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Petrov- Galerkin sonlu eleman yöntemi ile KdV-Burgers' denkleminin nümerik çözümü

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ÖZET

Bu bölümde KdVB denkleminin sayısal çözümleri şekil fonksiyonları kübik, ağırlık fonksiyonları kuadratik B-spline fonksiyonları alınarak Petrov-Galerkin yöntemi ile elde edildi. Yöntemin doğruluğu ve etkinliği için üç tane test problemi ele alındı. Elde edilen denklem sistemleri Thomas algoritması ile çözüldü. Ayrıca elde edilen sayısal sonuçlar literatürdeki sonuçlar ile karşılaştırıldı.

Anahtar Kelimeler: KdVB dnenklemi, Sonlu eleman metodu, Petrov- Galerkin, B-spline, Solitary dalgalar.

A Petrov-Galerkin finite element method for the numerical solution of the KdV-Burgers' equation

ABSTRACT

In this study, a Petrov-Galerkin finite element method, in which the element shape functions are cubic and weight functions are quadratic B-splines, is implemented to find the numerical solution of the Korteweg-de Vries-Burgers'(KdVB) equation. Accuracy of the presented method is demonstrated by three test problems. The obtained numerical results are compared with results given in the literature and shown graphically.

Key Words: KdVB equation, Finite element method, Petrov-Galerkin, B-splines, Solitary waves.

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1. Giriş

In this article, a numerical algorithm for solving to model both KdV and Burgers' equations and KdVB equation with a variant of initial and boundary conditions has been set up based on Petrov-Galerkin

The KdVB equation which we discuss in this paper is based upon both the KdV equation

$$U_t + \varepsilon . U . U_x + \mu U_{xxx} = 0, \qquad (1)$$

which plays a major role in the study of non-linear dispersive waves [1] and the Burgers' equation (BE)

$$U_t + \varepsilon . U . U_x - v . U_{xx} = 0, \qquad (2)$$

that its turbulence model is very important in fluid dynamics model [2], has the form

$$U_t + \varepsilon . U . U_x - v . U_{xx} + \mu U_{xxx} = 0, \qquad (3)$$

where ε, v and μ are positive parameters. The equation was first formulated by Su and Gardner. Since it involves both damping and dispersion, it has been a model equation for a wide class of nonlinear systems in the weak nonlinearity and long wavelength approximations [3]. The steady state solutions of the equation have been shown to model weak plasma shocks propagating perpendicularly to a magnetic field [4]. When diffusion dominates dispersion the steady state solutions of the KdVB equation are monotonic shocks, and when dispersion dominates, the shocks are oscillatory [5]. The equation has some other physical applications for instance study of wave propagation through liquid- filled elastic tubes [6] and for a description of shallow water waves on a viscous fluid [7]. Especially the travelling wave solution to the equation has been studied extensively. A travelling wave solution to the KdVB equation is presented by Demiray [8] by using the hyperbolic tangent method and an exponential rational function approach. Demiray [9], Antar and Demiray [10] derived KdVB equation as the governing evolution equation for wave propagation in fluid-filled elastic or viscoelastic tubes in which the effects of dispersion, dissipation and nonlinearity were present. Bona and Schonbek [11] studied the existence and uniqueness of bounded traveling wave solution to Eq.(3), which tends to constant states at plus and minus infinity. KdVB equation is composition of the KdV equation $(v = 0)$ and Burgers' equation($\mu = 0$). This equation has been solved analytically for a limited set of boundary and initial conditions. So, for the numerical treatment of the equation, some methods have been introduced with various boundary and initial conditions.Canosa and Gazdag [12], who discussed the evolution of non-analytic initial data into a monotonic shock, have used the accurate space derivative method to give the brief details of a numerical solution of the KdVB equation. A finite element solution of the KdVB equation based on Bubnov- Galerkin's method using cubic B-splines as element shape and weight functions, is set up by Zaki [5]. Zaki [13] also applied the collocation method with quintic B-spline finite element to obtain the numerical solution of the KdV, Burgers' and KdVB equations. KdVB equation solved numerically with the Galerkin's method using quadratic B-spline interpolation functions over the finite elements by Ali et al.[14] and using the quartic B-splines as both shape and weight functions over the finite intervals by Saka et al. [15]. S. Haq et al. [16] have used three radial basis functions (RBFs) collocation method to solve the KdVB equation. Karakoc et al. [17] have obtain the numerical solution of the KdVB by a new differential quadrature method based on quintic B-spline functions. Talat and El-Danaf [18] proposed a numerical solution for the KdVB equation by using the collocation method using the septic splines. Kaya [19] presented ADM to find the explicit and numerical solutions of the KdV, Burgers' and KdVB equation for the initial conditions. finite element method, in which the element shape functions are cubic and weight functions are quadratic B-splines. Rest of the paper is organized as follow. In Section 2, the numerical method is presented. Numerical results of KdVB equation with its special cases are presented in Section 3. In Section 4, the results are summarized.

