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m-quasi Einstein Metric and Paracontact Geometry

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ABSTRACT

The object of the upcoming article is to characterize paracontact metric manifolds admitting m-quasi Einstein metric. First we establish that if the metric g in a K-paracontact manifold is the m-quasi Einstein metric, then the manifold is of constant scalar curvature. Furthermore, we classify (k,μ)-paracontact metric manifolds whose metric is m-quasi Einstein metric. Finally, we construct a non-trivial example of such a manifold.

Keywords: *m-quasi Einstein metric, K-paracontact metric manifolds, (k,μ)-paracontact metric manifolds.*
AMS Subject Classification (2020): *Primary: 53C15 ; Secondary: 53C25; 53D10.*

1. Introduction

Kaneyuki and Williams [14] introduced a new subclass of contact metric manifolds called paracontact metric manifolds, in 1985. Several authors researched and generalized the manifold after that. Many geometers considered the manifold to be an interesting topic (see, [1], [2], [9], [10], [11],[15]and references therein). Many relationships exist between paracontact geometry and various disciplines of mathematics, mathematical physics, and material sciences. It became popular among notable geometers due to its wide range of uses.

The study of Einstein metric in Riemannian and contact geometry play a significant role in recent geometrical research. Several generalizations of Einstein manifolds have recently been researched, including Ricci solitons, gradient Ricci solitons, generalized quasi-Einstein solitons, and so on. Catino [8] introduced the fascinating concept of generalized quasi-Einstein metric for investigating harmonic Weyl tensor, which is defined as follows (see [8]):

If a C^∞ manifold M^n, $n>2$, admits a Riemannian metric g satisfying

$$S+\nabla^2\delta=\alpha d\delta\otimes d\delta+\gamma g,$$

for some smooth functions α, δ and γ, is called a generalized quasi-Einstein metric. Here, S denotes the Ricci tensor, d indicates the exterior derivative of g, ∇^2 being the Hessian operator and $\otimes$ is the tensor product, respectively. If $\alpha=\frac{1}{m}$ and $\gamma\in\mathbb{R}$, where m is an integer, then the above metric reduces to a m-quasi Einstein metric. Analogous to the definition in [13], a Riemannian metric g is called a m-quasi Einstein metric if there exists a smooth function $\delta: M^n\to\mathbb{R}$ such that

$$S+\nabla^2\delta-\frac{1}{m}d\delta\otimes d\delta=\gamma g, \tag{1.1}$$

where γ is a constant.

The m-Bakry-Emery Ricci tensor, which is proportional to the metric g and $\gamma=$ constant [17], is expressed as $S+\nabla^2\delta-\frac{1}{m}d\delta\otimes d\delta$. Here, g, the Riemannian metric with constant potential function δ is trivial and therefore, the manifold becomes an Einstein manifold. Moreover, the previous equation produces the gradient Ricci soliton for $m=\infty$. For $m=1$, m-quasi Einstein metrics restore static metrics. These metrics have been thoroughly examined in general relativity.

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In the year 2011, Case et al. [6] studied quasi-Einstein metrics and establish many rigidity results. In this connection, we may mention the study of Case [5], where some non-existence outcomes were procured for quasi-Einstein metric. Further, for a complete K-contact manifold, Ghosh [12] established that the manifold is compact, Einstein, and Sasakian. Chen [7] has recently investigated quasi-Einstein structures in the setting of almost Cosymplectic manifolds.

The above mentioned works inspire us to investigate a m-quasi Einstein metric in paracontact metric manifolds. Especially, we characterize the m-quasi Einstein metric on K-paracontact metric manifolds and (k,μ) -paracontact metric manifolds. Exactly, we prove the subsequent results:

Theorem 1.1. *If the metric g is the m-quasi Einstein metric in a K-paracontact metric manifolds (M^{2n+1},g), then the manifold is of constant scalar curvature.*

Theorem 1.2. *Let M^{2n+1} be a (k,μ)-paracontact metric manifolds with $k\neq -1$. If the metric g is a m-quasi Einstein metric, then either the manifold is locally isometric to a product of a flat manifold of dimension $(n+1)$ and a manifold of dimension n with negative constant curvature -4 or M^{2n+1} is Einstein.*

Theorem 1.3. *If the metric g in a (k,μ)-paracontact metric manifolds (M^3,g) is the m-quasi Einstein metric, then either the scalar curvature is constant or the potential function δ remains invariant under the Reeb vector field ξ.*

2. Preliminaries

If a C^∞ manifold M^{2n+1} is endowed with a Reeb vector field ξ , a (1, 1)tensor ϕ, and a 1-form η fulfilling the following conditions

$$\phi^2 = -I + \eta\otimes\xi, \tag{2.1}$$

$$\phi\xi = 0, \quad \eta\circ\phi = 0, \quad \eta(\xi) = 1, \tag{2.2}$$

then it has an almost paracontact metric structure (ϕ,ξ,η) (see [4], [14] and the tensor field ϕ induces an almost paracomplex structure on each fibre of $\mathcal{D} = ker\,\eta$.

