

$$B = 10k + 4 + (2i+1) = 10k + 2(5(\lfloor \frac{k}{2} \rfloor + t) + 3) + 5 = 10k + 10t + 5(2(\lfloor \frac{k}{2} \rfloor)) + 11$$

- If k is a positive odd integer, then $2\lfloor \frac{k}{2} \rfloor = k - 1$ and

$$B = 10k + 4 + (2i+1) = 10k + 10t + (k-1)5 + 11 = 3(5k+2) + (2t)5 \in \langle 5, 5k+2 \rangle$$

- If k is a positive even integer, then $2\lfloor \frac{k}{2} \rfloor = k$ and

$$B = 10k + 4 + (2i+1) = 15k + 10t + 11 = 3(5k+2) + (2t+1)5 \in \langle 5, 5k+2 \rangle$$

Thus, $M \in \Sigma$ for $(k \neq 1) i \neq 3, 5+3, \ldots, 5(\lfloor \frac{k}{2} \rfloor - 1) + 3$ and $k \in \mathbb{Z}^+, i \in \mathbb{N}$.

- If $i \equiv 4 \pmod 5$, then $i = 5u + 4$ for $u \in \mathbb{N}$.

$$B = 10k + 4 + (2i+1) = 10k + 10u + 13$$

Since M is telescopic, we can assume $B = 10k+10u+13 = 5x_1 + (5k+2)x_2$ for $u = 0, 1, \ldots, (k-1)$. Then $B = 10k+10u+13 = 5(2u+1)+2(5k+2)+3 = 5x_1 + (5k+2)x_2$. And so, $x_2 = 2$ and $5x_1 = 5(2u+1) + 3$. In this case we get $5(x_1 - 2u - 1) = 3$, which is a contradiction. On the other hand, $B = 10k + 4 + (2i+1) \in \langle 5, 5k+2 \rangle$ for $u \geq k$. Indeed, if $u \geq k$, then $u = k + t$, $t \in \mathbb{N}$ and

$$B = 10k + 4 + (2i+1) = 20k + 10t + 13 = (2t+1)5 + 4(5k+2) \in \langle 5, 5k+2 \rangle.$$

So, $M \in \Sigma$ for $i \neq 4, 5+4, \ldots, 5(k-1)+4$ and $k \in \mathbb{Z}^+, i \in \mathbb{N}$.

(c) Let $A = 10k+6$ for $k \in \mathbb{Z}^+$. By definition of minimal generators $10 < 10k+6 < B$ and $1 = gcd(10, 10k+6, B)$. So, $gcd(10, 10k+6, B) = gcd(2, B) = 1$, namely, $2 \nmid B$. In this case, it is obtained as $B = 10k + 6 + (2i+1)$ for $i \in \mathbb{N}$. Consider the values of B for $i \equiv r \pmod 5$.

- If $i \equiv 0 \pmod 5$, then $i = 5u$ for $u \in \mathbb{N}$.

$$B = 10k + 6 + (2i+1) = 10k + 10u + 7$$

Since M is telescopic, we may suppose $B = 10k+10u+7 = 5x_1 + (5k+3)x_2$ for $u = 0, 1, \ldots, k$. Then $B = 10k + 10u + 7 = 5(2u) + 2(5k+3) + 1 = 5x_1 + (5k+3)x_2$. And so, $x_2 = 2$ and $5x_1 = 5(2u) + 1$. In this case we get $5(x_1 - 2u) = 1$, which is a contradiction. On the other hand, $B = 10k + 6 + (2i+1) \in \langle 5, 5k+3 \rangle$ for $u \geq k+1$. Indeed, if $u \geq k+1$, then $u = k + t$, $t \in \mathbb{Z}^+$ and

$$B = 10k + 6 + (2i+1) = 20k + 10t + 7 = (2t-1)5 + 4(5k+3) \in \langle 5, 5k+3 \rangle.$$

So, $M \in \Phi$ for $i \neq 0, 5, \ldots, 5k$ and $k \in \mathbb{Z}^+, i \in \mathbb{N}$.