2. Petrov-Galerkin sonlu eleman yönteminin çözüm uygulaması

In this study, we will consider the KdVB Eq. (3) with the following boundary conditions

$$\begin{aligned} &U(a,t) = 0, && U(b,t) = 0, && \qquad (4)\\ &U_x(a,t) = 0, && U_x(b,t) = 0, && \\ &U_{xx}(a,t) = 0 && U_{xx}(b,t) = 0 && t > 0, \end{aligned}$$

and the initial condition

$$U(x,0) = f(x) \qquad a \le x \le b.$$

To implement the numerical methods, the space interval $a \le x \le b$ is discretized by uniform $(N+1)$ grid points $x_j = a + jh$ where $j = 0, 1, 2, \ldots\ldots, N$ and the grid spacing is given by $h = (b-a)/N$. Cubic B-splines $\phi_m(x), (m = -1(1)N + 1)$ at the knots x_m are defined over the interval $[a,b]$

$$\phi_m(x) = \frac{1}{h^3}\begin{cases} (x - x_{m-2})^3 & ,[x_{m-2}, x_{m-1}] \\ h^3 + 3h^2(x - x_{m-1}) + 3h(x - x_{m-1})^2 - 3(x - x_{m-1})^3 & ,[x_{m-1}, x_m] \\ h^3 + 3h^2(x_{m+1} - x) + 3h(x_{m+1} - x)^2 - 3(x_{m+1} - x)^3 & ,[x_m, x_{m+1}] \\ (x_{m+2} - x)^3 & ,[x_{m+1}, x_{m+2}] \\ 0 & ,otherwise \end{cases} \qquad (5)$$

which vanish outside of the interval $[x_{m-2}, x_{m+2}]$ [20]. The approximate solution $U_N(x,t)$ to the exact solution $U(x,t)$ is given by

$$U_N(x,t) = \sum_{j=-1}^{N+1} \delta_j(t)\, \phi_j(x) \qquad (6)$$

where $\delta_j(t)$ are time dependent parameters to be determined from the boundary and weighted residual conditions. The set of cubic B-spline functions $\{\phi_{-1}(x), \ldots\ldots\ldots, \phi_{N+1}(x)\}$ create a basis for functions defined over $[a, b]$. In each element, using the following local coordinate transformation

$$h\xi = x - x_m, \qquad 0 \le \xi \le 1 \qquad (7)$$

cubic B-spline shape functions in terms of ξ over the domain $[x_{m-1}, x_{m+2}]$ can be defined as

$$\begin{aligned} \phi_{m-1} &= (1-\xi)^3, \\ \phi_m &= 1 + 3(1-\xi) + 3(1-\xi)^2 - 3(1-\xi)^3, \\ \phi_{m+1} &= 1 + 3\xi + 3\xi^2 - 3\xi^3, \\ \phi_{m+2} &= \xi^3. \end{aligned} \qquad (8)$$

Variation of the function $U(x,t)$ over the element $[x_m, x_{m+1}]$ is approximated by

$$U_N(\xi,t) = \sum_{j=m-1}^{m+2} \delta_j(t)\, \phi_j(\xi) \qquad (9)$$

where $\delta_{m-1}, \delta_m, \delta_{m+1}, \delta_{m+2}$ act as element parameters and B-splines $\phi_{m-1}, \phi_m, \phi_{m+1}, \phi_{m+2}$ as element shape functions. Using expansion (6) and cubic B-splines (5), the nodal values $(U_N)_m$ and their first and second derivatives $(U_N)_x$, $(U_N)_{xx}$ can be calculated at the nodal points x_m in terms of nodal parameters following,

$(U_N)_m = U_N(x_m) = \delta_{m-1} + 4\delta_m + \delta_{m+1},$
$h\,(U_{Nx})_m = U_{Nx}(x_m) = 3(-\delta_{m-1} + \delta_{m+1}),$
$$h^2\,(U_{Nxx})_m = U_{Nxx}(x_m) = 6\,(\delta_{m-1} - 2\delta_m + \delta_{m+1}). \qquad (10)$$