Furthermore, if a semi-Riemannian metric g obeys

$$g(\xi,E) = \eta(E), \quad g(E,F) + g(\phi E,\phi F) = \eta(E)\eta(F), \tag{2.3}$$

then (ϕ,ξ,η,g) is claimed to be an almost paracontact metric-structure and M an almost paracontact metric manifolds [16], for any vector fields E and $F\in\chi(M)$, where $\chi(M)$ indicates the collection of all C^∞ vector fields of M.

The Nijenhuis torsion is defined by

$$[\phi,\phi](E,F) = [\phi E,\phi F] + \phi^2[E,F] - \phi[E,\phi F] - \phi[\phi E,F]$$

for any $E,\ F\in\chi(M)$. If $\mathbf{N}_\phi = [\phi,\phi] - 2d\eta\otimes\xi$ vanishes, then the almost paracontact metric manifolds is called normal. $\Phi(E,F) = g(E,\phi F)$ is the fundamental 2-form of the almost paracontact metric manifolds and M equipped with structure (ϕ,ξ,η,g) is known as a paracontact metric manifolds if $d\eta(E,F) = g(E,\phi F)$.

Two trace-free operator $h = \frac{1}{2}\pounds_\xi\phi$ and $l = R(.,\xi)\xi$ which are symmetric in a paracontact metric manifolds satisfies $h\phi = -\phi h$, $h\xi = 0 = l\xi$, $Trh = Trh\phi = 0$ and

$$\nabla_E\xi = -\phi E + \phi hE, \ \ \nabla_\xi h = -\phi + \phi h^2 - \phi l, \tag{2.4}$$

for all $E\in\chi(M)$. It should be noticed that ξ being Killing is identical to the condition $h=0$, and then the manifold becomes a K-paracontact metric manifolds. A para-Sasakian manifold is one that satisfies the normality criterion in a paracontact metric manifolds. All para-Sasakian manifold is necessarily K-paracontact metric manifolds whereas the converse is not always true, but it holds in three dimension [2].

In a K-paracontact metric manifolds the following equations hold :

$$R(E,\xi)\xi = -E + \eta(E)\xi, \tag{2.5}$$

$$\nabla_E\xi = -\phi E, \tag{2.6}$$

$$Q\xi = -2n\xi \tag{2.7}$$

$$(\nabla_E \phi)F = R(\xi, E)F, \tag{2.8}$$

$$(\nabla_{\phi E}\phi)\phi F - (\nabla_E \phi)F = 2g(E,F)\xi - \{E + \eta(E)\xi\}\eta(F), \tag{2.9}$$

for any $E,\ F \in \chi(M)$, where Ricci operator Q of M is given by $g(QE, F) = S(E, F)$.

In [4], the authors presented the idea of (k,μ)-nullity distribution in a paracontact metric manifolds, where k , $\mu \in \mathbb{R}$ are constants. $\mathbf{N}(k,\mu)$ is the (k,μ)-nullity distribution of M, which is defined by

$$\mathbf{N}(k,\mu) : p \to \mathbf{N}_p(k,\mu) = \{Z \in T_pM \mid R(E,F)Z = (\mu h + kI)(g(F,Z)E - g(E,Z)F)\},$$

for all vector fields E, $F \in \chi(M)$.

Definition 2.1. If R, the curvature tensor in a paracontact metric manifolds satisfies

$$R(E,F)\xi = k(\eta(F)E - \eta(E)F) + \mu(\eta(F)hE - \eta(E)hF), \tag{2.10}$$

then for any $E, F \in \chi(M)$, it is called a (k,μ)-paracontact metric manifolds. A (k,μ)-paracontact metric manifolds is said to be proper if $k \neq 0$ and $\mu \neq 0$.

It's worth noting [3] that whereas $k \leq 1$ is required in the contact case, there is no similar requirement for k in the paracontact case. Furthermore, whereas $k = 1$ implies that the manifold is Sasakian in the contact case, $k = -1$ does not imply that the manifold is para-Sasakian in the paracontact case.