- If $i \equiv 1 \pmod 5$, then $i = 5u + 1$ for $u \in \mathbb{N}$.

$$B = 10k + 6 + (2i+1) = 10k + 10u + 9$$

Since M is telescopic, we can assume $B = 10k+10u+9 = 5x_1 + (5k+3)x_2$ for $u = 0, 1, \ldots, (\lceil \frac{k}{2} \rceil - 1)$. Then $B = 10k + 10u + 9 = 5(2u) + 2(5k+3) + 3 = 5x_1 + (5k+3)x_2$. And so, $x_2 = 2$ and $5x_1 = 5(2u) + 3$. In this case we get $5(x_1 - 2u) = 3$, which is a conradiction. On the other hand,

$B = 10k + 6 + (2i + 1) \in \langle 5, 5k + 3\rangle$ for $u \geq \lceil \frac{k}{2} \rceil$. Indeed, if $u \geq \lceil \frac{k}{2} \rceil$, then $u = \lceil \frac{k}{2} \rceil + t$, $t \in \mathbb{Z}^+$ and

$$B = 10k + 6 + (2i + 1) = 10k + 10t + 5(2(\lceil \frac{k}{2} \rceil)) + 9.$$

In this case;

- If k is a positive odd integer, then

$$B = 10k + 6 + (2i + 1) = 15k + 10t + 14 = (2t + 1)5 + 3(5k + 3) \in \langle 5, 5k + 3\rangle$$

- If k is a positive even integer, then

$$B = 10k + 6 + (2i + 1) = 15k + 10t + 9 = (2t)5 + 3(5k + 3) \in \langle 5, 5k + 3\rangle$$

For that reason, $M \in \Phi$ for $i \neq 1, 5+1, \ldots, 5(\lceil \frac{k}{2} \rceil - 1) + 1$ and $k \in \mathbb{Z}^+, i \in \mathbb{N}$.

- If $i \equiv 2 \pmod 5$, then $i = 5u + 2$ for $u \in \mathbb{N}$ and

$$B = 10k + 6 + (2i + 1) = 10k + 10u + 11 = (2u + 1)5 + 2(5k + 3) \in \langle 5, 5k + 3\rangle.$$

Thus, $M \in \Phi$.

- If $i \equiv 3 \pmod 5$, then $i = 5u + 3$ for $u \in \mathbb{N}$.

$$B = 10k + 6 + (2i + 1) = 10k + 10u + 13 = (2u + k + 2)5 + (5k + 3) \in \langle 5, 5k + 3\rangle.$$

So, $M \in \Phi$.

- If $i \equiv 4 \pmod 5$, then $i = 5u + 4$ for $u \in \mathbb{N}$.

$$B = 10k + 6 + (2i + 1) = 10k + 10u + 15 = (2u + 2k + 3)5 \in \langle 5, 5k + 3\rangle.$$

Hence, $M \in \Phi$.

(d) Let $A = 10k+8$ for $k \in \mathbb{Z}^+$. By definition of minimal generators $10 < 10k+8 < B$ and $1 = gcd(10, 10k+8, B)$. So, $gcd(10, 10k+8, B) = gcd(2, B) = 1$, namely, $2 \nmid B$. In this case, it is obtained as $B = 10k + 8 + (2i + 1)$ for $i \in \mathbb{N}$. Consider the values of B for $i \equiv r \pmod 5$.

- If $i \equiv 0 \pmod 5$, then $i = 5u$ for $u \in \mathbb{N}$ and

$$B = 10k + 8 + (2i + 1) = 10k + 10u + 9 = (2u + k + 1)5 + 1(5k + 4) \in \langle 5, 5k + 4\rangle.$$

Thus, $M \in \Psi$

- If $i \equiv 1 \pmod 5$, then $i = 5u + 1$ for $u \in \mathbb{N}$.

$$B = 10k + 8 + (2i + 1) = 10k + 10u + 11 = (2u)5 + 2(5k + 4) + 3.$$

Since M is telescopic, we may assume $B = 10k + 10u + 11 \in \langle 5, 5k + 4\rangle$ for $u = 0, 1, \ldots, k$. Then $B = 10k + 10u + 11 = 5(2u) + 2(5k + 4) + 3 = 5x_1 + (5k + 4)x_2$. And so, $x_2 = 2$ and $5x_1 = 5(2u) + 3$. In this case we get $5(x_1 - 2u) = 3$, which is a contradiction. On the other hand, $B = 10k + 8 + (2i + 1) \in \langle 5, 5k + 4\rangle$ for $u \geq k$. Indeed, if $u \geq k$, then $u = k + t$, $t \in \mathbb{Z}^+$ and

$$B = 10k + 8 + (2i + 1) = 20k + 10t + 11 = (2t - 1)5 + 4(5k + 4) \in \langle 5, 5k + 4\rangle$$

So, $M \in \Psi$ for $i \neq 1, 5 + 1, \ldots, 5k + 1$ and $k \in \mathbb{Z}^+, i \in \mathbb{N}$.

- If $i \equiv 2 \pmod 5$, then $i = 5u + 2$ for $u \in \mathbb{N}$ and

$$B = 10k + 8 + (2i + 1) = 10k + 10u + 13 = (2u + 1)5 + 2(5k + 4) \in \langle 5, 5k + 4\rangle.$$

For that reason, $M \in \Psi$.