We have taken the weight function $W(x)$ as a quadratic B-spline $\psi_m(x)$. $\psi_m(x)$, at the knots x_m are defined over the interval $[a,b]$

$$\phi_m(x) = \frac{1}{h^2}\begin{cases}(x_{m+2}-x)^2 - 3(x_{m+1}-x)^2 + 3(x_m - x)^2 & ,[x_{m-1},x_m]\\ (x_{m+2}-x)^2 - 3(x_{m+1}-x)^2 & ,[x_m,x_{m+1}]\\ (x_{m+2}-x)^2 & ,[x_{m+1},x_{m+2}]\\ 0 & ,otherwise\,.\end{cases} \qquad (11)$$

If we also use the local coordinate transformation (5) over the typical finite element $[x_m, x_{m+1}]$, quadratic B-spline shape functions can be obtained as

$$\psi_{m-1} = (1-\eta)^2,$$
$$\psi_m = 1 + 2\eta - 2\eta^2,$$
$$\psi_{m+1} = \eta^2 \qquad (12)$$

Applying the Petrov-Galerkin technique to Eq.(3) with weight function $W(x)$, we get the weak form of Eq. (3)
$$\int_a^b W\,(U_t + \varepsilon\,U\,U_x - v\,U_{xx} + \mu\,U_{xxx})\,d_x = 0 \qquad (13)$$

When we take the approximation functions as cubic B- splines, weight functions with quadratic B- spline shape functions and substitute approximation (6) into integral Eq. (13), we obtain the following element contributions of the form
$$\sum_{j=m-1}^{m+2}\left(\int_0^1 \phi_i\phi_j d\xi\right)\dot{\delta}_j^e + \varepsilon Z_m \sum_{j=m-1}^{m+2}\left[\left(\int_0^1 \phi_i\phi_j' d\xi\right)\delta_j^e\right] + v\sum_{j=m-1}^{m+2}\left[\left(\left(\int_0^1 \phi_i'\phi_j' d\xi\right) - \phi_i\phi_j'\Big|_0^1\right)\delta_j^e\right] - \mu\sum_{j=m-1}^{m+2}\left[\left(\left(\int_0^1 \phi_i'\phi_j'' d\xi\right) - \phi_i\phi_j''\Big|_0^1\right)\delta_j^e\right]$$

$i = 1,2,3$ ve $j = m-2, m-1, \dots, m+2$ (14)
which can also be written in a matrix form as follows:

$$A^e\dot{\delta}^e + [\varepsilon\,Z_m B^e + v(C^e - E^e) - \mu(D^e - F^e)\delta^e] \qquad (15)$$

where $\delta^e = (\delta_{m-2},\ \delta_{m-1}, \delta_m, \delta_{m+1}, \delta_{m+2})^T$ are the element matrices and dot denotes differentiation with respect to t. The element matrices $A^e, B^e, C^e, D^e, E^e, F^e$ are rectangular 3 x 4 given by the following integrals:

$$A_{ij}^e = \int_0^1 \phi_i\phi_j d\xi = \frac{1}{60}\begin{bmatrix}10 & 71 & 38 & 1\\ 19 & 221 & 221 & 19\\ 1 & 38 & 71 & 10\end{bmatrix},$$
$$B_{ij}^e = \int_0^1 \phi_i\phi_j' d\xi = \frac{1}{10}\begin{bmatrix}-6 & -7 & 12 & 1\\ -13 & -41 & 41 & 13\\ -1 & -12 & 7 & 6\end{bmatrix},$$
$$C_{ij}^e = \int_0^1 \phi_i'\phi_j' d\xi = \frac{1}{2h}\begin{bmatrix}3 & 5 & -7 & -1\\ -2 & 2 & 2 & -2\\ -1 & -7 & 5 & 3\end{bmatrix},$$
$$D_{ij}^e = \int_0^1 \phi_i'\phi_j'' d\xi = \frac{1}{h^2}\begin{bmatrix}-4 & 6 & 0 & -2\\ 2 & -6 & 6 & -2\\ 2 & 0 & -6 & 4\end{bmatrix},$$
$$E_{ij}^e = \phi_i\phi_j'\Big|_0^1 = \frac{3}{h}\begin{bmatrix}1 & 0 & -1 & 0\\ 1 & -1 & -1 & 1\\ 0 & -1 & 0 & 1\end{bmatrix} \text{ and}$$
$$F_{ij}^e = \phi_i\phi_j''\Big|_0^1 = \frac{6}{h^2}\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 1 & -2 & 1\\ 0 & 1 & -2 & 1\end{bmatrix}$$