Presently we recollect some lemmas:

Lemma 2.1. *[4] The Ricci operator Q is written by*

$$\begin{aligned} QE =&[2-2n+n\mu]E - [2-2n-\mu]hE \\ &- [2-2n-2nk+n\mu)]\eta(E)\xi, \end{aligned} \tag{2.11}$$

for a (k,μ)-paracontact metric manifolds M^{2n+1} with $k \neq -1$.

Lemma 2.2. *[18, Theorem3.3] If a paracontact metric manifolds (M^{2n+1}, g) $n > 1$ obeys $R(E,F)\xi = 0$, then M^{2n+1}, is locally isometric to a product of a flat manifold of dimension $(n+1)$ and a manifold of dimension n with negative constant curvature -4.*

Lemma 2.3. *[12, Lemma 3.1] Every m-quasi Einstein metric satisfies the following:*

$$\begin{aligned} R(E,F)D\delta \quad = \quad & (\nabla_F Q)E - (\nabla_E Q)F + \frac{\gamma}{m}\{(F\delta)E - (E\delta)F\} \\ & + \frac{1}{m}\{(E\delta)QF - (F\delta)QE\}, \end{aligned} \tag{2.12}$$

for all $E,\ F \in \chi(M)$.

Zamkovoi [18] proved the following proposition :

Proposition 2.1. In a para-Sasakian manifold $\mathbf{M}^{2n+1}$, we have

$$S(E, \phi F) = -S(\phi E, F) - g(E, \phi F). \tag{2.13}$$

3. Proof of the main Theorems

3.1. Proof of Theorem 1.1

Here we first write the subsequent Lemma without proof (Since, in a K-paracontact metric manifolds ξ is Killing, the result can be obtained using (2.6)):

Lemma 3.1. *For a K-paracontact metric manifolds (M^{2n+1}, g), we have*

$$\nabla_\xi Q = Q\phi - \phi Q. \tag{3.1}$$

Hence, executing the inner product of (2.12) with the Reeb vector field ξ and using the above Lemma and (2.8) we obtain

$$\begin{aligned} -g((\nabla_F\phi)E, D\delta) &= g(\phi QF, E) + 2ng(\phi F, E) + \frac{\xi\delta}{m}g(QF, E) \\ &\quad -\frac{\gamma}{m}(\xi\delta)g(E, F) + \frac{\gamma+2n}{m}(F\delta)\eta(E). \end{aligned} \tag{3.2}$$

Using (2.9) in (3.2), we get

$$\begin{aligned} &2g(E, F)(\xi\delta) - [(F\delta) + \eta(F)(\xi\delta)]\eta(E) \\ &= g((\phi Q + Q\phi)F, E) + 4ng(\phi F, E) + \frac{\xi\delta}{m}g(QF + \phi Q\phi F, E) \\ &-2\frac{\gamma}{m}(\xi\delta)g(E, F) + \frac{\gamma+2n}{m}(F\delta)\eta(E) + \frac{\gamma}{m}(\xi\delta)\eta(E)\eta(F). \end{aligned} \tag{3.3}$$

Anti-symmetrizing the previous equation gives

$$\begin{aligned} &(E\delta)\eta(F) - (F\delta)\eta(E) \\ &= 2g((\phi Q + Q\phi)F, E) + 8ng(\phi F, E) \\ &+\frac{\gamma+2n}{m}[(F\delta)\eta(E) - (E\delta)\eta(F)]. \end{aligned} \tag{3.4}$$

Replacing F by ξ and using Proposition 2.1 the foregoing equation provides

$$\{\frac{\gamma+2n}{m} + 1\}[(\xi\delta)\eta(E) - (E\delta)] = 0. \tag{3.5}$$

This shows that either $\gamma + 2n + m \neq 0$ or $\gamma + 2n + m = 0$.

Case (i): If $\gamma + 2n + m \neq 0$, then we have $(E\delta) = (\xi\delta)\eta(E)$. From this we have $d\delta = (\xi\delta)\eta$, where d indicates the exterior differentiation. Again, exterior derivative of the last equation produces $d^2\delta = d(\xi\delta) \wedge \eta + (\xi\delta)d\eta$. Applying Poincare lemma $d^2 \equiv 0$ in the previous equation and then taking wedge product with η we get $(\xi\delta)\eta \wedge d\eta = 0$. Therefore, $\xi\delta = 0$, since $\eta \wedge d\eta \neq 0$ in M^{2n+1}. Thus we conclude that $d\delta = 0$ and hence δ is constant. Then from equation (1.1) we infer that M^{2n+1} is Einstein which implies that the scalar curvature is constant.