- If $i \equiv 3 \pmod 5$, then $i = 5u + 3$ for $u \in \mathbb{N}$ and

$$B = 10k + 8 + (2i + 1) = 10k + 10u + 15 = (2k + 2u + 3)5 \in \langle 5, 5k + 4\rangle.$$

Thus, $M \in \Psi$.

- If $i \equiv 4 \pmod 5$, then $i = 5u + 4$ for $u \in \mathbb{N}$.

$$B = 10k + 8 + (2i + 1) = 10k + 10u + 17 = (2u + 1)5 + 2(5k + 4) + 4.$$

Since M is telescopic, we can suppose $B = 10k + 10u + 17 \in \langle 5, 5k + 4\rangle$ for $u = 0, 1, \ldots, (\lfloor \frac{k}{2} \rfloor - 1)$. Then $B = 10k+10u+17 = 5(2u+1)+2(5k+4)+4 = 5x_1 + (5k + 4)x_2$. And so, $x_2 = 2$ and $5x_1 = 5(2u + 1) + 4$. In this case we get $5(x_1 - 2u - 1) = 4$, which is a contradiction. On the other hand, $B = 10k + 8 + (2i + 1) \in \langle 5, 5k + 4\rangle$ for $u \geq \lfloor \frac{k}{2} \rfloor$. Indeed, if $u \geq \lfloor \frac{k}{2} \rfloor$, then $u = \lfloor \frac{k}{2} \rfloor + t$, $t \in \mathbb{N}$ and

$$B = 10k + 8 + (2i + 1) = 10k + 10t + 5(2\lfloor \frac{k}{2} \rfloor) + 17$$

In this case,

- If k is a positive odd integer, then

$$B = 10k + 8 + (2i + 1) = 15k + 10t + 12 = (2t)5 + 3(5k + 4) \in \langle 5, 5k + 4\rangle$$

- If k is a positive even integer, then

$$B = 10k + 8 + (2i + 1) = 15k + 10t + 17 = (2t + 1)5 + 3(5k + 4) \in \langle 5, 5k + 4\rangle$$

So, $M \in \Psi$ for $i \neq 4, 5 + 4, \ldots, 5(\lfloor \frac{k}{2} \rfloor - 1) + 4$ and $k \neq 1, k \in \mathbb{Z}^+, i \in \mathbb{N}$.

($\Leftarrow$) Let the families $\Pi, \Sigma, \Upsilon, \Phi, \Psi$ be given as in Theorem. Suppose $M \in \Upsilon = \{\langle 10, 10k + 5, B\rangle : B > 10k + 5 \text{ and } 5 \nmid B \text{ for } k, B \in \mathbb{Z}^+\}$. Since $10 < 10k + 5 < B$ and $gcd(10, 10k + 5, B) = 1$, M is a numerical semigroup. Moreover, $gcd(10, 10k + 5) = 5$ and $B \in \langle \frac{10}{5}, \frac{10k+5}{5} \rangle = \langle 2, 2k + 1\rangle$. Indeed, since $B > 10k + 5$ and $5 \nmid B$, $B = 10k + 5 + r$ for $5 \nmid r$ and $r \in \mathbb{Z}^+$. By Proposition 1,

$$f(\langle 2, 2k + 1\rangle) = 2(2k + 1) - 2 - (2k + 1) = 2k - 1$$

Due to the fact that $B = 10k + 5 + r > 2k - 1 = f(\langle 2, 2k + 1\rangle)$, $B \in \langle 2, 2k + 1\rangle$. For that reason, M is telescopic. It can be similarly shown that Π, Σ, Φ, Ψ are also the families of the telescopic numerical semigroups. □

Proposition 2. [10] Let M be numerical semigroup with minimal generators $\{m_1, \ldots, m_r\}$. Also let $d = gcd(m_1, m_2 \ldots, m_{r-1})$ and $N = \langle \frac{m_1}{d}, \frac{m_2}{d}, \ldots, \frac{m_{r-1}}{d}, m_r\rangle$. Then the following conditions hold:

(1) $f(M) = df(N) + (d - 1)m_r$,
(2) $g(M) = dg(N) + \frac{(d-1)(m_r-1)}{2}$.

Henceforth, we will present some invariants of the telescopic numerical semigroups we obtained above.

Proposition 3. Let M be a member of the family Π given in Theorem 2.1. Then the following conditions hold:

(1) $f(M) = 40k + B - 2$,
(2) $g(M) = 20k + \frac{(B-1)}{2}$,
(3) $n(M) = 20k + \frac{(B-1)}{2}$.