Assembling contributions from all elements produce the first-order matrix equation system
$$A^e\dot{\delta}^e + [\varepsilon\,Z_m B^e + v(C^e - E^e) - \mu(D^e - F^e)\delta^e] = 0, \quad (16)$$

where $\delta = (\delta_{-1}, \delta_0, \dots, \delta_N, \delta_{N+1})^T$ are global element parameter and A, B, C, D, E and F are derived from the element matrices A^e, B^e, C^e, D^e and F^e, respectively.
Substituting the time derivative of the parameter $\dot{\delta}$ by usual finite difierence approximation $\dot{\delta} = \left(\frac{\delta^{n+1} - \delta^n}{\Delta t}\right)$ and parameter δ by the Crank- Nicholson formulation $\delta = \frac{\delta^n + \delta^{n+1}}{2}$ we obtain the $(N\text{x}2)$ x $(N \text{ x } 2)$ matrix system

$$[2A + \varepsilon Z_m B\Delta t + v\Delta t(C-E) - \mu\Delta t(D-F)]\,\delta^{n+1}$$
$$= [2A - \varepsilon Z_m B\Delta t - v\Delta t(C-E) + \mu\Delta t(D-F)]\,\delta^n \qquad (17)$$

where Δt is the time step. Applying the boundary conditions (4) to the system (17) the above matrix system is being square. The resulting matrices are asymmetrically banded but may be taken depleted septa-diagonal so are eficiently solved with a variant of the Thomas algorithm applying two or three inner iterations to $\delta^{n*} = \delta^n + \frac{(\delta^n - \delta^{n-1})}{2}$ at each time in order to improve the accuracy. A typical member of the matrix system(17) may be rewritten in terms of the nodal parameters δ_m^n as

$$\lambda_1\delta_{m-2}^{n+1} + \lambda_2\delta_{m-1}^{n+1} + \lambda_3\delta_m^{n+1} + \lambda_4\,\delta_{m+1}^{n+1} + \lambda_5\delta_{m+2}^{n+1} + \lambda_6\delta_{m+3}^{n+1} = \lambda_7\delta_{m-2}^n + \lambda_8\delta_{m-1}^n + \lambda_9\delta_m^n + \lambda_{10}\delta_{m+1}^n + \lambda_{11}\delta_{m+2}^n + \lambda_{12}\delta_{m+3}^n\,. \qquad (18)$$

The initial vector $\delta^0 = (\delta_{-1}^0, \delta_0^0, \dots\dots, \delta_{N+1}^0)$ is determined to iterate the system (17). So we rewrite the approximation over

the interval $[a,b]$ at time $t = 0$ as follows:
$U_N(x,0) = \sum_{m=-1}^{N+1}\phi_m(x)\delta_m^0.$

For this determination, the following two conditions are required:

$$U_N(x_m,0) = U(x_m,0), \qquad m = 0,1,\dots.,N\,,$$
$$U_{N_x}(x_0,0) = U_x(x_N,0) = 0\,.$$

Thus, the above conditions lead to a tridiagonal matrix system of the form

which can be solved using a variant of the Thomas algorithm.

$$\begin{bmatrix} -3 & 0 & 3 & & & & \\ 1 & 4 & 1 & & \cdots & & \\ & & \vdots & & \ddots & & \vdots \\ & & & & \cdots & & \\ & & & & 1 & 4 & 1 \\ & & & & -3 & 0 & 3 \end{bmatrix} \begin{bmatrix} \delta_{-1}^0 \\ \delta_0^0 \\ \vdots \\ \vdots \\ \vdots \\ \delta_N^0 \\ \delta_{N+1}^0 \end{bmatrix} = \begin{bmatrix} 0 \\ U(x_0) \\ \vdots \\ \vdots \\ \vdots \\ U(x_0) \\ 0 \end{bmatrix}$$

3. Sayısal sonuçlar

In this section, we have studied test problems concerning the KdV equation ($v = 0$), Burgers' equation ($\mu = 0$) and KdVB equation. L_2 error norms

$$L_2 = \|U^{exact} - U_N\|_2 \simeq \sqrt{h \sum_{j=1}^{N} \left|U_j^{exact} - (U_N)_j\right|^2} \; ,$$

and L_∞ error norms

$$L_\infty = \|U^{exact} - U_N\|_\infty \simeq \max_j \left|U_j^{exact} - (U_N)_j\right|,$$

$j = 1, 2, \dots, N-1,$

are used to measure the accuracy of the present algorithm and difference between analytical and numerical solutions at some specified times. We examine our results by calculating the following three conservative laws[13]:

$$I_1 = \int_a^b U\,dx \simeq h \sum_{J=1}^{N} U_j^n,$$

$$I_2 = \int_a^b U^{\,2} dx \simeq h \sum_{J=1}^{N} \left(U_j^n\right)^2,$$

$$I_3 = \int_a^b \left(U^3 - \frac{3\mu}{\varepsilon} U_x^2\right) dx \simeq h \sum_{J=1}^{N} \left[\left(U_j^n\right)^3 - \frac{3\mu}{\varepsilon} (U_x)_j^2\right],$$

which correspond to conversation of mass, momentum and energy, respectively. In the simulation of solitary wave motion, the invariants I_1, I_2 and I_3 are monitored to check the conversation of the numerical algorithm.