Case (ii): Now we consider the case $\gamma + 2n + m = 0$. Equation (1.1) can be written as

$$\nabla_E D\delta + QE = \frac{1}{m}(E\delta)D\delta + \gamma E. \tag{3.6}$$

Using (2.7) and (3.6), we can easily get

$$g(\nabla_\xi D\delta, \xi) = 2n + \frac{(\xi\delta)^2}{m} + \gamma. \tag{3.7}$$

Again, taking covariant differentiation of $g(\xi, D\delta) = (\xi\delta)$ along ξ, we easily infer $g(\nabla_\xi D\delta, \xi) = \xi(\xi\delta)$, where we have used $\nabla_\xi\xi = 0$. Therefore

$$\xi(\xi\delta) = 2n + \frac{(\xi\delta)^2}{m} + \gamma. \tag{3.8}$$

Now contracting (2.12) over E, we get

$$S(F, D\delta) = \frac{Fr}{2} + \frac{2n\gamma}{m}(F\delta) + \frac{1}{m}((QF)\delta) - \frac{r}{m}(F\delta). \tag{3.9}$$

Replacing $F = \xi$ in the foregoing equation, we obtain

$$(2n\gamma + 2nm - 2n - r)(\xi\delta) = 0. \tag{3.10}$$

Applying $\gamma = -(m + 2n)$, the previous equation reduces to $[r + 2n(2n + 1)](\xi\delta) = 0$. If $(\xi\delta) = 0$, then from (3.8) we have $\gamma = -2n$, which implies $m = 0$, a contradiction. Hence, we have $r = -2n(2n + 1) =$ constant.

Hence, the proof is finished.

3.2. *Proof of Theorem* 1.2

First, from the equation (2.11) we obtain $Q\xi = 2nk\xi$. Using (2.4) and differentiating $Q\xi = 2nk\xi$ along the vector field E, we infer

$$(\nabla_E Q)\xi = Q(\phi - \phi h)E - 2nk(\phi - \phi h)E. \tag{3.11}$$

Executing inner product operation of (2.12) and applying $Q\xi = 2nk\xi$, we lead

$$\begin{aligned} g(R(E,F)D\delta,\xi) &= g((\nabla_F Q)E - (\nabla_E Q)F,\xi) \\ &\quad + \frac{\gamma - 2nk}{m}\{(F\delta)\eta(E) - (E\delta)\eta(F)\}. \end{aligned} \tag{3.12}$$

Again, from (2.10), we get

$$g(R(E,F)D\delta,\xi) = -k\{\eta(F)(E\delta) - \eta(E)(F\delta)\} - \mu\{\eta(F)(hE\delta) - \eta(E)(hF\delta)\}. \tag{3.13}$$

Combining equation (3.12) and (3.13) reveal that

$$\begin{aligned} &-k\{\eta(F)(E\delta) - \eta(E)(F\delta)\} - \mu\{\eta(F)((hE)\delta) - \eta(E)((hF)\delta)\} \\ &= g((\nabla_F Q)E - (\nabla_E Q)F,\xi) \\ &+ \frac{\gamma - 2nk}{m}\{(F\delta)\eta(E) - (E\delta)\eta(F)\}. \end{aligned} \tag{3.14}$$

Now from (3.11), we can easily get $g((\nabla_F Q)\xi,\xi) = 0$ and $g((\nabla_\xi Q)F,\xi) = 0$. Using this and replacing F by ξ in the preceding equation, we obtain

$$hE\delta = \frac{km - \gamma + 2nk}{m\mu}[\eta(E)(\xi\delta) - (E\delta)]. \tag{3.15}$$

Differentiating the foregoing equation and applying (2.4), (3.6) and (3.15), we infer

$$\begin{aligned} &(\nabla_E h)D\delta - hQE + \frac{k_1}{m}(E\delta)(\xi\delta)\xi + \gamma hE \\ &= k_1\{(-\phi E + \phi hE) - \gamma E + QE + (E(\xi\delta))\xi\}, \end{aligned} \tag{3.16}$$

where $k_1 = \frac{km-\gamma+2nk}{m\mu}$ = constant. Also, equation (3.6) reveals that

$$\xi(\xi\delta) + 2nk - \frac{(\xi\delta)^2}{m} - \gamma = 0, \tag{3.17}$$

where we have used $Q\xi = 2nk\xi$.