Proof. Assume that M is a member of the family Π given in Theorem 2.1. Then $2 = gcd(10, 10k+2)$. Now take $N = \langle \frac{10}{2}, \frac{10k+2}{2}, B\rangle = \langle 5, 5k+1, B\rangle$. $B \in \langle 5, 5k+1\rangle$ from Theorem 2.1. For that reason, $N = \langle 5, 5k+1\rangle$. By Proposition 1, $f(N) = 20k-1$ and $g(N) = 10k$. By Proposition 2 and the equality $g(M) + n(M) = f(M) + 1$,
$f(M) = 2(20k-1) + (2-1)B = 40k + B - 2$,
$g(M) = 2(10k) + \frac{(2-1)(B-1)}{2} = 20k + \frac{(B-1)}{2}$,
$n(M) + (20k + \frac{(B-1)}{2}) = (40k + B - 2) + 1 \Rightarrow n(M) = 20k + \frac{(B-1)}{2}$. □

The proof of each of the following propositions can be done in a similar way to the proof of Proposition 3.

Proposition 4. Let M be a member of the family Σ given in Theorem 2.1. Then the following conditions hold:
(1) $f(M) = 40k + B + 6$,
(2) $g(M) = 20k + 4 + \frac{(B-1)}{2}$,
(3) $n(M) = 20k + 4 + \frac{(B-1)}{2}$.

Proposition 5. Let M be a member of the family Υ given in Theorem 2.1. Then the following conditions hold:
(1) $f(M) = 10k + 4B - 5$,
(2) $g(M) = 5k + 2B - 2$,
(3) $n(M) = 5k + 2B - 2$.

Proposition 6. Let M be a member of the family Φ given in Theorem 2.1. Then the following conditions hold:
(1) $f(M) = 40k + B + 14$,
(2) $g(M) = 20k + 8 + \frac{(B-1)}{2}$,
(3) $n(M) = 20k + 8 + \frac{(B-1)}{2}$.

Proposition 7. Let M be a member of the family Ψ given in Theorem 2.1. Then the following conditions hold:
(1) $f(M) = 40k + B + 22$,
(2) $g(M) = 20k + 12 + \frac{(B-1)}{2}$,
(3) $n(M) = 20k + 12 + \frac{(B-1)}{2}$.

Example 2.2. Let $M = \langle 10, 34, B\rangle$. Let's find all the telescopic numerical semigroups written in this form. Since $34 = 3 \cdot 10 + 4$, $k = 3$ and $M = \langle 10, 34, B\rangle \in \Sigma$. Thus, $B \notin \{41, 43, 53, 63\}$ and $B \in \{35, 37, 39, 45, 47, 49, 51, 55, 57, 59, 61, 65, \longrightarrow\}$. Let's consider the numerical semigroup $M = \langle 10, 34, 39\rangle$.
(1) $f(M) = 40k + B + 6 = 40 \cdot 3 + 39 + 6 = 165$,
(2) $g(M) = 20k + 4 + \frac{(B-1)}{2} = 20 \cdot 3 + 4 + \frac{(39-1)}{2} = 83$,
(3) $n(M) = 20k + 4 + \frac{(B-1)}{2} = 20 \cdot 3 + 4 + \frac{(39-1)}{2} = 83$.
Indeed, when $M = \langle 10, 34, 39\rangle = \{0, 10, 20, 30, 34, 39, 40, 44, 49, 50, 54, 59, 60, 64, 68, 69, 70, 73, 74, 78, 79, 80, 83, 84, 88, 89, 90, 93, 94, 98, 99, 100, 102, 103, 104, 107, 108, 109, 110, 112, 113, 114, 117, 118, 119, 120, 122, 123, 124, 127, 128, 129, 130, 132, 133, 134, 136, 137, 138, 139, 140, 141, 142, 143, 144, 146, 147, 148, 149, 150, 151, 152, 153, 154, 156, 157, 158, 159, 160, 161, 162, 163, 164, 166, \longrightarrow\}$ is written, it can be seen that the above values can be found.

3. Conclusion

The concept of telescopic semigroups was first used by Kirfeland and Pellikaan. They tried to determine numerical semigroups associated with algebraic-geometric codes with some nice properties [11]. The study of numerical semigroups has significant applications in algebraic error coding theory. Besides, it is known that every telescopic numerical semigroup is a free numerical semigroup [14]. Free semigroups, on the other hand, have an important place in the studies of symmetric numerical semigroups [18]. Symmetric numerical semigroups have been the focus of attention in algebraic geometry studies because of their connections with curves and coordinate rings [12]. Recently, researchers have been trying to establish a relationship between the studies in their field and the studies of numerical semigroups. This study is particularly concerned with telescopic numerical semigroup families, which other researchers have been the focus of attention.

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