3.1 KdV tipi çözümler

If we take the parameters $v = 0,\ \mu = -1$ and $\varepsilon = -6$ in Eq. (3), the equation returns to KdV equation. As a first test problem, we will consider the KdV equation with the boundary conditions $U(0, t) = U(2, t) = 0$ and the initial condition

$$U(x,0) = 3C\,\mathrm{sec}\,h^2[Ax + D]$$

$$U_N(x_m, 0) = U(x_m, 0), \qquad m = 0, 1, \dots, N,$$

where $A = \frac{1}{2}\left(\varepsilon C/\mu\right)^{1/2}$, $C = 0.3,\ D = -6.$

An analytic solution of the KdV equation is given by
$$U(x,t) = 3C\,\mathrm{sec}\,h^2[Ax - Bt + D]$$

where $B = \varepsilon CA$. This solution corresponds to a single soliton with amplitude $3C$, locates initially at x_0 and moves to the right at speed εC in the positive x-direction. As it is known, the KdV equation describes the theory of water waves in shallow channels and exhibits special solutions, known as solitons, which are stable and do not disperse with time [21].

For comparison with earlier results, we take $\varepsilon = 1$, $\mu = 4.84 \times 10^{-4}$, $C = 0.3, D = -6$, $h = 0.001$, $\Delta t = 0.005$ and $\Delta t = 0.0005$. The run of the algorithm is continued up to time $t = 3$ over the problem region $0 \le x \le 2$. Values of the three invariants as well as L_2 and L_∞ error norms obtained from present method have been reported in Table 1.

As seen in Table 1, the error norms L_2 and L_∞ are found to be small enough, and the computed values of invariants are in good agreement with their analytical values $I_1 = 0.144598, I_2 = 0.086759, I_3 = 0.046850$. The percentage of the relative error of the conserved quantities I_1, I_2 and I_3 is calculated with respect to the conserved quantities at $t = 0$. Percentage of relative changes of I_1, I_2 and I_3 for $\Delta t = 0.005$ are obtained by $0.03\%, 0.00\%, 0.008\%$ respectively. Thus, the quantities in the invariants remain almost constant during the computer run. Perspective views of the traveling solitons are graphed at time $t = 0, 1, 2$ and 3 in Figure 1.

Table 1: Invariants and error norms for single solitary wave with $C = 0.3, h = 0.001, \Delta t = 0.005, 0 \le x \le 2$.

t	I_1	I_2	I_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
0	0.14459	0.08676	0.04685	0.00000	0.00000
1	0.14460	0.08676	0.04685	0.07839	0.20894
2	0.14460	0.08676	0.04685	0.13744	0.38602
3	0.14460	0.08676	0.04685	0.15426	0.42837
$t = 3$[13]	0.14460	0.08676	0.04685	0.02984	0.07525
$t = 3$[15]	0.14459	0.08675	0.04685	0.2516	0.6603
$t = 3$[24]	-	-	-	0.28	-

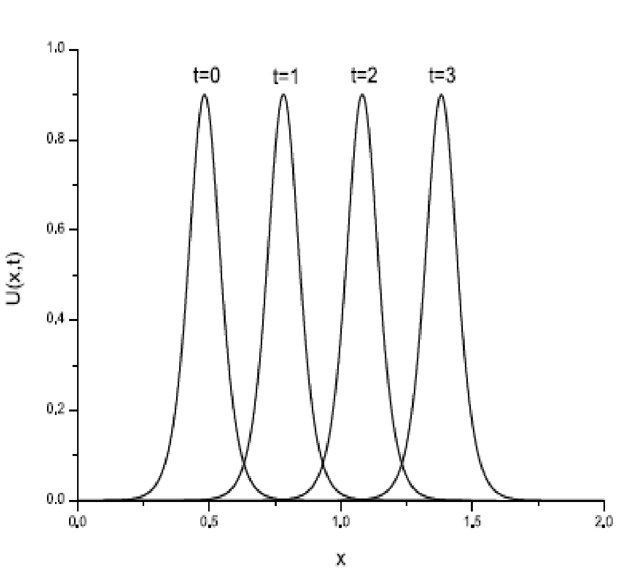

Figure 1: Single solitary wave with $C = 0.3,\ h = 0.001,\ \Delta t = 0.005,\ 0 \le x \le 2\ ,\quad t = 0, 1, 2$ and 3.