Parallelly, from (2.10) we have $l = k\phi^2 + \mu h$. Using this and $h^2 = (k+1)\phi^2$ in (2.4), we obtain $\nabla_\xi h = -\mu h\phi$. Putting E by ξ in (3.16) and using (3.17), $Q\xi = 2nk\xi$, $\nabla_\xi h = -\mu h\phi$ it readily follows $\mu hD\delta = 0$. This shows that either $\mu = 0$ or $\mu \neq 0$.

Case (i): In this case $\mu = 0$. Replacing E, F by ϕE and ϕF respectively in (3.12) and using (2.11), $h^2 = (k+1)\phi^2$ we easily get $k = \frac{\mu(n+1)}{2-\mu}$. Applying $\mu = 0$, it readily follows $k = 0$. If $k = 0$, then the equation (2.10) yields $R(E,F)\xi = 0$. Therefore, from Lemma 2.2 we state that M^{2n+1}, $n > 1$ is locally isometric to a product of a flat manifold of dimension $(n+1)$ and a manifold of dimension n with negative constant curvature -4.

Case (ii): If $\mu \neq 0$, then $hD\delta = 0$. Now, operating h and using $h^2 = (k+1)\phi^2$, we have $(k+1)\phi^2 D\delta = 0$. Since $k \neq -1$, we obtain $d\delta = (\xi\delta)\xi$ and hence δ is constant, following the proof of the previous theorem (in case (i)). Hence the manifold is Einstein.

Thus the proof is completed.

3.3. *Proof of Theorem* 1.3

Let us consider M^3, a (k,μ)-paracontact metric manifolds of dimension 3. In M^3, the subsequent relations hold [2]:

$$(\nabla_\xi h)E = \mu h(\phi E), \tag{3.18}$$

$$(\nabla_E \eta)F = -g(\phi E, F) + g(\phi hE, F), \tag{3.19}$$

$$\phi\xi = 2k\xi. \tag{3.20}$$

It is well known that in a Riemannian or a semi-Riemannian manifold of dimension 3, the curvature tensor takes the form

$$\begin{aligned} R(E,F)Z &= g(F,Z)QE-g(E,Z)QF+S(F,Z)E-S(E,Z)F \\ &\quad -\frac{r}{2}[g(F,Z)E-g(E,Z)F]. \end{aligned} \tag{3.21}$$

Replacing Z by ξ in (3.21) and using (2.10) we get

$$\begin{aligned} &k(\eta(F)E-\eta(E)F)+\mu(\eta(F)hE-\eta(E)hF) \\ &=\eta(F)QE-\eta(E)QF+(2k-\frac{r}{2})[\eta(F)E-\eta(E)F]. \end{aligned} \tag{3.22}$$

Putting $F=\xi$ in (3.22) and applying (3.20) and $h\xi=0$, we obtain

$$QE=(\frac{r}{2}-k)E+(3k-\frac{r}{2})\eta(E)\xi+\mu hE, \tag{3.23}$$

that is,

$$S(E,F)=(\frac{r}{2}-k)g(E,F)+(3k-\frac{r}{2})\eta(E)\eta(F)+\mu g(hE,F). \tag{3.24}$$

Now we establish the following lemma:

Lemma 3.2. *In $M^3(\eta,\xi,\phi,g)$, we have*

$$\xi r=0 \tag{3.25}$$

Proof. Differentiating (3.23) covariantly in the direction of E and using (2.4) and (3.19), we infer

$$\begin{aligned} (\nabla_E Q)F &= \frac{dr(E)}{2}(F-\eta(F)\xi) \\ &\quad (-\frac{r}{2}+3k)[-g(\phi E,F)\xi+g(\phi hE,F)\xi \\ &\quad -\eta(F)\phi E+\eta(F)\phi hE]+\mu(\nabla_E h)F. \end{aligned} \tag{3.26}$$

From (3.26), we can write

$$\begin{aligned} g((\nabla_E Q)F,Z) &= \frac{dr(E)}{2}[g(F,Z)-\eta(F)\eta(Z)] \\ &\quad (-\frac{r}{2}+3k)[-g(\phi E,F)\eta(Z)+g(\phi hE,F)\eta(Z) \\ &\quad -\eta(F)g(\phi E,Z)+\eta(F)g(\phi hE,Z)]+\mu g((\nabla_E h)F,Z). \end{aligned}$$