As seen from the figure that the solitons moves to the right at a constant speed and preserves its amplitude and shape with increasing time as expected.

As a second test problem, evolution of a train of solitary waves of the KdV equation has been studied using the Maxwellian initial condition

$$\mathrm{U}(\mathrm{x}, 0) = \exp(-\mathrm{x}^2) \qquad (19)$$

with boundary conditions
$\mathrm{U}(-15, \mathrm{t}) = \mathrm{U}(15, \mathrm{t}) = 0, \qquad \mathrm{t} > 0$

for different values of μ. For this problem, the behavior of the solution depends on the value of μ [22]. For $\mu \gg \mu c$, the Maxwellian initial condition does not break up into solitons but exhibits rapidly oscillating wave packets. When $\mu \simeq \mu c$ a mixed type of solution is found, which consists of a leading soliton and an oscillating tail. For $\mu < \mu c$, the Maxwellian breaks up into a number of solitons according to the values of μ.

For the purpose of comparison with results of recent works [13, 21], computations are carried out for the cases $\mu = 0.04, 0.01, 0.001$ and 0.0005 and simulations are run up to time $t = 12$ with $\varepsilon = 1, h = 0.02$ and $\Delta t = 0.03$. A single solitary wave and an oscillating tail is formed when $\mu = 0.04$ as shown in Fig.2(a). For $\mu = 0.01$, the Maxwellian initial pulse breaks up into a train of three solitons as drawn in Fig.2(b).

When $\mu = 0.001$ nine solitons are formed as depicted Fig.2(c). Finally for $\mu = 0.0005$ twelve solitons are formed as shown in Fig.2(d). These graphs are in agreement with the ones found in Refs. [13, 21].

The computed values of the invariants of motion for different values of μ are tabulated in Table 2. It is observed that the obtained values of the invariants remain almost constant during the computer run which are all in good agreement with the Refs. [13, 21].

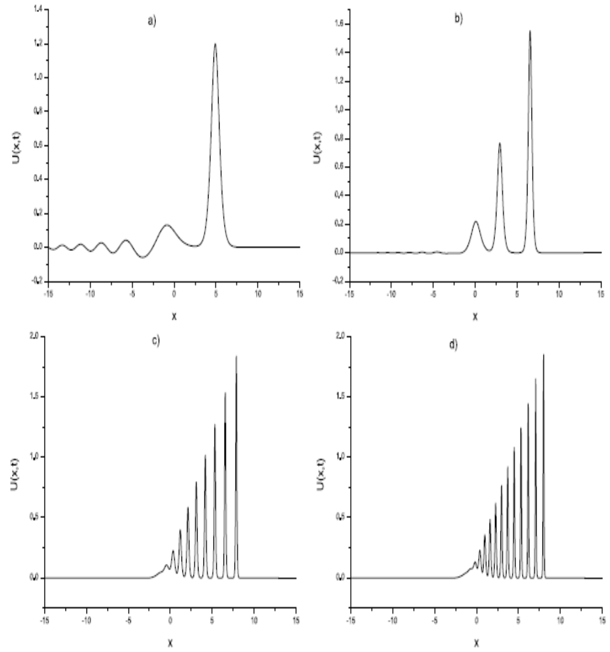

Figure 2: Maxwellian initial condition for KdV type solutions.

3.2 Burgers' tipi çözümler

By taking $\mu = 0$ and $\varepsilon = 1$ in Eq.(3), we obtain Burgers' equation. We consider the Burgers' equation with the boundary conditions
$U(a,\ t)\ =\ U(b,\ t)\ =\ 0 \qquad\qquad t \ge 1,$

and the initial condition at time $t\ =\ 1$ given by

$$U(x,1) = \frac{x}{1 + \exp\left[\frac{1}{4v}(x^2 - 1/4)\right]}$$

Burgers' equation has the following analytic solution [23],

$$U(x,t) = \frac{x/t}{1 + \sqrt{t/t_0}\, exp[x^2/4vt]},$$

where $t_0 = \exp(1/8v)$. This solution represents shock-like solutions of the Burgers' equation. For comparison with the relevant known results in [13, 15] we have used three set of the values $v = 0.5, h = \Delta t = 0.01$ and $0 \le x \le 10, v = 0.05, h = \Delta t = 0.01$ and $0 \le x \le 3,\ \ v = 0.005, h = 0.005, \Delta t = 0.01$ and $0 \le x \le 1.4$.