Putting $E=Z=e_i$ in the foregoing equation and summing over i $(1\le i\le 2n+1)$ and using $divQ=\frac{1}{2}grad\ r$ and (3.18), where $\{e_i\}$ is the orthonormal basis for the tangent space of M, we get $\xi r=0$, the required result. □

Now contracting (2.12) over the vector field E, we obtain

$$S(F,D\delta)=\frac{Fr}{2}+\frac{2\gamma}{m}(F\delta)+\frac{1}{m}((QF)\delta)-\frac{r}{m}(F\delta). \tag{3.27}$$

Again, from (3.24) we infer that

$$S(F,D\delta)=(\frac{r}{2}-k)(F\delta)+(3k-\frac{r}{2})\eta(F)(\xi\delta)+\mu((hF)\delta). \tag{3.28}$$

Combining the previous two equations and putting $F=\xi$ reveal that

$$(r+2mk-2k-2\gamma)(\xi\delta)=0. \tag{3.29}$$

This shows that either $r = 2k(1-m) + 2\gamma$ or $r \neq 2k(1-m) + 2\gamma$.

Case (i): If $r = 2k(1-m) + 2\gamma$, then M^3 is of constant scalar curvature, since k, m, γ are constant.

Case (ii): If $r \neq 2k(1-m) + 2\gamma$, then $\xi\delta = 0$.

This finishes the proof.

The m-quasi Einstein metric becomes gradient Ricci soliton when $m = \infty$, as we know. By putting $m = \infty$ into (3.27), we get

$$S(F, D\delta) = \frac{Fr}{2}. \tag{3.30}$$

Combining the last equation with the equation (3.28) and replacing $F = \xi$ infer that

$$k(\xi\delta) = 0. \tag{3.31}$$

Hence, we state:

Corollary 3.1. *If in a proper (k, μ)-paracontact metric manifolds (M^3, g), the metric g be the gradient Ricci soliton, then the potential function δ remains invariant under the Reeb vector field ξ.*

4. Example of a proper (k, μ)-paracontact manifold

We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3\}$. Let the three linearly independent vector fields e_1, e_2 and e_3 obeying

$$[e_1, e_2] = -2e_3 \quad , [e_1, e_3] = 2e_2$$

and

$$[e_2, e_3] = 2e_1,$$

generates M.

Let g be the semi-Riemannian metric defined by

$$g(e_1, e_1) = 1, g(e_2, e_2) = -1, g(e_3, e_3) = 1,$$

$$g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0.$$

Here, the 1-form η is defined by $\eta(E) = g(E, e_1)$ for any vector field $E \in \chi(M)$.

Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi(e_1) = 0, \quad \phi(e_2) = e_3, \quad \phi(e_3) = e_2.$$

Then using the linearity of ϕ and g we obtain

$$\eta(e_1) = 1,$$
$$\phi^2 E = E - \eta(E)e_1,$$
$$g(\phi E, \phi F) = -g(E, F) + \eta(E)\eta(F)$$

for any vector fields $E, F \in \chi(M)$.

Then for $e_1 = \xi$, the structure (η, ξ, ϕ, g) defines a paracontact structure on M.

Let $e_1 = \xi$ and making use of Koszul's formula, we calculate the following

$$\begin{gathered} \nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_2 = -e_3, \quad \nabla_{e_1} e_3 = -e_2, \\ \nabla_{e_2} e_1 = e_3, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_3 = -e_1, \\ \nabla_{e_3} e_1 = -3e_2, \quad \nabla_{e_3} e_2 = -3e_1, \quad \nabla_{e_3} e_3 = 0. \end{gathered} \tag{4.1}$$

Applying (2.4) from the above we get $he_1 = 0$, $he_2 = 2e_2$ and $he_3 = -2e_3$.

With the help of the (4.1) it is simple to verify that

$$\begin{gathered} R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_3 = -e_2, \quad R(e_1, e_3)e_3 = -e_1, \\ R(e_1, e_2)e_2 = -7e_1, \quad R(e_2, e_3)e_2 = -e_3, \quad R(e_1, e_3)e_2 = 0, \\ R(e_1, e_2)e_1 = -7e_2, \quad R(e_2, e_3)e_1 = 0, \quad R(e_1, e_3)e_1 = e_3. \end{gathered}$$

Using (2.10), from the above we easily infer that the constructed manifold is a (k, μ)-paracontact manifold where $k = 3$ and $\mu = 2$.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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