The computations are carried out for times up to time $t\ =\ 5$. Table 3 displays a comparison of the values of the error norms obtained by the present method with those obtained in Ref. [13, 15]. It can be seen from the Table 3 that the error norms L_2 and L_∞ are found to be small enough and they provide better results for the Ref. [15].

Figure 3 illustrates the propagation of shock for $v = 0.5, h = \Delta t = 0.01$ and $0 \le x \le 10, v = 0.05, h = \Delta t = 0.01$

And

$0 \le x \le 3,\ v = 0.005, h = 0.005, \Delta t = 0.01$ and $0 \le x \le 1.4$ at some different times, respectively.

Table 2: Invariants of Maxwellian initial condition with $\mu = 0.04, 0.01, 0.001$ and 0.0005.

t	μ	I_1	I_2	I_3	μ	I_1	I_2	I_3
0	0.4	1.7724545	1.2533143	0.8729292	0.001	1.7724545	1.0195668	1.0195668
3		1.7725009	1.2532528	0.8728428		1.7724323	1.2477440	1.0138905
6		1.7721406	1.2532358	0.8709166		1.7723277	1.2459644	1.0121144
9		1.7730989	1.2532011	0.7871368		1.7722212	1.2455925	1.0116252
12		1.7672991	1.2531067	0.7337504		1.7721151	1.2453424	1.0112155
0	0.01	1.7724545	1.2533143	0.9857274	0.00005	1.7724545	1.2533143	1.0214468
3		1.7724545	1.2528271	0.9852049		1.7721824	1.2419779	1.0099962
6		1.7724530	1.2526859	0.9850665		1.7710074	1.2364783	1.0029630
9		1.7724540	1.2526720	0.9850535		1.7698574	1.2368802	0.9985923
12		1.7724932	1.2526694	0.9848915		1.7687514	1.2316253	0.9845714

In this figure, both numerical and analytical solutions visualized at some times for $v = 0.5$ and $v = 0.005$ from which it has seen that initial shock becomes steadier for the larger viscosity $v = 0.5$ and sharpness remains the same for the smaller viscosity $v = 0.005$ as program runs. These graphs are in complete agreement with those reported by Refs.[15] and profiles of those solutions are in distinguishable.

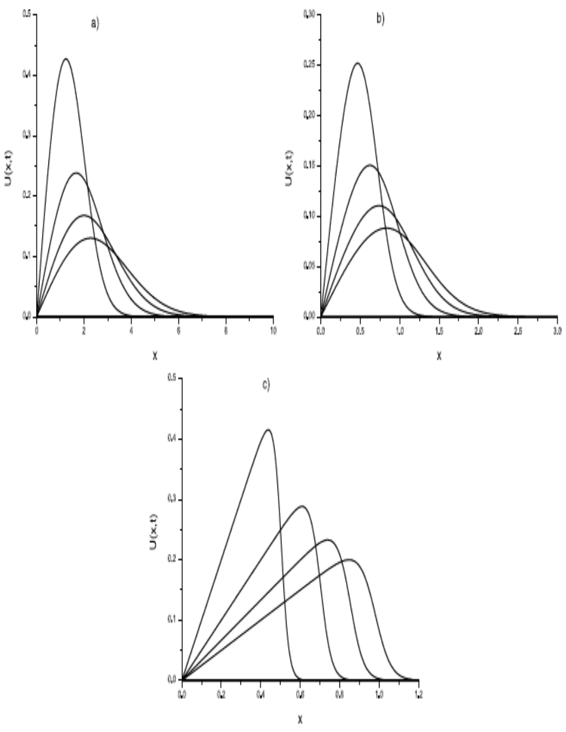

Figure 3: Burgers' type solutions different values for $v = 0.5$, 0.05 and 0.005 at time $t = 2, 3, 4, 5$.

3.3 KdVB tipi çözümleri

As a last problem, we have considered the behavior of the KdVB equation and have studied the effect of using different values of v on the solution vector. For this problem we use the initial condition [14].

$$U(x,0) = 0.5\left[1 - \tan h\,\frac{|x| - x_0}{d}\right]$$

and boundary conditions $U(-50,t) = U(150,t) = 0$. We have chosen an interval $-50 \le x \le 150$, in the time period $t \in [0, 800]$ for a simulation with $x_0 = 25$ and $d = 5$.

We have taken $\varepsilon = 0.2$, $\mu = 0.1, h = 0.05$, $\Delta t = 0.4$ and $v = 0,\ 0.0001,\ 0.005,\ 0.01,\ 0.03,\ 0.05,\ 0.2$ and 0.4 respectively, to see the effects of viscosity in Eq.(3).

Figure 4 illustrates the behaviour of the solutions for different values of v at time $t = 800$. As shown in Figure 4a, 4b, 4c, 4d solution of the KdVB equation behaves similarly to that of the KdV equation when small viscosities are used. In these cases Eq. (3) is a KdV type equation and a train of $10, 10, 9$ and 8 solitons are formed, respectively.

It is clearly observed from Figures 4g and 4h that as viscosity v increases the solution of KdVB equation tends to behave like the solution of Burgers' equation and solutions behaves like a travelling wave for which the amplitude is damped.

The values of the invariants, amplitude and peak position at time $t = 800$ for $v = 0$ are displayed in Table 4. It is clear from the Table that the conservation quantities are satisfactorily constant with the proposed algorithms.

Table 3: Error norms for different values of v.

t	$v = 0.5$		$v = 0.05$		$v = 0.005$	
	$L_2 \, x \, 10^5$	$L_\infty \, x \, 10^5$	$L_2 \, x \, 10^5$	$L_\infty \, x \, 10^5$	$L_\infty \, x \, 10^5$	$L_\infty \, x \, 10^5$
2	0.194	0.120	0.249	0.528	0.837	0.390
3	0.121	0.060	0.215	0.333	0.238	0.263
4	0.369	0.527	0.473	1.639	0.381	0.564
$t = 4[13]$	0.116	0.389	0.092	0.101	0.212	1.423
$t = 4[15]$	0.245	0.528	0.461	1.703	0.208	0.692

Table 4: Invariants for KdV type simulation with $h = 0.05$, $\Delta t = 0.4$, $\varepsilon = 0.2, \mu = 0.1, -50 \le x \le 150$ at time $t = 800$.

Present method			
t	I_1	I_2	I_3
0	50.00030	45.00057	42.30076
100	50.00041	45.00000	42.30013
200	50.00233	44.99839	42.29833
300	50.00386	44.99733	42.29503
400	49.99742	44.99689	42.29549
500	49.98584	44.99668	42.29716
600	49.97126	44.99660	42.29716
700	49.96681	44.99653	42.29153
800	49.97301	44.99635	42.28482
800[13]MQ	49.96331	44.99803	42.29974
800[15]	49.97291	45.00011	42.30072

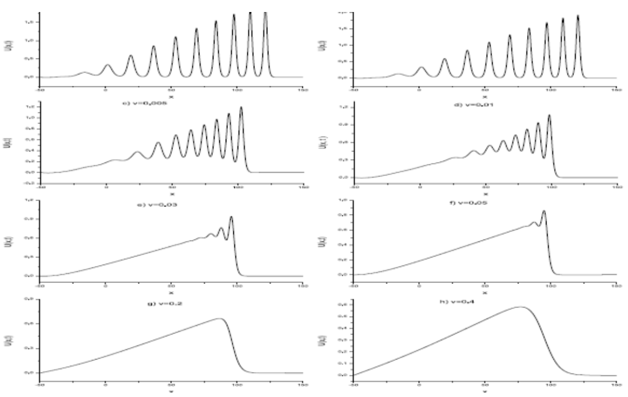

Figure 4: KdVB type solutions at time $t = 800$, different values for $a)\ v = 0\ b)\ v = 0.0001\ c)\ v = 0.005,\ d)\ v = 0.01,\ e)\ v = 0.03;\ f)\ v = 0.05,\ g)\ v = 0.2,\ h)\ v = 0.4$.

4. Sonuç

In this article, a numerical algorithm based on a Petrov-Galerkin method using quadratic weight functions and cubic B-spline shape functions has been successfully presented to obtain the numerical solutions of KdVB equation. To show the validity of the method and compare with earlier works we choose the appropriate test problems and observe the solutions under the different values of v and μ. It is shown that our scheme is accurate and efficient. It has been concluded that the numerical solutions tend to behave like Burgers' equation when diffusion dominates whereas KdV type behavior has been obtained when dispersion dominates. Our scheme for KdV and Burgers' equation is agreement with earlier schemes in the literature. The numerical method has been shown for the long runs, $t = 800$, considered with the simulations of the KdVB equation have assured us that the present method can be effectively used for long runs of the KdVB equation. So the method can be also used efficiently for solving a large number of physically important nonlinear problems.

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